Problem 2.4 Sigma-Delta Modulation. 20 points.

Shown below is a type of sigma-delta modulator called a noise-shaping feedback coder.

We can approximate the effect of the quantizer as a gain $K$, which would make the overall system linear and time-invariant. Replace the quantizer with a gain of $K$ and derive the transfer function from input $x[n]$ to output $y[n]$.

\[
Y(z) = K x'[z]
\]
\[
x'[z] = x[z] - H(z) e[z]
\]
\[
e[z] = Y[z] - x'[z]
\]
\[
x'[z] = x[z] - H(z) \left[ Y[z] - x'[z] \right]
\]
\[
x[z] = \frac{x[z] - H(z) Y[z]}{1 - H(z)}
\]
\[
Y[z] = K \left[ \frac{x[z] - H(z) Y[z]}{1 - H(z)} \right]
\]
\[
Y[z] \left[ 1 + \frac{K H(z)}{1 - H(z)} \right] = \frac{K x[z]}{1 - H(z)}
\]
\[
Y[z] = \frac{K x[z]}{(1 - H(z))} \frac{1 - H(z) + K H(z)}{(1 - H(z))}
\]
\[ Y[z] = \frac{k \times [z]}{1 + H(z)(k-1)} \]

Therefore
\[ \frac{Y[z]}{X[z]} = \frac{K}{1 + (k-1)H(z)} \]
Problem 2.4 Sigma-Delta Modulation. 15 points.

Shown below is a type of continuous-time sigma-delta modulator:

We can approximate the effect of the quantizer as a gain $K$, which would make the overall system linear and time-invariant. Replace the quantizer with a gain of $K$ and derive the transfer function from input $x(t)$ to output $y(t)$. The LTI system shown graphically as an integral sign is an integrator.

$$R(s) = \overline{X}(s) - \overline{V}(s)$$

$$\overline{V}(s) = \frac{1}{s} R(s)$$

$$\overline{Y}(s) = K \overline{V}(s)$$

Combining these three equations:

$$\overline{Y}(s) = K \cdot \frac{1}{s} \left( \overline{X}(s) - \overline{V}(s) \right)$$

$$(1 + \frac{K}{s}) \overline{Y}(s) = \frac{K}{s} \overline{X}(s)$$

$$\frac{\overline{Y}(s)}{\overline{X}(s)} = \frac{\frac{K}{s}}{1 + \frac{K}{s}} = \frac{K}{s + K}$$