Homework: Logic Functions

1. We say a variable x is in the **support** of a Boolean function f if $f_x \neq f_{\bar{x}}$. How many Boolean functions of three variables have three variables in their support?

   Note that a Boolean formula involving three variables does not necessarily imply that all three variables are in the support of the function denoted by the formula; consider $f = xyz + xy\bar{z}$ which can be simplified to $f = xy$.

   15 marks

2. Consider a single output completely specified Boolean function f. Let F be a cover of F. If F is unate in a variable, must f be unate in that variable? What about the converse?

   5 marks

3. Consider the completely specified function whose on-set is $\{xyzw\} = \{0110, 0111, 1000, 1001, 1101, 1110, 1111\}$.

   (a) List all the implicants of the function.

   (b) List all the primes of the function.

   (c) List all the essential primes of the function if they exist. For each essential prime, list all the essential vertices.

   (d) The set of all the primes forms a cover of this function. Is this cover irredundant? If not, make it so by removing some primes.

   Repeat the above questions except (a) for the incompletely specified function whose on-set is the same as the above and don't care set is $\{xyzw\} = \{1010, 1011\}$.

   25 marks

4. Prove the following properties of cofactors.

   (a) $\big((f)_x\big)_y = \big((f)_y\big)_x$

   (b) $(f+g)_x = f_x + g_x$

   (c) $(fg)_x = f_x g_x$

   (d) $(\overline{f})_x = \overline{(f_x)}$

   (e) $x \cdot f = x \cdot f_x$

   15 marks

5. The functions f_x and $f_{x'}$ can be combined logically in many ways. Two combinations that are extremely useful are based on OR and AND. Specifically, define $\exists x.f \triangleq f_x + f_{x'}$, and $\forall x.f \triangleq f_x \cdot f_{x'}$.

   (a) Prove that the operations are dual in that $\exists.f = \big(\forall x.(f)'\big)'$

   (b) Prove that $\exists x.f$ is the smallest function independent of x which contains f, and that $\forall x.f$ is the largest function independent of x which is contained in f. **15 marks**

6. A function $f : 2^N \mapsto 2$ is said to be *symmetric with respect to variables* x_i and x_j, if for every $(\alpha_0, \alpha_1, \ldots, \alpha_{N-1} \in 2^N$, it is the case that

$$f(\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_j, \ldots, \alpha_{N-1}) = f(\alpha_0, \alpha_1, \ldots, \alpha_j, \ldots, \alpha_i, \ldots, \alpha_{N-1})$$

. For example $f = x_0 x_1 + x_2$ is symmetric with respect to x_0 and x_1; however, it is not symmetric with respect to x_0 and x_2 (because for example, $f(0,0,1) \neq f(1,0,0)$).

We say f is *symmetric* if it is symmetric with respect to every pair of variables. For example, $f = x_0 + x_1 + \cdots + x_{N-1}$ is symmetric; AND, and parity are also symmetric.

Show that if f is symmetric, then it can always be implemented with a logic circuit containing $O(N \cdot \lg N)$ 2-i/p NAND gates.

15 marks

7. ($\star$-credit) Let A be a set of m distinct Boolean vectors of dimension n. A set $S \subset \{1, \ldots, n\}$ is defined to be a *witness* for a vector u in A if for every other $v \in A$, there is a coordinate in S on which u differs from v.

As an example, consider the set $\{0000, 0010, 1001\}$. Then $\{1, 3\}$ is a witness for 0000, since 0000 differs from 0010 in coordinate 3, and from 1001 in coordinate 1. (Since 0000 differs from 1001 in coordinate 4 in addition to coordinate 1, $\{3, 4\}$ is also a witness for 0000.)

Prove the following:

(a) A always contains a vector whose witness is of size no more than $\lceil \log_2 m \rceil$.

(b) Show that every vector in A has a witness of size no more than $m - 1$.

(c) Construct a set A of n dimensional vectors such that $\mid A \mid = n$ and A contains a vector whose witness is of size $n - 1$.

(d) (Bondy's Theorem) Prove that if $\mid A \mid \leq n$ then there is a set S of at most $\mid A \mid$ coordinates such that S is a witness for each vector in A.

25 marks

8. Hardware Prelim Question, UC Berkeley 1989

Given only 2 component types, specifically 2-input NOR gates and tri-state bus drivers, but as many as you want, show a hardware design for a 32-bit processor with the following instruction set:

- `subtr(x)`: AC $\leftarrow$ AC $-$ $m(x)$
- `store(x)`: $m(x) \leftarrow$ AC
- `brnch(x)`: **if** AC ≥ 0 **then** PC $\rightarrow x$

A large part of your score will depend on you organization and neatness in presenting a solution. Use hierarchical descriptions. Use good engineering practice in your decisions. If an assumption is needed, make the simplest assumption and state clearly what it is.

Not to be turned in

The theory of Boolean functions is very deep; many key questions are still unanswered. You can get an idea of the techniques used for advanced analysis by reading the following book:

- "Gems of Theoretical Computer Science," U. Schning and R. Pruim, Springer-Verlag, 1998.

In particular, Chapter 11 gives a proof that the parity function on N variables cannot be implemented with an AND-OR circuit that has $O\big(P(N)\big)$ gates.