**Solution Set for Homework #3 on Fourier Series and Sampling**

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***PROBLEM 1: FOURIER ANALYSIS AND SYNTHESIS***

**Prologue:** The purpose of this problem is to use properties of the continuous-time Fourier series in computing the Fourier series coefficients. Throughout the remainder of the course, we’ll be using properties of continuous-time Fourier transforms and other transforms to simplify the computation of the transform.

**Problem:** *Signal Processing First*, problem P-3.14, page 67. The problem gives an example of a signal that has period *T*0 and another signal . The Fourier series coefficients for can be computed from the Fourier series coefficients for using where .

**Solution for part (a):** Here are two different solutions for .

Solution #2 for part (a)

Let :

Solution #1 for part (a)

Let :

When scaling any signal in amplitude, the Fourier Series coefficients are scaled by the same amount.

**Solution for part (b):** Here are two different solutions for .

Solution #1 for part (b)

Let :

Solution #2 for part (b)

Let :

Using a substitution of variables with and . The limits of integration becomes and becomes ,

When delaying a signal, the Fourier Series coefficients are multiplied by . This is another example of a shift in time causing shift in phase.

Using the conclusion derived in parts (a) and (b) with *A* = 2 and *td* = ¼ *T*0,

Given

1. Below, the plots of and are plotted for two periods to better show the shift in time . Note the doubling in amplitude for .

Graphical user interface, application, Word

Description automatically generated

% Fourier synthesis for square wave

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% The University of Texas at Austin

% Written in Fall 2017

% Version 2.0

%

% Fourier series coefficients ak for a square

% wave with period T0 that is

% 1 for 0 <= t < T0/2

% 0 for T0/2 <= t < T0

%

% Derivation is in Sec. 3-6.1 in Signal

% Processing First (2003) on pages 52-53

% Pick a value for the period of x(t)

T0 = 1;

f0 = 1 / T0;

% Pick number of terms for Fourier synthesis

N = 10;

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Description automatically generatedfmax = N \* f0;

% Define a sampling rate for plotting

fs = 24 \* fmax;

Ts = 1 / fs;

% Define samples in time for one period

%t = -0.5\*T0 : Ts : 0.5\*T0;

t = -T0 : Ts : T0;

% First Fourier synthesis term

a0 = 0.5;

b0 = 2\*a0;

x = a0 \* ones(1, length(t));

y = b0 \* ones(1, length(t));

figure;

plot(t, y);

ylabel('Square Wave Delayed by T0/4 and scaled by 2')

hold on;

% Generate each pair of synthesis terms

for k = 1 : N

% Define Fourier coefficients at k and -k

akpos = (1 - (-1)^k) / (j\*2\*pi\*k);

akneg = (1 - (-1)^(-k)) / (j\*2\*pi\*(-k));

bkpos = 2\*(exp(-j\*2\*pi\*k\*(1/4)\*T0))\*akpos;

bkneg = 2\*(exp(-j\*2\*pi\*(-k)\*(1/4)\*T0))\*akneg;

theta = j\*2\*pi\*k\*f0\*t;

x = x + akpos \* exp(theta) + akneg \* exp(-theta);

y = y + bkpos \* exp(theta) + bkneg \* exp(-theta);

% Plot Fourier synthesis for indices -k ... k

plot(t, y);

pause(0.5);

end

hold off;

figure;

plot(t, x);

ylabel('Original Square Wave')

***PROBLEM 2: SAMPLING***

**Prolog:** Periodicity is a bit different for discrete-time signals than continuous-time signals because the discrete-time domain is on an integer grid whereas the continuous-time domain is on a real number line.

**Problem:** *Signal Processing First*, problem P-4.2, page 96, with an additional part (d).

In the continuous-time domain, the fundamental period is (2/11) seconds:

Due to sampling at *f*s = 10 Hz, = :

A = 7, ϕ =

Due to sampling at *f*s = 5 Hz, = :

This signal is undersampled, because *f*0 > *f*s / 2. The following equation shows the effect of aliasing (but not related to folding) caused by the undersampling:

A = 7, ϕ =

This signal is 15/11 times oversampled because *f*0 < *f*s / 2

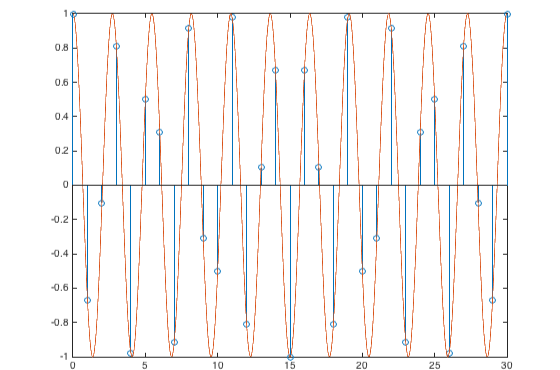
A = 7, ϕ =

1. As shown at the beginning of this problem’s solution:

and

According to the hint that is provided for this solution, which comes from [Handout D on Discrete-Time Periodicity](http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20D%20Discrete-Time%20Periodicity.pdf), *x*[*n*] is periodic with a discrete-time period of *N*0 samples if  
 for all possible integer values of .

Because 11 and 30 are relatively prime, the smallest possible positive integer for is 30 samples. Therefore, the fundamental period of *x*[*n*] is 30 samples. Those 30 samples contain 11 continuous-time periods, which corresponds to 2.67 samples in each continuous-time period.

Although not required, here’s a way to visualize differences in periodicity by superimposing plots of *x*(*t*) and *x*[*n*]. In *x*[*n*], the amplitude of 1 at  
*n* = 0 does not repeat until *n* = 30.

fs = 15;  
Ts = 1/fs;  
wHat = 2\*pi\*f0/fs;

N0 = 30;  
n = 0 : N0;  
yofn = cos(wHat\*n);

t = 0 : 0.01 : N0;  
yoft = cos(wHat\*t);  
figure;  
stem(n, yofn);  
hold;  
plot(t, yoft);

**Epilogue:** For a sinusoidal signal with discrete-time frequency where the common factors in and have been removed so that *N* and *L* are relatively prime, the discrete-time signal has a fundamental period of *L* samples. The fundamental period of *L* samples contains *N* periods of a continuous-time sinusoid with frequency . Please see [Handout D on Discrete-Time Periodicity](http://users.ece.utexas.edu/~bevans/courses/signals/handouts/Appendix%20D%20Discrete-Time%20Periodicity.pdf).