Some Numerical Aspects of Approximate Linearization of Single Input Nonlinear Systems

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Abstract

We characterize approximate linearization to order m of a single input nonlinear system in terms of the existence of an integrating factor to order $m-2$, which is equivalent to the well known condition that a system of vector fields be order $m-1$ involutive. We present a constructive method to obtain the required transformation for approximate linearization, which results in substantial computational savings.

1. Introduction

Linearization of nonlinear systems by coordinate transformation and state feedback has been one of the most active research topics in recent years. Su (1982) and Jakubczyk and Respondek (1980) characterized feedback linearizability, i.e., the ability to linearize a system by a nonlinear state feedback and coordinate change, in terms of the involutiveness of vector fields, while Hermann (1982) and Gardner (1987) studied the dual characterizations in terms of differential forms. The linearizability of nonlinear discrete-time systems has also been studied extensively (Grizzle 1985), (Lee, Arapostathis, and Marcus 1987), (Nam 1987). Feedback linearization

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offers a method of building a controller for a nonlinear system by designing one for the equivalent linear system and utilizing the transformation (from linear to nonlinear) along with its inverse. These linearization techniques, being robust, tolerate some deviation from perfect linearization (Charlet 1987). This approach has already been applied to the design of automatic flight-control systems for aircraft (Meyer, Su, and Hunt 1984), motor controller design (Spong et al. 1987), etc.

On the other hand, since less restrictive conditions are required for approximate linearization, this technique offers the means of enlarging the class of nonlinear systems to which linearizing techniques are applicable. It was shown in (Krener 1984, 1987) that approximate linearization could be obtained by weakening the hypotheses required for feedback linearization. Lee and Marcus (1986) obtained conditions for the approximate linearization of discrete-time systems through the expansion of higher order derivative terms into planar matrices.

In this work, we first show that a system is feedback linearizable to order m if and only if there exists an integrating factor to order $m-2$. Of course, this is equivalent to Krener's (Krener 1984) condition that a system of vector fields is order $m-1$ involutive. It appears, though, that the approach to the problem via one-forms results in substantial computational savings when it comes to computing the required transformation (see Remark 3).

2. Preliminary Remarks and Definitions

Consider a single-input, single-output system

$$\dot{x} = f(x) + ug(x), \quad x \in M\,, \tag{1}$$

where M is an n-dimensional analytic manifold and $f,\ g$ are analytic vector fields on M. Since the questions addressed in this work are local in nature, we identify M with an open neighborhood of the origin in $\mathbb{R}^n$. We also assume that f has an equilibrium (or fixed) point at $x=0$. It is well known that (1) is feedback linearizable if and only if the vector fields $\{g, ad_f g, \ldots, ad_f^{n-1} g\}$ are linearly independent, and $span\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is an involutive distribution. The involutiveness of $span\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is equivalent to the existence of a nonzero scalar function $h : M \to \mathbb{R}$ satisfying $\langle dh,\, ad_f^i g\rangle = 0,\ i = 0, 1, \ldots, n-2$. Hence, feedback linearizability of the system in (1) reduces to the existence of a function $h : M \to \mathbb{R}$ such that

$$\begin{cases} \langle dh,\, ad_f^i g\rangle = 0\,, & i = 0, 1, \ldots, n-2, \\ \langle dh,\, ad_f^{n-1} g\rangle \neq 0\,. \end{cases} \tag{2}$$

If such a function h is available, one may directly obtain a linearizing feedback u and coordinate transformation map T. Specifically, if we let

$$u(t) \equiv \left(v(t) - L_f^n h\right) \Big/ L_g L_f^{n-1} h\,, \tag{3}$$

$$T(x) \equiv \left[h \;\; L_f h \;\cdots\; L_f^{n-1} h\right]^T (x) \tag{4}$$

the system in (1) is transformed into a linear system $\dot{\xi} = A\xi + bv$, where $\xi = T(x)$, v is the new input coordinate, and (A, b) is a Brunovsky controllable pair. Obtaining a function h satisfying (2), is not an easy task; in general, in order to do so, one needs to solve a set of partial differential equations.

We will need the following definitions. Let $x = \{x_1, \ldots, x_n\}$ be coordinate functions in $\mathbb{R}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, i.e., a n-tuple of non-negative integers. The monomial x^α and the differential operator D^α are defined by

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}\,,$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}\,, \qquad \text{where} \;\; D_j = \frac{\partial}{\partial x_j}.$$

We also let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \ldots \alpha_n!$.

With these definitions, for a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, Taylor's expansion formula takes the form

$$\varphi(x) = \varphi(0) + \sum_{|\alpha|=1}^{m} \frac{1}{\alpha!} D^\alpha \varphi(0)\, x^\alpha + R_{m+1}(x)\,, \tag{5}$$

where $R_{m+1}(x)$ is the remainder term.

Definition: (Krener 1984) System (1) is called *feedback linearizable to order* i or i^{th} *order linearizable*, if there exists a coordinate transformation map $\xi = T(x)$ and a feedback $u = \beta(x)v + \alpha(x)$ such that, in the new coordinates,

$$\dot{\xi} = \left(A\xi + \mathcal{O}^{i+1}(\xi)\right) + \left(b + \mathcal{O}^{i}(\xi)\right) v\,,$$

where $\mathcal{O}^i(\xi)$ is the class of functions f such that $\limsup_{||\xi||\to 0} \dfrac{||f(\xi)||}{||\xi||^i} < \infty$.

Remark 1. From the infinitesimal linear approximation of (1) around the equilibrium point, we obtain the system

$$\dot{x} = Fx + Gu\,,$$

where $F = \partial f/\partial x|_{x=0}$ and $G = g(0)$. Hence, this linear system approximates (1) to order 1.

3. Approximate Linearization

We define the function matrix $U(x)$ by

$$U(x) = \left[ad_f^{n-1}g \mid \cdots \mid ad_f g \mid g\right] \tag{6}$$

and we let

$$\omega(x) = \left[\omega_1(x)\ \ldots\ \omega_n(x)\right] = \left[1\ \ 0\ \ldots\ 0\right] U^{-1}(x)\,. \tag{7}$$

Since $\omega(x)$ is defined to be the first row of the inverse of the matrix $U(x)$, it follows that

$$\begin{cases} \langle \omega,\, ad_f^i g\rangle = 0\,, & i = 0,1,\ldots,n-2, \\ \langle \omega,\, ad_f^{n-1} g\rangle = 1\,. & \end{cases} \tag{8}$$

With a slight abuse of notation, we define a one-form ω by

$$\omega = \sum_{i=1}^{n} \omega_i dx_i\,. \tag{9}$$

System (1) is feedback linearizable, if and only if there exists a scalar function, namely an integrating factor, $r : M \to \mathbb{R}$ such that

$$dh = r\omega\,. \tag{10}$$

A necessary and sufficient condition for the exactness of the one-form $r\omega$ is $d(r\omega) = 0$. Hence, feedback linearizability is equivalent to the existence of an integrating factor $r : M \to \mathbb{R}$ such that

$$\frac{\partial r\omega_i}{\partial x_j} = \frac{\partial r\omega_j}{\partial x_i}, \qquad 1 \le i < j \le n\,. \tag{11}$$

Since $d\omega \wedge \omega = 0$ for a manifold whose dimension is 2, it is always possible to find an integrating factor r for 2-dimensional manifolds. Hereafter, we assume that the dimension of the manifold is greater than 2.

Given an integrating factor r, one can obtain h from (10) and, thus, the linearizing feedback (3) and the coordinate transformation map (4) can be easily constructed. Hence, the problem of obtaining the desired coordinate change and feedback reduces to that of obtaining an integrating factor. But, solving (11) for r is a difficult problem. However, in the case of approximate linearization, the situation is quite different.

Proposition 1. *System (1) is feedback linearizable to order $m \geq 2$ if and only if there exists a function $r : M \to \mathbb{R}$ such that*

$$\left. \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \frac{\partial r\omega_i}{\partial x_j} \right|_{x=0} = \left. \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}} \frac{\partial r\omega_j}{\partial x_i} \right|_{x=0}, \tag{12}$$

for all $i, j = 1, 2, \ldots, n$ and $|\alpha| \leq m - 2$.

Note that any m^{th} order linearizable system can be represented as

$$\dot{x} = Ax + bv + \mathcal{O}^{m+1}(x) + \mathcal{O}^m(x)v\,.$$

Then, $\omega(x) = dx_1 + \mathcal{O}^m(x)$. Hence, with $r(x) = 1$, the necessity of (12) follows. Sufficiency follows from the proof of the following Lemma.

Lemma 1. *Suppose that there is a function r which satisfies (12). We let $\tilde{\omega}(x) = r(x)\omega(x)$ and define a scalar function $\tilde{h} : M \to \mathbb{R}$ by*

$$\tilde{h}(x) = \sum_{i=1}^{n} \Big\{ \tilde{\omega}_i(0) + \sum_{|\alpha|=1}^{m-1} \frac{1}{\big(1 + |\alpha|\big)\, \alpha!} D^\alpha \tilde{\omega}_i(0)\, x^\alpha \Big\} x_i\,. \tag{13}$$

Then, utilizing the feedback $u(t) = \big(v(t) - L_f^n \tilde{h}\big) \Big/ L_g L_f^{n-1} \tilde{h}$, and the coordinate transformation map $\tilde{T} : M \to \mathbb{R}^n$, $\tilde{T}(x) = \Big[\tilde{h} \;\; L_f \tilde{h} \; \cdots \; L_f^{n-1} \tilde{h} \Big]^T (x)$, we obtain the m^{th} order linearized system

$$\dot{\xi} = A\xi + bv + \mathcal{O}^{m+1}(\xi) + \mathcal{O}^m(\xi)v\,,$$

where $\xi = \tilde{T}(x)$,

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Proof: For $i = 1, \ldots, n$, let δ_i denote the multi-index of length n defined by $(\delta_i)_i = 1$ and $(\delta_i)_j = 0$, $i \neq j$. Then

$$\frac{\partial \tilde{h}(x)}{\partial x_k} = \tilde{\omega}_k(0) + \sum_{|\alpha|=1}^{m-1} \frac{\alpha_k + 1}{\big(1 + |\alpha|\big)\, \alpha!} D^\alpha \tilde{\omega}_k(0)\, x^\alpha$$
$$+ \sum_{i \neq k} \sum_{\substack{|\alpha|=1 \\ \alpha_k > 0}}^{m-1} \frac{\alpha_k}{\big(1 + |\alpha|\big)\, \alpha!} D^\alpha \tilde{\omega}_i(0)\, x^{(\alpha + \delta_i - \delta_k)}. \tag{14}$$

Observe that, if $i \neq k$, then, for every multi-index α, with $\alpha_k > 0$, there corresponds a multi-index $\tilde{\alpha} \equiv \alpha + \delta_i - \delta_k$, with $\tilde{\alpha}_i > 0$, and vice-versa. Then, by (12), if $\alpha_k > 0$,

$$D^\alpha \tilde{\omega}_i(0) = D^{\alpha + \delta_i - \delta_k} \tilde{\omega}_k(0) = D^{\tilde{\alpha}} \tilde{\omega}_k(0)\,, \tag{15}$$

and, furthermore,

$$\frac{\alpha_k}{\big(1 + |\alpha|\big)\, \alpha!} = \frac{\tilde{\alpha}_k + 1}{\big(1 + |\tilde{\alpha} - \delta_i + \delta_k|\big)\, (\tilde{\alpha} - \delta_i + \delta_k)!} = \frac{\tilde{\alpha}_i}{\big(1 + |\tilde{\alpha}|\big)\, \tilde{\alpha}!}\,. \tag{16}$$

Therefore, by (15) and (16) the second sum on the right hand side of (14) reduces to

$$\sum_{i \neq k} \sum_{\substack{|\alpha|=1 \\ \alpha_k > 0}}^{m-1} \frac{\alpha_k}{\big(1 + |\alpha|\big)\, \alpha!} D^\alpha \tilde{\omega}_i(0)\, x^{(\alpha + \delta_i - \delta_k)} = \sum_{i \neq k} \sum_{\substack{|\tilde{\alpha}|=1 \\ \tilde{\alpha}_i > 0}}^{m-1} \frac{\tilde{\alpha}_i}{\big(1 + |\tilde{\alpha}|\big)\, \tilde{\alpha}!} D^{\tilde{\alpha}} \tilde{\omega}_k(0)\, x^{\tilde{\alpha}}. \tag{17}$$

The second sum on the right hand side of (17) is taken over all multi-indices of order 1 to $m-1$ whose i^{th} coordinate is positive. Thus, using (17) in (14), replacing the variable $\tilde{\alpha}$ with α, and combining the two summations in (14), we obtain

$$\frac{\partial \tilde{h}(x)}{\partial x_k} = \tilde{\omega}_k(0) + \sum_{|\alpha|=1}^{m-1} \frac{1}{\alpha!} D^\alpha \tilde{\omega}_k(0)\, x^\alpha\,. \tag{18}$$

Hence, by (18) and (5),

$$d\tilde{h}(x) = \tilde{\omega}(x) + \mathcal{O}^m(x)\,.$$

Evaluating the Lie derivatives, we obtain, in view of (8),

$$L_g\tilde{h} = \langle d\tilde{h},\, g\rangle = \langle\tilde{\omega},\, g\rangle + \mathcal{O}^m(x) = \mathcal{O}^m(x)\,,$$

$$L_gL_f\tilde{h} = L_f\langle d\tilde{h},\, g\rangle - \langle d\tilde{h},\, ad_f g\rangle = \mathcal{O}^m(x)\,,$$

and proceeding, in a similar fashion, $L_gL_f^i\tilde{h} = \mathcal{O}^m(x)$ for $i = 0, \ldots, n-2$. Therefore,

$$\begin{aligned} \dot{\xi} &= A\xi + bv + \mathcal{O}^m(x)u \\ &= A\xi + bv + \mathcal{O}^{m+1}(\xi) + \mathcal{O}^m(\xi)v\,. \end{aligned}$$

■

Since Proposition 1 characterizes approximate linearizability in terms of the existence of an integrating factor, it is not readily verifiable. The corollaries that follow offer a method of verifying approximate linearizability directly. We define $\omega^{(m-1)}$ to be the one-form obtained from ω by truncating after the $(m-1)^{th}$ order term in x, i.e., we let

$$\omega^{(m-1)}(x) = \sum_{i=1}^{n}\omega_i^{(m-1)}(x)dx_i = \sum_{i=1}^{n}\Big\{w_i(0) + \sum_{|\alpha|=1}^{m-1}\frac{1}{\alpha!}D^\alpha w_i(0)x^\alpha\Big\}dx_i\,.$$

Corollary 1. *System (1) is feedback linearizable to order $m \geq 2$ if and only if*

$$d\omega^{(m-1)}(x) \wedge \omega^{(m-1)}(x) = 0 + \mathcal{O}^{m-1}(x)\,.$$

Proof: Since $\omega^{(m-1)}$ is defined to be the $(m-1)^{th}$ order approximation of ω, (12) is equivalent to

$$\frac{\partial}{\partial x_i}r\omega_j^{(m-1)}(x) = \frac{\partial}{\partial x_j}r\omega_i^{(m-1)}(x) + \mathcal{O}^{m-1}(x)\,, \qquad 1 \leq i, j \leq n\,.$$

Thus, it follows from Proposition 1 that m^{th} order linearizability implies that $d\omega^{(m-1)}(x) \wedge \omega^{(m-1)}(x) = 0 + \mathcal{O}^{m-1}(x)$. The converse also holds. ■

Define

$$\mathcal{X}_{ijk}(x) = \omega_k^{(m-1)}(x)\big(D_j\omega_i^{(m-1)}(x) - D_i\omega_j^{(m-1)}(x)\big)$$
$$+ \omega_j^{(m-1)}(x)\big(D_i\omega_k^{(m-1)}(x) - D_k\omega_i^{(m-1)}(x)\big)$$
$$+ \omega_i^{(m-1)}(x)\big(D_k\omega_j^{(m-1)}(x) - D_j\omega_k^{(m-1)}(x)\big)\,.$$

In view of the identity

$$d\omega^{(m-1)}(x) \wedge \omega^{(m-1)}(x) = \sum_{\substack{1\le i,j,k\le n \\ i\ne j\ne k\ne i}} \mathcal{X}_{ijk}(x)\, dx_j \wedge dx_i \wedge dx_k\,,$$

we obtain the following Corollary.

Corollary 2. *System (1) is feedback linearizable to order $m \ge 2$ if and only if for $i, j, k \in \{1, \ldots, n\}$, $i \ne j \ne k \ne i$,*

$$D^\alpha \mathcal{X}_{ijk}(0) = 0\,, \qquad \forall\, |\alpha| \le m-2\,. \tag{19}$$

4. Solving for the Integrating Factor

Proposition 1 does not offer a method of obtaining the integrating factor r. In this Section, we will develop a constructive algorithm for finding an integrating factor r for those systems which are m^{th} order linearizable. We first consider a system which is linearizable to order 2. By Proposition 1, there is an integrating factor $r(x)$ such that for all $1 \le i, j \le n$

$$\left.\frac{\partial r\omega_i}{\partial x_j}\right|_{x=0} = \left.\frac{\partial r\omega_j}{\partial x_i}\right|_{x=0}\,. \tag{20}$$

In such a case, it is always possible to satisfy (20) with a first order polynomial

$$r_1(x) = 1 + \overline{r}_1 x_1 + \cdots + \overline{r}_n x_n\,, \tag{21}$$

since the differentials of higher order terms vanish at $x = 0$. We express $\omega(x)$ as

$$\omega(x) = \omega(0) + D_1\omega(0)\, x_1 + \cdots + D_n\omega(0)\, x_n + \mathcal{O}^2(x)\,.$$

Simplifying the notation, let $\overline{\omega}_k = \omega_k(0)$, , $1 \leq k \leq n$. Then, equation (20) reduces to

$$\overline{r}_i \overline{\omega}_j - \overline{r}_j \overline{\omega}_i = D_j \omega_i(0) - D_i \omega_j(0)\,, \qquad 1 \leq i, j \leq n\,. \tag{22}$$

Representing (22) in matrix form, we obtain, letting $y_{ij} := D_j \omega_i(0) - D_i \omega_j(0)$,

$$\begin{bmatrix} \overline{\omega}_2 & -\overline{\omega}_1 & 0 & 0 & \cdots & 0 \\ \overline{\omega}_3 & 0 & -\overline{\omega}_1 & 0 & \cdots & 0 \\ 0 & \overline{\omega}_3 & -\overline{\omega}_2 & 0 & \cdots & 0 \\ \overline{\omega}_4 & 0 & 0 & -\overline{\omega}_1 & \cdots & 0 \\ 0 & \overline{\omega}_4 & 0 & -\overline{\omega}_2 & \cdots & 0 \\ 0 & 0 & \overline{\omega}_4 & -\overline{\omega}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\omega}_n & 0 & 0 & 0 & \cdots & -\overline{\omega}_1 \\ 0 & \overline{\omega}_n & 0 & 0 & \cdots & -\overline{\omega}_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\overline{\omega}_{n-1} \end{bmatrix} \begin{bmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \\ \vdots \\ \\ \overline{r}_{n-1} \\ \overline{r}_n \end{bmatrix} = \begin{bmatrix} y_{12} \\ y_{13} \\ y_{23} \\ y_{14} \\ y_{24} \\ y_{34} \\ \vdots \\ y_{1n} \\ y_{2n} \\ \vdots \\ y_{(n-1)n} \end{bmatrix}. \tag{23}$$

Let $\Omega_n = [\overline{\omega}_1 \ldots \overline{\omega}_n]^T$ and I_n denote the identity matrix of dimension n. For each $n = 2, 3, \ldots$, we define a matrix H_n of dimension $\binom{n}{2} \times n$ as follows: For $n = 2$, let $H_2 = [\overline{\omega}_2 \quad -\overline{\omega}_1]$ and inductively for $n \geq 3$ by

$$H_n = \left(\begin{array}{c|c} H_{n-1} & 0 \\ \hline \overline{\omega}_n\, I_{n-1} & -\Omega_{n-1} \end{array} \right).$$

Also let $R_n = [\overline{r}_1 \ldots \overline{r}_n]^T$ and Y_n denote the vector on the right hand side of (23). Then, equation (23) takes the form $H_n\, R_n = Y_n$. Observe that (23) is invariant under permutations of the subscripts $\{1, \ldots, n\}$. Therefore, since $\Omega_n \neq 0$, by (7), we may assume, without loss of generality, that $\overline{\omega}_n \neq 0$. Then, since $\overline{\omega}_n\, I_{n-1}$ is a minor of H_n, the rank of H_n is at least $n-1$. On the other hand, $H_n\, \Omega_n = 0$ and it follows that $rank\,(H_n) = n-1$. The matrix $K_n = \begin{pmatrix} \overline{\omega}_n\, I_m \\ -H_{n-1}^T \end{pmatrix}$, where $m = \binom{n-1}{2}$, has rank $\binom{n-1}{2}$ and satisfies $K_n^T\, H_n = 0$. Thus, the span of K_n is precisely the kernel of H_n^T. For (23) to have a solution R_n it is necessary and sufficient that Y_n be orthogonal to the kernel of H_n^T or, equivalently, $K_n^T\, Y_n = 0$. This reduces to the requirement that, for all $1 \leq i, j, k \leq n$, $i \neq j \neq k \neq i$,

$$\overline{\omega}_k y_{ij} + \overline{\omega}_j y_{ki} + \overline{\omega}_i y_{jk} = 0\,. \tag{24}$$

Equation (24) is equivalent to the condition in Corollary 2 for second order linearizable systems. We have the following Lemma.

Lemma 2. *The linear system in (23) has a solution $R_n = [\overline{r}_1 \, \ldots \, \overline{r}_n]^T$ if and only if (24) holds. Also, assuming that $\overline{\omega}_n \neq 0$, the family of solutions of (23) can be parameterized by*

$$\overline{r}_i = \frac{y_{in}}{\overline{\omega}_n} + a\,\overline{\omega}_i\,, \qquad i = 1, \ldots, n\,.$$

where a is an arbitrary constant.

Remark 2. The first part of Lemma 2 follows essentially from Proposition 1 and Corollary 2. The second part results from the structure of (23) as analyzed above.

Remark 3. By Lemma 2 one choice for the coefficients $\overline{r}_1, \cdots, \overline{r}_n$ is

$$\overline{r}_i = \frac{D_n\omega_i(0) - D_i\omega_n(0)}{\overline{\omega}_n}\,, \qquad i = 1, \ldots, n\,.$$

With these values, the transformation which linearizes the system to order 2 may be easily obtained from Lemma 1. Note that in Krener's work (1984, 1987), this task requires the solution of $n^2(n+3)/2$ linear equations in $n(n+1)^2/2 + n$ unknowns.

This approach can be extended to higher order linearizable systems. In the third order case, we need to find an integrating factor $r(x)$ such that (20) holds and, for $1 \leq i, j, k \leq n$.

$$D_k \frac{\partial r\omega_i}{\partial x_j}\bigg|_{x=0} = D_k \frac{\partial r\omega_j}{\partial x_i}\bigg|_{x=0}\,. \tag{25}$$

We let

$$r_2(x) = r_1(x) + \sum_{i,j=1}^{n} \overline{r}_{ij} x_i x_j\,,$$

where $\overline{r}_{ij} = \overline{r}_{ji}$ and $r_1(x)$ is a first order polynomial whose coefficients satisfy (23). Clearly, the second order linearizability condition (20) is satisfied automatically. Also for each k, (25) reduces to

$$\begin{aligned}\overline{r}_{ki}\overline{\omega}_j - \overline{r}_{kj}\overline{\omega}_i =& D_{kj}\omega_i(0) - D_{ki}\omega_j(0) + \overline{r}_j D_k\omega_i(0) - \overline{r}_i D_k\omega_j(0) \\ &+ \overline{r}_k\big(D_j\omega_i(0) - D_i\omega_j(0)\big)\,, \qquad 1 \leq i, j, k \leq n\,. \end{aligned} \tag{26}$$

Let $z_{ij}^{(k)}$ denote the right hand side of (26). For each fixed k, (26) is a system of linear equations which has the same structure as (23). An inductive argument shows that if $\overline{r}_1, \cdots, \overline{r}_n$ solve (23) and the system is third order linearizable (i.e., it satisfies the condition in Corollary 2 with $m = 2$), then there exist constants $\overline{r}_{ki}$ which solve (26). Moreover, these can be obtained as follows: first solve for $\overline{r}_{ni}$, $i = 1, \ldots, n$ as dictated by Lemma 2 to obtain

$$\overline{r}_{ni} = \frac{z_{in}^{(n)}}{\overline{\omega}_n}\,, \qquad i = 1, \ldots, n\,, \tag{27}$$

and then solve recursively for each $k = n-1, n-2, \ldots, 1$ using the previously computed coefficients to obtain

$$\overline{r}_{ki} = \frac{z_{in}^{(k)} - \overline{r}_{nk}\overline{\omega}_i}{\overline{\omega}_n}\,, \qquad i = 1, \ldots, k\,. \tag{28}$$

The number of coefficients computed in (27)–(28) is $n(n+1)/2$.

The same procedure can be used to obtain integrating factors to an arbitrary order. In passing from one order to the next one computes recursively for the coefficients of the next order term utilizing all the previously computed coefficients. This claim, which is summarized in the following Proposition, can be derived by means of a straightforward induction, which is unfortunately very messy and will be skipped. The analogous observation for vector fields has already been made by Krener (1984).

Proposition 2. *Consider an ν^{th} order linearizable system defined on an n-dimensional ($n \geq 3$) analytic manifold M and $\omega(x)$ as defined in (7). Suppose that $r_{\nu-2}(x)$ is an integrating factor polynomial of degree $\nu - 2$ which satisfies for $|\alpha| \leq \nu - 3$*

$$D^\alpha \frac{\partial r_{\nu-2}\omega_i}{\partial x_j}\bigg|_{x=0} = D^\alpha \frac{\partial r_{\nu-2}\omega_j}{\partial x_i}\bigg|_{x=0}\,.$$

Defining

$$r_{\nu-1}(x) = r_{\nu-2}(x) + \sum_{|\gamma|=\nu-1} \overline{r}_\gamma x^\gamma\,,$$

one can find coefficients $\overline{r}_\gamma$ such that for $|\beta| = \nu - 2$,

$$D^\beta \frac{\partial r_{\nu-1}\omega_i}{\partial x_j}\bigg|_{x=0} = D^\beta \frac{\partial r_{\nu-1}\omega_j}{\partial x_i}\bigg|_{x=0}\,,$$

by solving the linear system of equations

$$\overline{r}_{\beta+\delta_j}\overline{\omega}_i - \overline{r}_{\beta+\delta_i}\overline{\omega}_j = D^\beta \frac{\partial \omega_j}{\partial x_i}\bigg|_{x=0} - D^\beta \frac{\partial \omega_i}{\partial x_j}\bigg|_{x=0}$$
$$+ \sum_{\substack{q>0 \\ p+q=\beta}} \frac{\beta!}{p!q!}\big(\overline{r}_{p+\delta_i} D^q w_j(0) - \overline{r}_{p+\delta_j} D^q w_i(0)\big)$$
$$+ \sum_{\substack{q>0 \\ p+q=\beta}} \frac{\beta!}{p!q!}\overline{r}_p\big(D^{q+\delta_i} w_j(0) - D^{q+\delta_j} w_i(0)\big)\,.$$

5. An Example

Consider the nonlinear system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \sin x_2 + x_2 x_4 \\ -x_3 + x_2^2 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u\,. \tag{29}$$

Then,

$$U(x) = \Big[ad_f^3 g \ \big|\ ad_f^2 g \ \big|\ ad_f g \ \big|\ g\Big] = \begin{bmatrix} \begin{smallmatrix} 2x_2(x_3-x_2^2) \\ +2x_4+\cos x_2 \end{smallmatrix} & (x_3 - x_2^2) & -x_2 & 0 \\ 2x_2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\langle ad_f g,\, ad_f^2 g\rangle = [-2 \ \ 0 \ \ 0 \ \ 0]^T$ does not lie in $span\{g, ad_f g, ad_f^2 g\}$, this distribution is not involutive. Therefore, system (29) is not feedback linearizable. In the following we establish that the system is feedback linearizable to order 2. We have,

$$\omega(x) = [1 \ \ 0 \ \ldots \ 0]\, U^{-1}(x) = \frac{1}{\det U(x)}[-1 \ \ x_3 - x_2^2 \ \ x_2 \ \ 0],$$

where $\det U(x) = 4x_2(x_3 - x_2^2) + 2x_4 + \cos x_2$. Since $\overline{\omega}_1 = -1 \neq 0$, it follows that the system is feedback linearizable to order 2 if and only if for $(i,j) = (2,3),\ (2,4),\ (3,4)$,

$$\overline{\omega}_1\big(D_j\omega_i(0) - D_i\omega_j(0)\big) + \overline{\omega}_j\big(D_i\omega_1(0) - D_1\omega_i(0)\big) + \overline{\omega}_i\big(D_1\omega_j(0) - D_j\omega_1(0)\big) = 0\,. \tag{30}$$

However, since the only nonzero derivative terms are $D_3\omega_2(0) = D_2\omega_3(0) = 1$ and $D_4\omega_1(0) = 2$, condition (30) is satisfied. Solving (23), we obtain

$$[r_1 \;\; r_2 \;\; r_3 \;\; r_4] = [0 \;\; 0 \;\; 0 \;\; -2],$$

or, equivalently, an integrating factor $r(x) = 1 - 2x_4$. Finally, with

$$\tilde{\omega}(x) = \frac{(1-2x_4)}{\det U(x)}[-1 \;\; x_3 - x_2^2 \;\; x_2 \;\; 0],$$

and $\tilde{h}(x)$ as defined in (13), we can linearize the system to order 2.

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