An all-pass filter has a magnitude response that is constant for all frequencies. The phase response may or may not be linear.

A simple all-pass filter is the gain, i.e. \( y(t) = g \times x(t) \) or \( y[n] = g \times x[n] \) where \( x \) is the input and \( y \) is the output. Impulse response is \( h(t) = g \delta(t) \) or \( h[n] = g \delta[n] \). The frequency response is simply equal to \( g \).

Another simple all-pass filter is the ideal delay, i.e. \( y(t) = x(t-t_0) \) or \( y[n] = x[n-n_0] \) where \( t_0 \) and \( n_0 \) are constants. Impulse response is \( h(t) = \delta(t-t_0) \) or \( h[n] = \delta[n-n_0] \). The frequency response is \( H \text{ freq}(t) = e^{-j\omega t_0} \) or \( H \text{ freq}(\omega) = e^{-j\omega n_0} \). The magnitude response is equal to one in either case. Phase response is linear.

A cascade of a gain and an ideal delay also has an all-pass response.

A first-order IIR filter with one real-valued pole and one real-valued zero is all-pass if the zero location is equal to the reciprocal of the pole location:

\[
H(z) = \frac{z - \frac{1}{r}}{z - r} \quad \Rightarrow \quad H_{\text{freq}}(\omega) = \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r}
\]

assuming that \( |r| < 1 \) for asymptotic stability. Magnitude response is

\[
|H_{\text{freq}}(\omega)| = \left| \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r} \right| = \left| \frac{e^{j\omega} - \frac{1}{r}}{e^{j\omega} - r} \right|
\]

Here, \( |e^{j\omega} - a| = \sqrt{(\cos\omega - a)^2 + \sin^2\omega} = \sqrt{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega} = \sqrt{a^2 - 2a\cos\omega + 1} \).
A first-order IIR filter with one complex-valued pole and one complex-valued zero is all-pass if the zero radius is the reciprocal to the pole radius and if the angles are the same:

\[
H(z) = \frac{z - \frac{1}{r_0} e^{j\omega_0}}{z - r_0 e^{j\omega_0}} \Rightarrow H_{\text{freq}}(\omega) = \frac{e^{j\omega} - \frac{1}{r_0} e^{j\omega_0}}{e^{j\omega} - r_0 e^{j\omega_0}}
\]

assuming that \( r_0 < 1 \) for asymptotic stability. The magnitude response is

\[
\left| H_{\text{freq}}(\omega) \right| = \frac{|e^{j\omega} - \frac{1}{r_0} e^{j\omega_0}|}{|e^{j\omega} - r_0 e^{j\omega_0}|}
\]

Here, \( |e^{j\omega} - (a+jb)| = |(\cos \omega - a) + j(\sin \omega - b)| \)

\[
= \sqrt{(\cos \omega - a)^2 + (\sin \omega - b)^2}
\]

\[
= \sqrt{\cos^2 \omega - 2ac \cos \omega + a^2 + \sin^2 \omega - 2b \sin \omega + b^2}
\]

\[
= \sqrt{(a^2 + b^2) - 2 \frac{ac + b}{2} \cos (\omega + \theta) + 1}
\]

where \( \theta = \arctan\left(-\frac{b}{a}\right) \).

\[
\left| H_{\text{freq}}(\omega) \right| = \frac{\sqrt{\frac{1}{r_0^2} - 2r_0 \cos (\omega + \theta) + 1}}{\sqrt{r_0^2 - 2r_0 \cos (\omega + \theta) + 1}}
\]

where \( \theta = \arctan\left(-\frac{\sin \omega_0}{r_0 \cos \omega_0}\right) = -\omega_0 \)

\( \varphi = \arctan\left(-\frac{\sin \omega_0}{r_0 \cos \omega_0}\right) = \omega_0 \)

Therefore,

\[
\left| H_{\text{freq}}(\omega) \right| = \frac{\sqrt{\frac{1}{r_0^2} - 2r_0 \cos (\omega - \omega_0) + 1}}{\sqrt{r_0^2 - 2r_0 \cos (\omega - \omega_0) + 1}}
\]

\[
= \frac{1}{r_0} \left( \frac{r_0^2 - 2r_0 \cos (\omega - \omega_0) + 1}{\sqrt{r_0^2 - 2r_0 \cos (\omega - \omega_0) + 1}} \right) = \frac{1}{r_0}
\]