Course notes for EE394V Restructured Electricity Markets: Locational Marginal Pricing

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Unit commitment

(i) Temporal issues,
(ii) Formulation,
(iii) Mixed-integer programming,
(iv) Make-whole payments,
(v) Lagrangian relaxation,
(vi) Duality gaps,
(vii) Role of prices and implications for investment decisions,

(viii) Transmission constraints,
(ix) Robust and stochastic unit commitment,
(x) Homework exercises.

10.1 Temporal issues

- So far we have considered particular dispatch intervals.
- Demand has been represented by its assumed known average value over the dispatch interval, or its value at the end of the interval, ignoring whether this demand was occurring:
  - “now” (that is, in the next few minutes or next dispatch interval), or
  - in the future (such as during an hour of tomorrow).
- Supply has been represented by assuming that unit commitment decisions had already been taken:
  - each generator’s commitment status is fixed.
- In this section we will generalize this in several ways, by considering:

  (i) variation of demand over time,
  (ii) ramp rates,
  (iii) unit commitment, and
  (iv) day-ahead and real-time markets.
- We will discuss the relationship between day-ahead and real-time markets in Section 11.

10.1.1 *Variation of demand over time*

- Suppose that we are considering the average demand in each dispatch interval or period, say each hour, for tomorrow:
  - (as in Section 8.3.2, in some formulations we might prefer to consider the demand power level at the *end* of each interval instead of representing the average level, with ramping of the demand assumed to be linear between boundaries of intervals.)
- We are planning **day-ahead**.
- For now, we will continue to ignore unit commitment decisions.
- For each hour $t = 1, \ldots, n_T$, we have a specification or a forecast of the average power demand, $\overline{D}_t$ in dispatch interval t.
- We need to satisfy average power balance each hour (and, of course, continuously, but this will be achieved by the **real-time** market).

10.1.2 Ramp-constrained economic dispatch

10.1.2.1 Decision variables

- We generalize our previous formulation so that P_{kt} represents the average power generated by generator $k = 1, \ldots, n_P$ in hour $t = 1, \ldots, n_T$:
  - (if we are considering a formulation where we are targetting the demand at the end of each interval, then we would similarly consider the generation level at the end of each interval instead of representing the average level of generation, with ramping of generation assumed to be linear between boundaries of intervals.)
- We collect the entries P_{kt} together into a vector $P_k \in \mathbb{R}^{n_T}$.
- As previously, we can also consider the spinning reserve and let S_{kt} be the amount of spinning reserve provided by generator k in hour t.
- We collect the entries S_{kt} together into a vector $S_k \in \mathbb{R}^{n_T}$.
- We collect P_k and S_k together into a vector $x_k \in \mathbb{R}^{2n_T}$.
- We collect the vectors x_k together into a vector $x \in \mathbb{R}^{2n_P n_T}$.
- In some examples, we will only consider energy and not reserve, in which case, we re-define $x = P \in \mathbb{R}^{n_P n_T}$ and re-define any associated functions, matrices, and vectors appropriately.

10.1.2.2 System constraints

- Typical system equality constraints would include average power balance in each hour of tomorrow, which we will represent in the form $Ax = b$.
- For example:
  - for simplicity, if we ignore reserve, then $x = P = \begin{bmatrix} P_1 \\ \vdots \\ P_{n_P} \end{bmatrix} \in \mathbb{R}^{n_P n_T}$, with

    $P_k = \begin{bmatrix} P_{k1} \\ \vdots \\ P_{kn_T} \end{bmatrix} \in \mathbb{R}^{n_T}$,
  - let $\overline{D} \in \mathbb{R}^{n_T}$ be a vector of forecasts of average demand in each hour,
  - let $A = \begin{bmatrix} -\mathbf{I} & \cdots & -\mathbf{I} \end{bmatrix}$ and $b = -\overline{D}$,
  - then $Ax = b$ represents average power balance in each hour.
- Typical system inequality constraints would include reserve requirements and transmission constraints in each hour, which we will represent in the form $Cx \leq d$.

System constraints, continued

- We will continue to use $\lambda^\star$ and $\mu^\star$ for the Lagrange multipliers on the system equality and inequality constraints, respectively.
- However, we have changed the definition of the system constraints:
  - in Section 9, the vector $\lambda^\star$, for example, represented the Lagrange multipliers on the system constraints of power balance at each *location*, but was implicitly applying for just one given time or interval, so its subscript k related to location, whereas
  - for most of the development of temporal issues, the vector $\lambda^\star$ will represent the Lagrange multipliers on the system constraints of power balance for each time, so it will have subscript t relating to time, but not location,
  - in Section 10.8, we will consider both location and temporal issues, so $\lambda^\star$ will have subscripts for both location and time.

10.1.2.3 Generator constraints including ramp-rate constraints

- Each generator k has a feasible operating set $\mathbb{S}_k$.
- In addition to minimum and maximum generation and spinning reserve constraints, there can be **inter-temporal constraints** in the specification of $\mathbb{S}_k$ that limit the change in average production from hour to hour.
- For example, if the ramp-rate limit is 1 MW per minute then the generator constraints for generator k could be:

$$\begin{aligned}
\forall t = 1,\ldots,n_T, &\quad \underline{P}_k \leq P_{kt} \leq \overline{P}_k,\\
\forall t = 1,\ldots,n_T, &\quad 0 \leq S_{kt} \leq 10,\\
\forall t = 1,\ldots,n_T, &\quad \underline{P}_k \leq P_{kt} + S_{kt} \leq \overline{P}_k,\\
\forall t = 1,\ldots,n_T, &\quad P_{k,(t-1)} - 60 \leq P_{kt} \leq P_{k,(t-1)} + 60 - S_{kt},
\end{aligned}$$

- where P_{k0} and S_{k0} are the power and reserve for the last hour of today, and
- where we have required that procured spinning reserve be available for deployment within any one 10 minute period throughout the hour.

Generator constraints including ramp-rate constraints, continued

- As previously, we can specify the feasible operating set for generator k in the form:

$$\mathbb{S}_k = \{x_k \in \mathbb{R}^{2n_T} | \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\},$$

- where $\Gamma_k \in \mathbb{R}^{r_k \times 2n_T}$, $\underline{\delta}_k \in \mathbb{R}^{r_k}$, and $\overline{\delta}_k \in \mathbb{R}^{r_k}$.
- Other formulations of generator constraints besides our example also fit into this form.

10.1.2.4 Generator costs

- Generator k has a cost function f_k for its generation over the hours $t = 1, \ldots, n_T$.
- Typically, if a unit is committed then the production in one hour does not (directly) affect the costs in another hour so that the costs are additively separable across time:

$$\forall x_k, f_k(x_k) = \sum_{t=1}^{n_T} f_{kt}(x_{kt}),$$

- where $x_{kt} = \begin{bmatrix} P_{kt} \\ S_{kt} \end{bmatrix}$.
- Typically, we would expect that f_{kt} does not vary significantly from hour to hour, except for:
  - temperature and pressure related changes, and
  - significant change in fuel availability or cost.
- This formulation ignores start-up and min-load costs:
  - will be included later when we explicitly consider unit commitment.

- The resulting ramp-constrained economic dispatch problem is in the form of our generalized economic dispatch problem:

$$\begin{aligned} \min_{\forall k, x_k \in \mathbb{S}_k} \{f(x) | Ax = b, Cx \leq d\} \\ = \min_{x \in \mathbb{R}^{2n_P n_T}} \{f(x) | Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}. \end{aligned}$$

- If f is convex then the problem is convex and can be solved with standard algorithms for minimizing convex problems.
- For example, if f is linear then the problem is a linear program:

$$\min_{x \in \mathbb{R}^{2n_P n_T}} \left\{ c^\dagger x \middle| Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \right\}. \quad (10.1)$$

- Note that this formulation does not exactly match any specific market design, but will illustrate temporal coupling:
  - is similar to some European day-ahead markets, including the EUPHEMIA market coupling algorithm, but those markets also include other features such as “minimum income condition constraints.”

10.1.2.6 Ramp-constrained example

- Suppose that we have two generators, $n_P = 2$, with costs:

$$\begin{aligned} \forall t, f_{1t}(P_{1t}) &= 2P_{1t}, 100 \leq P_{1t} \leq 300, \\ \forall t, f_{2t}(P_{2t}) &= 5P_{2t}, 100 \leq P_{2t} \leq 300. \end{aligned}$$

- The generators have ramp-rate limits of $\Delta_1 = 200$ MW/h and $\Delta_2 = 100$ MW/h, respectively.
- We consider day-ahead dispatch across two hours, $n_T = 2$, with demands:

t	0	1	2
$\overline{D}_t$	200	400	600

- The $t = 0$ entry in the table is the demand for the last hour of today.
- The $t = 1, 2$ entries are the demands for the first two hours of tomorrow.
- Also, $P_{10} = 100$ MW and $P_{20} = 100$ MW are the generations in the last hour of today.
- We ignore reserve requirements so that the only system constraint is supply-demand balance for power.
- We solve the ramp-constrained economic dispatch problem.

- The generator constraints for generator $k = 1,2$ are:

$$\begin{aligned} \forall t = 1,2, \quad & 100 = \underline{P}_k \leq P_{kt} \leq \overline{P}_k = 300, \\ \forall t = 1,2, \quad & P_{k,(t-1)} - \Delta_k \leq P_{kt} \leq P_{k,(t-1)} + \Delta_k, \end{aligned}$$

- which we can represent in the form $\mathbb{S}_k = \{x_k \in \mathbb{R}^2 | \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k\}$,
- by defining $\underline{\delta}_k \in \mathbb{R}^4$, $\Gamma_k \in \mathbb{R}^{4\times 2}$, and $\overline{\delta}_k \in \mathbb{R}^4$ as:

$$\underline{\delta}_k = \begin{bmatrix} \underline{P}_k \\ P_{k,0} - \Delta_k \\ \underline{P}_k \\ -\Delta_k \end{bmatrix}, \Gamma_k = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \overline{\delta}_k = \begin{bmatrix} \overline{P}_k \\ P_{k,0} + \Delta_k \\ \overline{P}_k \\ \Delta_k \end{bmatrix}.$$

- We label the Lagrange multipliers on these generator inequality constraints as, respectively, $\underline{\mu}_k = \begin{bmatrix} \underline{\mu}^\star_{k1\text{capacity}} \\ \underline{\mu}^\star_{k1\text{ramp}} \\ \underline{\mu}^\star_{k2\text{capacity}} \\ \underline{\mu}^\star_{k2\text{ramp}} \end{bmatrix}, \overline{\mu}_k = \begin{bmatrix} \overline{\mu}^\star_{k1\text{capacity}} \\ \overline{\mu}^\star_{k1\text{ramp}} \\ \overline{\mu}^\star_{k2\text{capacity}} \\ \overline{\mu}^\star_{k2\text{ramp}} \end{bmatrix}.$

Ramp-constrained example, continued

- Since generator 1 has lower costs, we would prefer to use it instead of generator 2.
- Since the ramp-rate limit for generator 1 is $\Delta_1 = 200$, for hour $t = 1$, we consider setting:

$$\begin{aligned} P_{11} &= P_{10} + \Delta_1, \\ &= 100 + 200, \\ &= 300, \\ &= \overline{P}_1. \end{aligned}$$

- With $P_{11} = 300 = \overline{P}_1$, to meet demand we would have:

$$\begin{aligned} P_{21} &= \overline{D}_1 - P_{11}, \\ &= 400 - 300, \\ &= 100. \end{aligned}$$

- However, we now have a problem in hour $t = 2$, since:
  - generator 1 would be at its maximum $\overline{P}_1$,
  - generator 2 can only increase by $\Delta_2 = 100$ from hour 1 to hour 2, so that $P_{22} \leq P_{21} + \Delta_1 = 100 + 100 = 200$ MW, and
  - supply would then be 100 MW less than demand in hour 2.
- Setting $P_{11} = 300$ does not work!

Ramp-constrained example, continued

- Instead, we need both generators $k = 1, 2$ each producing at their capacity of $\overline{P}_k = 300$ MW in hour 2 to meet the demand, so that $P_{12} = P_{22} = 300$ MW.
  - Working backwards in time, generator 2 must be producing at least 200 MW in hour 1 because of its ramp rate constraint, so $P_{21} \geq 200$ MW.
  - Since generator 2 has higher costs, we do not want it to produce more than necessary, and so we will try to see if we can set $P_{21} = 200$ MW.
  - In this case, generator 1 must produce $P_{11} = 200$ MW in hour 1 to meet demand of $\overline{D}_1 = 400$.
  - This solution satisfies the ramp-rate constraints and is optimal.
- The ramp-constrained economic dispatch solution is:

t	0	1	2
$\overline{D}_t$	200	400	600
$P^{\star}_{1t}$	100	200	300
$P^{\star}_{2t}$	100	200	300

Ramp-constrained example, continued

- What are the values of the Lagrange multipliers?
- Standard linear programming software would provide the values.
- However, to answer this question without linear programming software, we will consider several of the first-order necessary conditions.
- Generator $k = 1$:
  - neither at its maximum nor minimum in hour 1, is at its maximum in hour 2, and no ramp constraints binding across any hours,
- Generator $k = 2$:
  - neither at its maximum nor minimum in hour 1, is at its maximum in hour 2, and the increasing ramp constraints are binding across two successive pairs of dispatch intervals, from $t = 0$ to $t = 1$ and from $t = 1$ to $t = 2$.

Ramp-constrained example, continued

- Generator 1:

  is at its maximum in hour 2, and
  no other binding constraints.

- That is the binding generator constraint for generator 1 is:

$$P_{12} \leq 300, \text{ (Lagrange multiplier } \overline{\mu}^{\star}_{12\text{capacity}}\text{)},$$

- By complementary slackness, all Lagrange multipliers on generator constraints for generator 1 are zero, except for the Lagrange multiplier on this one binding constraints, so that
  $\underline{\mu}^{\star}_{1} = \mathbf{0}, \overline{\mu}^{\star}_{11\text{capacity}} = 0, \overline{\mu}^{\star}_{11\text{ramp}} = 0, \overline{\mu}^{\star}_{12\text{ramp}} = 0.$

Ramp-constrained example, continued

- By the first-order necessary conditions for generator 1 in hour 1 associated with P_{11}:

$$\begin{aligned} 0 &= \nabla f_{11}(P_{11}^{\star}) - \lambda_1^{\star} - [\Gamma_{11}]^{\dagger}\underline{\mu}_1^{\star} + [\Gamma_{11}]^{\dagger}\overline{\mu}_1^{\star}, \\ &= \nabla f_{11}(P_{11}^{\star}) - \lambda_1^{\star}, \\ &= 2 - \lambda_1^{\star}, \end{aligned}$$

- where:

  Γ_1 is the generator constraint matrix for generator 1,

  $\Gamma_{11} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is the column of Γ_1 associated with P_{11}, and

  $\underline{\mu}_1^{\star} = \mathbf{0}$, while the only non-zero entry of $\overline{\mu}_1^{\star}$ is $\overline{\mu}_{12\text{capacity}}^{\star}$.
- That is, $\lambda_1^{\star} = 2$.

- Generator 2:

  is at its maximum in hour 2,
  has its ramp rate constraint binding from hour 0 to hour 1, and
  has its ramp rate constraint binding from hour 1 to hour 2.

- That is the binding generator constraints for generator 2 are:

$$
\begin{aligned}
P_{22} &\leq 300, \text{ (Lagrange multiplier } \overline{\mu}^{\star}_{22\text{capacity}}), \\
P_{21} &\leq P_{20} + \Delta_2, \text{ (Lagrange multiplier } \overline{\mu}^{\star}_{21\text{ramp}}), \\
P_{22} &\leq P_{21} + \Delta_2, \text{ (Lagrange multiplier } \overline{\mu}^{\star}_{22\text{ramp}}).
\end{aligned}
$$

- By complementary slackness, all Lagrange multipliers on generator constraints for generator 2 are zero, except for the Lagrange multipliers on these three binding constraints, so that $\underline{\mu}^{\star}_{2} = \mathbf{0}, \overline{\mu}^{\star}_{21\text{capacity}} = 0$.

Ramp-constrained example, continued

- By the first-order necessary conditions for generator 2 associated with P_{21}:

$$
\begin{aligned}
0 &= \nabla f_{21}(P_{21}^\star) - \lambda_1^\star - [\Gamma_{21}]^\dagger \underline{\mu}_2^\star + [\Gamma_{21}]^\dagger \overline{\mu}_2^\star, \\
&= \nabla f_{21}(P_{21}^\star) - \lambda_1^\star - \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}^\dagger \begin{bmatrix} \underline{\mu}_{21\text{capacity}}^\star \\ \underline{\mu}_{21\text{ramp}}^\star \\ \underline{\mu}_{22\text{capacity}}^\star \\ \underline{\mu}_{22\text{ramp}}^\star \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}^\dagger \begin{bmatrix} \overline{\mu}_{21\text{capacity}}^\star \\ \overline{\mu}_{21\text{ramp}}^\star \\ \overline{\mu}_{22\text{capacity}}^\star \\ \overline{\mu}_{22\text{ramp}}^\star \end{bmatrix}, \\
&= \nabla f_{21}(P_{21}^\star) - \lambda_1^\star - \underline{\mu}_{21\text{capacity}}^\star - \underline{\mu}_{21\text{ramp}}^\star + \underline{\mu}_{22\text{ramp}}^\star \\
&\qquad + \overline{\mu}_{21\text{capacity}}^\star + \overline{\mu}_{21\text{ramp}}^\star - \overline{\mu}_{22\text{ramp}}^\star, \\
&= \nabla f_{21}(P_{21}^\star) - \lambda_1^\star + \overline{\mu}_{21\text{ramp}}^\star - \overline{\mu}_{22\text{ramp}}^\star, \\
&\quad \text{by complementary slackness, since } \underline{\mu}_2^\star = \mathbf{0}, \overline{\mu}_{21\text{capacity}}^\star = 0, \\
&= 5 - 2 + \overline{\mu}_{21\text{ramp}}^\star - \overline{\mu}_{22\text{ramp}}^\star,
\end{aligned}
$$

- where:

  Γ_2 is the generator constraint matrix for generator 2,

  $\Gamma_{21} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$ is the column of Γ_2 associated with P_{21}, and

  $\underline{\mu}_2^\star = \begin{bmatrix} \underline{\mu}_{21\text{capacity}}^\star \\ \underline{\mu}_{21\text{ramp}}^\star \\ \underline{\mu}_{22\text{capacity}}^\star \\ \underline{\mu}_{22\text{ramp}}^\star \end{bmatrix}$ and $\overline{\mu}_2^\star = \begin{bmatrix} \overline{\mu}_{21\text{capacity}}^\star \\ \overline{\mu}_{21\text{ramp}}^\star \\ \overline{\mu}_{22\text{capacity}}^\star \\ \overline{\mu}_{22\text{ramp}}^\star \end{bmatrix}$ are the Lagrange multipliers on the generator constraints for generator 2, and we know that:

$$\underline{\mu}_2^\star = \mathbf{0}, \overline{\mu}_{21\text{capacity}}^\star = 0.$$

- Therefore, $\overline{\mu}_{22\text{ramp}}^\star = \overline{\mu}_{21\text{ramp}}^\star + 3$.

Ramp-constrained example, continued

- By the first-order necessary conditions for generator 2 associated with P_{22}:

$$\begin{aligned} 0 &= \nabla f_{22}(P_{22}^{\star}) - \lambda_2^{\star} - [\Gamma_{22}]^{\dagger}\underline{\mu}_2^{\star} + [\Gamma_{22}]^{\dagger}\overline{\mu}_2^{\star}, \\ &= \nabla f_{22}(P_{22}^{\star}) - \lambda_2^{\star} + \overline{\mu}_{22\text{capacity}}^{\star} + \overline{\mu}_{22\text{ramp}}^{\star}, \\ & \quad \text{by complementary slackness,} \\ &= 5 - \lambda_2^{\star} + \overline{\mu}_{22\text{capacity}}^{\star} + \overline{\mu}_{22\text{ramp}}^{\star}, \end{aligned}$$

- where:

  Γ_2 is the generator constraint matrix for generator 2,

  $\Gamma_{22} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is the column of Γ_2 associated with P_{22}, and

  $\underline{\mu}_2^{\star}$ and $\overline{\mu}_2^{\star}$ are the Lagrange multipliers on the generator constraints for generator 2.

- Therefore, $\lambda_2^{\star} = 5 + \overline{\mu}_{22\text{capacity}}^{\star} + \overline{\mu}_{22\text{ramp}}^{\star}$.

Ramp-constrained example, continued

- Summarizing:

$$\begin{aligned} \overline{\mu}^{\star}_{22\text{ramp}} &= \overline{\mu}^{\star}_{21\text{ramp}} + 3, \\ \lambda^{\star}_{2} &= 5 + \overline{\mu}^{\star}_{22\text{capacity}} + \overline{\mu}^{\star}_{22\text{ramp}}. \end{aligned}$$

- These are two equations in four variables.
- Let's try to find a non-negative solution for these two equations in the four variables $\overline{\mu}^{\star}_{22\text{ramp}}, \overline{\mu}^{\star}_{21\text{ramp}}, \overline{\mu}^{\star}_{22\text{capacity}}$, and $\lambda^{\star}$:

$$\begin{aligned} \text{We set } \overline{\mu}^{\star}_{21\text{ramp}} &= 0, \text{ hypothesizing that constraint is ``just'' binding,} \\ \text{Therefore: } \overline{\mu}^{\star}_{22\text{ramp}} &= \overline{\mu}^{\star}_{21\text{ramp}} + 3, \\ &= 3, \\ \text{We set } \overline{\mu}^{\star}_{22\text{capacity}} &= 0, \text{ hypothesizing that constraint is ``just'' binding,} \\ \text{Therefore: } \lambda^{\star}_{2} &= 5 + \overline{\mu}^{\star}_{22\text{capacity}} + \overline{\mu}^{\star}_{22\text{ramp}}, \\ &= 5 + 0 + 3, \\ &= 8. \end{aligned}$$

Ramp-constrained example, continued

- The solution is:

$$\begin{aligned} \overline{\mu}^{\star}_{21\text{ramp}} &= 0, \\ \overline{\mu}^{\star}_{22\text{ramp}} &= 3, \\ \overline{\mu}^{\star}_{22\text{capacity}} &= 0, \\ \lambda^{\star}_{2} &= 8. \end{aligned}$$

- These particular values constitute one of multiple solutions for the Lagrange multipliers.
- Any other solution of the two equations having non-negative values for the Lagrange multipliers on the inequality constraints also provides Lagrange multipliers for this problem.

10.1.3 Ramp-constrained offer-based economic dispatch

10.1.3.1 Generator offers

- Generator k makes an offer for its generation.
- The offer is usually required to be separable across hours.
- Sometimes market rules require the offer for each hour t to be fixed independent of t (as in PJM) and sometimes the offer can vary from hour to hour (as in ISO-NE, NYISO, and ERCOT):
  - market rules on fixed versus varying offers can affect the exercise of market power,
  - discussed in market power course,

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- Assuming that offers reflect marginal costs, the offer for generator k is:

$$\nabla f_{kt}, t = 1, \ldots, n_T,$$

- where $x_{kt} = [P_{kt}]$ for simplicity, ignoring reserve and where we will typically assume that the marginal costs do not vary with time, even though the notation allows for such variation.

10.1.3.2 Offer-based economic dispatch and prices

- Using the offers, we can solve the first-order necessary and sufficient conditions for offer-based ramp-constrained economic dispatch:

$$\min_{x \in \mathbb{R}^{2n_P n_T}} \{ f(x) | Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k x_k \leq \overline{\delta}_k \}.$$

- The solution involves dispatch $x_k^\star$ for each generator k and Lagrange multipliers:

  $\lambda^\star$ and $\mu^\star$ on system constraints, and

  $\underline{\mu}_k^\star$ and $\overline{\mu}_k^\star$ on generator constraints for each generator k.
- By Theorem 8.1 in Section 8.12.4.4, dispatch-supporting prices can be constructed as previously: $\pi_{x_k}^{\text{LMP}} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star$.
- To summarize: the generalization of the problem to include more complicated generator constraints and more complicated system constraints does not fundamentally complicate the pricing rule, so long as the generalized economic dispatch problem is convex:
  - we will qualify this statement in the context of *anticipating* prices.

10.1.3.3 Ramp-constrained example

- Continuing with the previous example from Section 10.1.2.6, assume that the generators offer at their marginal costs in each hour:

$$\begin{aligned} \nabla f_{1t}(P_{1t}) &= 2, 100 \leq P_{1t} \leq 300, t = 1, 2, \\ \nabla f_{2t}(P_{2t}) &= 5, 100 \leq P_{2t} \leq 300, t = 1, 2. \end{aligned}$$

- From the previous analysis, we have that $\pi_{P_k}^{\text{LMP}} = \lambda^\star$ and:

t	1	2
$\overline{D}_t$	400	600
$P_{1t}^\star$	200	300
$P_{2t}^\star$	200	300
$\pi_{P_{kt}}^{\text{LMP}}$	2	8

Ramp-constrained example, continued

- The price for energy in hour $t = 1$ is $\pi^{\text{LMP}}_{P_{k1}} = \lambda^\star_1 = \$2/\text{MWh}$:
  - generator 1 with offer price $\nabla f_{11}(P^\star_{11}) = \$2/\text{MWh}$ is marginal, but
  - the price is *lower* than the offer price of $\nabla f_{21}(P^\star_{21}) = \$5/\text{MWh}$ for generator 2, even though this generator is dispatched above its minimum.

Ramp-constrained example, continued

- Generator 2 is operating *above* its minimum in hour $t = 1$, so it is operating at a loss in hour $t = 1$ and could reduce its operating losses if it operated at its minimum in hour $t = 1$.
- Why would generator operate above its minimum in hour $t = 1$ when the price is only \$2/MWh?
- The price for energy in hour $t = 2$ is $\pi_{P_{k2}}^{\text{LMP}} = \lambda_2^{\star} =$ \$8/MWh, which is higher than the higher offer price of both generators!
- The price in hour $t = 2$ is necessary to induce generator 2 to produce at a loss in hour $t = 1$:
  - The infra-marginal rent in hour $t = 2$ equals the loss in hour $t = 1$ for generator 2.
  - Generator 2 is indifferent to any levels of production that involve $P_{22} - P_{21} = \Delta_1$.
  - The prices support the dispatch but do not strictly support dispatch.

Ramp-constrained example, continued

- Generator 2 is marginal in hour $t = 2$ in that changes to its offer price $\nabla f_{22}(P_{22}^\star)$ in hour $t = 2$ would affect the price $\lambda_2^\star$ in hour $t = 2$:
  - the price in hour $t = 2$ is $\lambda_2^\star = \nabla f_{22}(P_{22}^\star) + (\nabla f_{21}(P_{21}^\star) - \nabla f_{11}(P_{11}^\star))$.
- But note that offers of generators 1 and 2 in hour $t = 1$ also affect the price in hour $t = 2$:
  - we might say that generators 1 and 2 are also "marginal" in hour 1, but this sense is somewhat different to the earlier use of "marginal" since offer prices $\nabla f_{11}(P_{11}^\star)$ and $\nabla f_{21}(P_{21}^\star)$ of generators 1 and 2 in hour $t = 1$ are both involved in setting the price for hour $t = 2$.
- Prices are above the highest marginal cost because there are binding ramp-rate constraints.
  - We also saw in Homework Exercise 9.2 that prices can also rise above the highest offer price in the presence of binding transmission constraints.

10.1.3.4 Discussion

- This example is somewhat unrealistic for several reasons:
  - Ramp-rate constraints are typically not binding across multiple hours in markets such as ERCOT (but increased wind generation may change this in the morning ramp-up of demand and the evening ramp-down of demand, and evening ramp up of net load in California already involves large ramps across multiple hours).
  - The more expensive generator has the tighter ramp-rate constraint.
  - Some day-ahead markets, such as the ERCOT market, do not represent ramp-rate constraints (several other US ISOs do represent ramp rates in day-ahead).
- This particular example requires *anticipation* across multiple intervals (in this case hours) to find the optimal solution:
  - Anticipation across multiple intervals is not always necessary for finding the ramp-constrained optimum.
  - See homework exercise 10.5.

Discussion, continued

- As will be discussed in Section 10.2, **day-ahead** markets provide all prices to market participants for a full day at once and can therefore support anticipation:
  - but, as mentioned, the ERCOT day-ahead market, for example, does not (currently) represent ramp-rate constraints,
  - several other markets do represent ramp-rate constraints in day-ahead.
- Some **real-time** markets do represent ramp-rate constraints across several (five minute) dispatch intervals in so-called **lookahead** dispatch:
  - California market, PJM, and MISO,
  - The typical arrangement with lookahead dispatch in the real-time market is to solve multi-interval dispatch (and in some cases unit commitment) for several intervals but to only commit to prices and dispatch for the next interval.

Discussion, continued

- If market participants do not anticipate prices in subsequent intervals (or if these prices are not implemented) then the market cannot incentivize sequences of dispatch through time that involve anticipation:
  - Real-time markets can represent ramp-rate constraints on change in generation between most recent interval and the next interval (see Homework Exercise 10.2), but
  - Anticipation is required to incentivize actions when, for example, there are binding ramp rate constraints between *two or more* successive *pairs* of dispatch intervals (as was necessary in the ramp-constrained example in Section 10.1.3.3).
- Despite the implications of anticipation, the example illustrates that inter-temporal constraints do not *per se* present fundamental difficulties for pricing so long as future prices are anticipated:
  - ramp-constrained economic dispatch is convex.

Discussion, continued

- In the next section, we will see that non-convexities introduced by our formulation of unit commitment decisions do pose difficulties for pricing.
- Analogously, “minimum income condition constraints” in some European market designs such as EUPHEMIA also pose difficulties for pricing.

10.2 Formulation of unit commitment

- Now we consider the commitment of generators.
- In US day-ahead markets, the ISO makes decisions today about commitment, dispatch, and prices for tomorrow, solving the **day-ahead** unit commitment problem, resulting in:
  - a commitment decision for each participating generator for each hour of tomorrow,
  - an energy dispatch decision and ancillary services decisions for each generator for each hour of tomorrow, and
  - prices for energy and ancillary services for each hour of tomorrow.
- That is, day-ahead prices are announced for all hours of tomorrow, allowing for anticipation.

Formulation of unit commitment, continued

- In contrast, in several European markets and the Australian market, “decentralized” commitment decisions are typically made by generation owners:
  - the optimization formulation we will develop would typically be solved by individual owners for their own portfolio, even if there is also a day-ahead economic dispatch market, while the day-ahead EU market itself is similar to, but not exactly the same as, the formulation of the ramp-constrained economic dispatch formulation in the last section,
  - in US markets it is also generally possible for individual generation owners to make such decentralized commitment decisions.
- Our motivation for developing centralized unit commitment is that the cost of incorrect decentralized commitment decisions could be large, particularly when transmission constraints are binding.
  - However, the cost of incorrect decentralized commitment decisions is an empirical question that has not been studied in a systematic way, except for particular case studies such as in the ERCOT “backcast” study, which estimated hundreds of millions of dollars in savings.

Formulation of unit commitment, continued

- Unlike the economic dispatch problems and the generalizations we have considered so far, unit commitment requires **integer** variables to represent the decisions.
- The integer variables present difficulties in two related ways:

  (i) solving the problem, and

  (ii) non-existence of dispatch- (and commitment-) supporting prices.

- In Section 10.3, we will briefly describe mixed-integer linear programming software for solving these problems, as now used by all ISOs in the US.
- In Section 10.4, we will introduce make-whole payments as an approach to provide incentives to generators to commit and dispatch consistent with the commitment and dispatch solution obtained by the ISO.

Formulation of unit commitment, continued

- In Section 10.5, we will then apply **Lagrangian relaxation** (introduced in Section 4.7.4), by dualizing the supply–demand constraints and seeking the dual maximizer, as an approach to approximately solving the unit commitment problem.
- Lagrangian relaxation will help us to understand:
  - the difficulty in solving unit commitment problems, and
  - why the previous approach to finding dispatch-supporting prices for convex problems using Theorem 8.1 from Section 8.12.4.4 works for intertemporal issues such as ramping constraints, but does not (quite) work in the context of unit commitment.
  - the discussion generalizes the case considered in Section 4.8.3.
- In the exercises, we will also explore alternative formulations of unit commitment and more computationally efficient approaches to finding the dual maximizer:
  - see Exercises 10.3, 10.4, and 10.6.

10.2.1 Decision variables

- We will consider a typical unit commitment formulation where decisions are made for each hour over a time horizon:
  – day-ahead unit commitment involves 24 hours for tomorrow.
- As in the ramp-constrained economic dispatch formulation in Section 10.1.2, assume that generators can provide energy and one type of reserve, so the continuous decision variables for generator k in hour $t = 1, \ldots, n_T$, are $x_{kt} = \begin{bmatrix} P_{kt} \\ S_{kt} \end{bmatrix}$, typically with $n_T = 24$.
- We collect the entries x_{kt} together into a vector $x_k \in \mathbb{Z}^{2n_T}$ and collect the vectors x_k together into a vector $x \in \mathbb{Z}^{2n_P n_T}$.
- In addition to these continuous decision variables, we must consider representation of the decision of a generator to be on or off.
- We will represent this with **binary** variables:

$$z_{kt} = \begin{cases} 0, & \text{if generator } k \text{ is off in hour } t, \\ 1, & \text{if generator } k \text{ is on in hour } t. \end{cases}$$

- We collect the entries z_{kt} together into a vector $z_k \in \mathbb{Z}^{n_T}$ and collect the vectors z_k together into a vector $z \in \mathbb{Z}^{n_P n_T}$.

Decision variables, continued

- Other more general representations may be necessary in some cases:
  - **combined-cycle** generators typically have multiple operating modes, requiring **integer** or several binary variables to represent the commitment decision for each hour,
  - additional continuous generator variables may be defined to allow for convenient representation of the objective or constraints.
- Other market designs, such as EUPHEMIA also use binary variables to represent some issues.
- Various “tricks” are typically used in the specification of problems with integer and binary variables in order to facilitate solution:
  - some of these tricks are proprietary or not widely known, and
  - we will simply consider a straightforward formulation in the main discussion,
  - we will explore some of the tricks in Exercises 10.3, 10.4, and 10.6:
    - will involve expanding the decision vector to include additional continuous generator variables u.

10.2.2 *Generator constraints*

- We can consider the requirement for z_{kt} to be binary as consisting of two requirements:

$$\begin{aligned} z_{kt} &\in \{z_{kt} \in \mathbb{R} | 0 \leq z_{kt} \leq 1\}, \\ z_{kt} &\in \mathbb{Z}. \end{aligned}$$

- The first requirement that z_{kt} be between 0 and 1 is an example of a generator constraint that can be represented with linear inequalities.
  - This fits our previous formulation for economic dispatch.
  - As previously, suitable $\underline{\delta}_k$, $\overline{\delta}_k$, and Γ_k can be found to express such generator constraints in the form:

$$\underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k.$$

  - For example, the constraint $0 \leq z_{kt} \leq 1, \forall t$ could be expressed as:

$$\mathbf{0} \leq [\mathbf{I} \ \ \mathbf{0}] \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \mathbf{1}.$$

Generator constraints, continued

- The requirement that generator k is either off (and not producing) or on (and producing between minimum and maximum capacity limits) can also be expressed with linear inequalities:
  - ignoring reserves, the requirements are:

$$\underline{P}_k z_{kt} \leq P_{kt} \leq \overline{P}_k z_{kt}, \forall t,$$

    where $\underline{P}_k$ and $\overline{P}_k$ are the minimum and maximum production capacities;
  - including one type of reserve specified by S_{kt}, the requirements are:

$$\begin{aligned} \underline{P}_k z_{kt} \leq P_{kt} \leq \overline{P}_k z_{kt}, \forall t, \\ \underline{S}_k z_{kt} \leq S_{kt} \leq \overline{S}_k z_{kt}, \forall t \\ \underline{P}_k z_{kt} \leq P_{kt} + S_{kt} \leq \overline{P}_k z_{kt}, \forall t, \end{aligned}$$

    where $\underline{S}_k$ and $\overline{S}_k$ are the lower and upper limits on reserve; and
  - both of these requirements can be expressed in the form:

$$\underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k.$$

Generator constraints, continued

- For example, consider a simplified single interval model including energy and reserve, with $P_k = [P_{k1}]$, $S_k = [S_{k1}]$, and $x_k = \begin{bmatrix} P_{k1} \\ S_{k1} \end{bmatrix}$.
- We can express the generator constraints in the form $\underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k$ by defining $\Gamma_k \in \mathbb{R}^{r_k \times 3}$, $\underline{\delta}_k \in \mathbb{R}^{r_k}$, and $\overline{\delta}_k \in \mathbb{R}^{r_k}$, with $r_k = 6$, as follows:

$$\Gamma_k = \begin{bmatrix} -\underline{P}_k & 1 & 0 \\ -\overline{P}_k & 1 & 0 \\ -\underline{S}_k & 0 & 1 \\ -\overline{S}_k & 0 & 1 \\ -\underline{P}_k & 1 & 1 \\ -\overline{P}_k & 1 & 1 \end{bmatrix}, \underline{\delta}_k = \begin{bmatrix} 0 \\ -M \\ 0 \\ -M \\ 0 \\ -M \end{bmatrix}, \overline{\delta}_k = \begin{bmatrix} M \\ 0 \\ M \\ 0 \\ M \\ 0 \end{bmatrix},$$

- where M is a sufficiently large number (and the constraints corresponding to these entries are effectively ignored).
- With n_T periods, $r_k = 6n_T$.

Generator constraints, continued

- Summarizing, the requirement that z_{kt} be integer-valued and the requirements on x_k yields a non-convex feasible operating set for each generator:

$$\mathbb{S}_k = \left\{ \begin{bmatrix} z_k \\ x_k \end{bmatrix} \in \mathbb{Z}^{n_T} \times \mathbb{R}^{2n_T} \middle| \underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k \right\}.$$

- Although the constraints $\underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k$ are convex, the integrality of z_k makes the feasible set $\mathbb{S}_k$ non-convex, as in the example in Section 4.8.3.
- This means that the unit commitment problem is a non-convex problem.
- The non-convexity makes solution difficult and complicates the pricing rule as discussed in Section 4.8.

10.2.3 Generator costs

- We now assume that the cost function for generator k depends on both z_k and x_k, so that $f_k : \mathbb{Z}^{n_T} \times \mathbb{R}^{2n_T} \to \mathbb{R}$.
- For convenience, we will sometimes assume that f_k has been extrapolated to a function $f_k : \mathbb{R}^{n_T} \times \mathbb{R}^{2n_T} \to \mathbb{R}$.
- The cost function for generator k represents:

  the cost of producing energy and of providing reserve (already considered in the dispatch problem),

  start-up costs, and

  no-load or **min-load costs** (typically associated with auxiliary costs as illustrated in Figure 5.2).

- Because start-up costs can depend on *changes* in commitment status, the cost function is no longer (completely) additively separable.

Generator costs, continued

- However, costs can usually be considered to be the sum of costs associated with:

  start-up costs, expressible in terms of the integer variables z_k, (but not additively separable across time in the most straightforward formulation),

  no-load or min-load costs, additively separable across time, and expressible in terms of the integer variables $z_{kt}, t = 1, \ldots, n_T$, and

  incremental energy and reserves costs, additively separable across time, and expressible in terms of the continuous production variables x_{kt} in each interval $t = 1, \ldots, n_T$ for which the unit is running.

10.2.3.1 Start-up costs

- For a generator with a steam boiler, start-up costs include the cost of the energy needed to warm up the boiler:
  - this cost will vary with the time since last shut-down, but
  - we will ignore the variation of start-up costs with the time since last shut-down.
- Start-up costs could also vary with t because of variation in fuel costs:
  - the formulation developed here will allow for start-up costs that vary with t, but
  - all examples will have start-up costs that are independent of t.

- Start-up costs can be expressed in terms of z_k:

$$\sum_{t=1}^{n_T} s_{kt} z_{kt} \left(1 - z_{k,(t-1)}\right), \tag{10.2}$$

where:

s_{kt} are the start-up costs for starting up in interval t, ignoring variation of start-up cost with time since last shutdown, and

z_{k0} is the commitment status at the end of today.

- That is, start-up costs are only incurred when a generator was off in hour $t-1$ (so that $\left(1 - z_{k,(t-1)}\right) = 1$) and on in hour t (so that $z_{kt} = 1$).
- This formulation is non-linear and non-separable across time:
  - by defining additional variables $u_k \in \mathbb{R}^{n_T}$ and constraints, a linear re-formulation is possible that is more suitable for use with standard software (see in Exercise 10.4);
  - for now, we will continue with the non-linear formulation since it emphasizes the coupling of decisions between hours, but we will use the linear re-formulation in Section 10.3.

10.2.3.2 Minimum-load costs

- Minimum-load (Min-load) costs are the costs to operate at the minimum capacity, $P_k = \underline{P}_k, S_k = 0$ during an interval when the unit is committed.
- Min-load costs depend on z_k and are additively separable across time and can be expressed in the form:

$$\sum_{t=1}^{n_T} \underline{f}_{kt} \times z_{kt},$$

where $\underline{f}_{kt}$ is the min-load costs per interval for operating at $\underline{P}_k$.
- In some markets, including ERCOT, the min-load costs $\underline{f}_{kt}$ are expressed as the product of:

  a min-load average energy cost per unit energy, multiplied by
  the minimum capacity $\underline{P}_k$.
- In some markets, such as MISO, **no-load costs** are specified, in which case energy offers are interpreted as specifying costs for production above zero:
  - actual values of production are still required to be at or above minimum capacity $\underline{P}_k$.

10.2.3.3 Incremental energy and reserves costs

- Incremental energy and reserves costs for operating above minimum production depend on the value of x_{kt} in each interval for which the unit is running, and are additively separable across time.
- We will again assume that energy and reserves costs are themselves additively separable as the sum of terms due to producing energy and providing reserves, as in Section 8.12.1.3.
- Consider the marginal costs ∇f_{ktP} for generator k to produce energy and the marginal costs ∇f_{ktS} to provide reserve in interval t, assuming that the generator is in-service.
- The operating cost during an interval t when generator k is in-service is equal to the sum of the min-load costs and the incremental energy and reserve costs for operation above minimum capacity $\underline{P}_k$.

Incremental energy and reserves costs, continued

- The incremental energy and reserve costs for operation above minimum capacity $\underline{P}_k$ in interval t can be evaluated from the sum of the two integrals:

$$\int_{P'_{kt}=\underline{P}_k}^{P'_{kt}=P_{kt}} \nabla f_{ktP}(P'_{kt})\, dP'_{kt} + \int_{S'_{kt}=0}^{S'_{kt}=S_{kt}} \nabla f_{ktS}(S'_{kt})\, dS'_{kt},$$

- where P_{kt} is the generation level and S_{kt} the reserve contribution.
- Min-load costs (and start-up costs) must be added to the incremental energy and reserves costs to evaluate the cost function for generator k.

10.2.4 Objective

- Adding together the start-up costs, the min-load costs, and the incremental energy and reserve costs, the cost function of generator k is therefore:

$$\forall z_k \in \mathbb{Z}^{n_T}, \forall x \in \mathbb{R}^{n_T}, f\left(\begin{bmatrix} z_k \\ x_k \end{bmatrix}\right) = \sum_{t=1}^{n_T} \left[s_{kt}(1 - z_{k,(t-1)}) + \underline{f}_{kt} \right.$$
$$\left. + \int_{P'_{kt}=\underline{P}_k}^{P'_{kt}=P_{kt}} \nabla f_{ktP}(P'_{kt})\, dP'_{kt} + \int_{S'_{kt}=0}^{S'_{kt}=S_{kt}} \nabla f_{ktS}(S'_{kt})\, dS'_{kt} \right] z_{kt}. \quad (10.3)$$

- Typically, the incremental reserve costs $\int_{S'_{kt}=0}^{S'_{kt}=S_{kt}} \nabla f_{ktS}(S'_{kt})\, dS'_{kt}$ are zero.
- This function is non-linear in $\begin{bmatrix} z_k \\ x_k \end{bmatrix}$.
- By considering the generator constraints, and by including some additional variables u_k and constraints, the form of (10.3) can be re-formulated so that it is linear in $\begin{bmatrix} z_k \\ x_k \\ u_k \end{bmatrix}$ (see in Exercises 10.3 and 10.4).

Objective, continued

- As previously, we define the objective of the unit commitment problem to be the sum of the cost functions of all of the generators:

$$\forall z \in \mathbb{Z}^{n_P n_T}, x \in \mathbb{R}^{2 n_P n_T}, f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) = \sum_{k=1}^{n_P} f_k\left(\begin{bmatrix} z_k \\ x_k \end{bmatrix}\right).$$

10.2.5 System constraints

- Typical system equality constraints would include average power balance in each hour of tomorrow, which we will represent in the general form $Ax = b$.
- For example, as in Section 10.1.2.2:
  - if we ignore reserve, then $x = P = \begin{bmatrix} P_1 \\ \vdots \\ P_{n_P} \end{bmatrix} \in \mathbb{R}^{n_P n_T}$, with

    $$P_k = \begin{bmatrix} P_{k1} \\ \vdots \\ P_{kn_T} \end{bmatrix} \in \mathbb{R}^{n_T},$$
  - let $\overline{D} \in \mathbb{R}^{n_T}$ be a vector of forecasts of average demand in each hour,
  - let $A = \begin{bmatrix} -\mathbf{I} & \cdots & -\mathbf{I} \end{bmatrix}$ and $b = -\overline{D}$,
  - then $Ax = b$ represents average power balance in each hour.
- Typical system inequality constraints would include reserve requirements and transmission constraints in each hour, which we will represent in the general form $Cx \leq d$.

10.2.6 Problem

- The unit commitment problem is:

$$\min_{\forall k, \begin{bmatrix} z_k \\ x_k \end{bmatrix} \in \mathbb{S}_k} \left\{ f\left(\begin{bmatrix} z \\ x \end{bmatrix} \right) \middle| Ax = b, Cx \leq d \right\}$$
$$= \min_{z \in \mathbb{Z}^{n_P n_T}, x \in \mathbb{R}^{2 n_P n_T}} \left\{ f\left(\begin{bmatrix} z \\ x \end{bmatrix} \right) \middle| Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \end{bmatrix} \leq \overline{\delta}_k \right\}. \tag{10.4}$$

- In principle, the ISO obtains offers from the market participants that specify f, and then solves Problem (10.4) for **optimal commitment and dispatch**, which we will denote by $z^\star$ and $x^\star$, respectively.
- In some examples and some of the development, we will only consider energy and not reserve, in which case, $x = P \in \mathbb{R}^{n_P n_T}$, as in the example in Section 10.2.5.

10.2.7 Generator offers

- How to specify the offers from generators to the ISO?
- Building on offer-based economic dispatch, we will still assume that the dependence of offers on power and reserves are specified as the gradient of costs with respect to power and reserves.
- We will assume that the dependence of offers on power and reserves are required to be separable across time, so that the offers are specified by:

$$\nabla f_{kt} = \begin{bmatrix} \nabla f_{ktP} \\ \nabla f_{ktS} \end{bmatrix}, t = 1, \ldots, n_T,$$

- with the understanding that the offer function dependence on power and reserves is only meaningful in interval t if $z_{kt} = 1$, and where we are considering only one type of reserve for simplicity.
- We will call this collection of functions $\nabla f_{kt}, t = 1, \ldots, n_T$, the *incremental* energy and reserve offers, to emphasize that there are other components of the cost, namely start-up costs and min-load costs.
- Although the notation allows for different incremental energy and reserve costs for each interval, market rules may restrict this choice.

Generator offers, continued

- To specify the start-up costs, the values of $s_{kt}, t = 1, \ldots, n_T$ are required.
- To specify the min-load costs, the values of $\underline{f}_{kt}, t = 1, \ldots, n_T$ are required.
- We will assume that the generator specifies:

  a **start-up offer** equal to its start-up costs,
  a **min-load offer** equal to its min-load costs, and
  an **incremental energy and reserve offer** equal to its incremental energy and reserve costs.

- The **offer cost function** can then be reconstructed from the start-up offer, the min-load offer, and and the incremental energy and reserve offers using (10.3), given that the minimum production level $\underline{P}_k$ is known.
- Assuming that the incremental reserve offer costs are zero, the offer cost function is:

$$\forall z_k \in \mathbb{Z}^{n_T}, \forall x \in \mathbb{R}^{n_T}, f\left(\begin{bmatrix} z_k \\ x_k \end{bmatrix}\right)$$
$$= \sum_{t=1}^{n_T} \left[s_{kt}(1 - z_{k,(t-1)}) + \underline{f}_{kt} + \int_{P'_{kt}=\underline{P}_k}^{P'_{kt}=P_{kt}} \nabla f_{ktP}(P'_{kt}) \, dP'_{kt} \right] z_{kt}.$$

Generator offers, continued

- In contrast to the economic dispatch problem, it is necessary to explicitly represent the cost function (and not just its derivative) in the unit commitment problem in order to:
  - compare alternative costs of committing and dispatching different combinations of generators in Problem (10.4), and
  - (as we will see in Section 10.4) to calculate **make-whole** costs.
- The assumption that costs are truthfully revealed by the offers is not innocuous:
  - the analysis of energy offers in Section 8.11.2 does not apply to start-up and minimum-load offers, even if each generator cannot affect the *energy* prices.
  - markets such as ERCOT have additional procedures to verify that start-up and minimum-load offers reflect costs.

10.3 Mixed-integer programming

- Commercial software for solving mixed-integer programming problems has become much more capable in the last two decades.
- The highest performance algorithms are for mixed-integer linear programming (MILP).
- Exercises 10.3 and 10.4 show how to re-formulate the unit commitment Problem (10.4) so that it has a linear objective by incorporating additional continuous variables and linear constraints into the problem.
- This allows the unit commitment problem to be re-formulated into a mixed-integer linear program of the form of problem (4.44).

10.3.1 Mixed-integer programming formulation of unit commitment

- That is, unit commitment can be formulated as:

$$\min_{\substack{z \in \mathbb{Z}^{n_P n_T}, \\ x \in \mathbb{R}^{2n_P n_T}, \\ u \in \mathbb{R}^{n_P n_T}}} \left\{ c^{\dagger} \begin{bmatrix} z \\ x \\ u \end{bmatrix} \middle| Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \\ u_k \end{bmatrix} \leq \overline{\delta}_k \right\}, \qquad (10.5)$$

  where:
  - the decision variables can now include additional continuous variables u besides the energy generation and reserve contribution in order to represent start-up issues (see Exercises 10.3 and 10.4),
  - the generator constraints $\underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k \\ x_k \\ u_k \end{bmatrix} \leq \overline{\delta}_k$ now include additional constraints to represent start-up issues (and also to represent minimum up- and down-times; see Exercise 10.4), and
  - the integer variables z and the additional continuous variables u to represent start-up issues do not appear in the system constraints $Ax = b, Cx \leq d$.

Mixed-integer programming formulation of unit commitment, continued

- All US ISOs now use mixed-integer programming algorithms for solving unit commitment.
- In principle, MILP algorithms can exactly solve the unit commitment problem.
- In practice requirements on the time-to-solve may require that a feasible but sub-optimal solution be accepted.
- We will nevertheless suppose that the ISO can solve Problem (10.4) (or its linear re-formulation, Problem (10.5)), and that the minimizer is $z^\star$ and $x^\star$ (together with $u^\star$ in the case of the linear re-formulation).

10.3.2 Unit commitment example

- Consider the previous example in Section 4.8.3 where a single generator was available to meet a demand of $\overline{D} = 3$ MW in the single interval $n_T = 1$.
- The generator had two variables associated with its operation:
  - the "unit commitment" variable $z \in \mathbb{Z}$, and
  - the "production" variable $x = P \in \mathbb{R}$.
- The cost function $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ for the generator and its generator constraints are:

$$f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) = 4z + x, z \in \{0,1\}, 2z \leq x \leq 4z,$$

- with min-load costs of 6, and marginal cost of incremental energy of 1.
- This unit commitment problem is in the form of a mixed-integer linear program, which we repeat from (4.45):

$$\min_{z \in \mathbb{Z}, x \in \mathbb{R}} \{4z + x | -x = -3, 0 \leq z \leq 1, 2z \leq x \leq 4z\}, \qquad (10.6)$$

- The solution is $z^\star = 1$ and $x^\star = 3$, with generator cost $4z^\star + x^\star = 7$.

10.3.3 Unit commitment example with two generators

- Now suppose that there are two generators available to meet a demand of $\overline{D}$ MW in the single interval $n_T = 1$.
- Generator $k = 1,2$ each has two variables associated with its operation:
  - the "unit commitment" variable $z_k \in \mathbb{Z}$, and
  - the "production" variable $x_k = P_k \in \mathbb{R}$.
- The cost functions $f_k : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}, k = 1,2$ for the generators and their generator constraints are:

$$
\begin{aligned}
f_1\left(\begin{bmatrix} z_1 \\ x_1 \end{bmatrix}\right) &= 4z_1 + x_1, z_1 \in \{0,1\}, 2z_1 \leq x_1 \leq 4z_1, \\
f_2\left(\begin{bmatrix} z_2 \\ x_2 \end{bmatrix}\right) &= z_2 + 2x_2, z_2 \in \{0,1\}, 0.5z_2 \leq x_2 \leq 4z_2.
\end{aligned}
$$

- This unit commitment problem is also in the form of a mixed-integer linear program:

$$
\min_{z \in \mathbb{Z}^2, x^2 \in \mathbb{R}} \left\{ 4z_1 + z_2 + x_1 + 2x_2 \,\middle|\, \begin{array}{l} -x_1 - x_2 = -\overline{D}, \mathbf{0} \leq z \leq \mathbf{1}, \\ 2z_1 \leq x_1 \leq 4z_1, 0.5z_2 \leq x_2 \leq 4z_2 \end{array} \right\}.
$$

Unit commitment example with two generators, continued

- Minimum capacity of generator 1 is 2, while minimum capacity of generator 2 is 0.5:
  - For $\overline{D} < 0.5$ there is no feasible solution,
  - For $0.5 \leq \overline{D} < 2$, the only feasible (and therefore optimal) solution is $z_1^\star = x_1^\star = 0, z_2^\star = 1, x_2^\star = \overline{D}$.
- Maximum capacity of generator 1 and of generator 2 is 4:
  - For $\overline{D} > 8$ there is no feasible solution.
  - For $4 < \overline{D} \leq 8$, both generators must be on, generator 1 has the lower marginal cost, so $z_1^\star = z_2^\star = 1, x_1^\star = 4, x_2^\star = \overline{D} - 4$.
- For $2 \leq \overline{D} < 3$, generator 2 is cheapest to meet demand.
- For $3 \leq \overline{D} < 4$, generator 1 is cheapest to meet demand.
- For $\overline{D} = 3$, generator 1 and 2 have the same cost of 7 to meet demand.
- If there was more than one interval, if the generators had start-up costs, and if demand varied across intervals, then the problem would be more difficult to solve because of the interaction between start up costs and the min-load and incremental energy costs. (See Exercise 10.5.)

10.4 Make-whole costs

10.4.1 Implementing the results of unit commitment

- We now consider payments to the generators.
- Based on the discussion in Sections 8.10 and 8.11 and based on Theorem 8.1 in Section 8.12.4.4, we might consider setting prices for energy based on the Lagrange multipliers on the supply–demand balance constraint and other system constraints from a continuous optimization problem.
- In most US ISOs, the practice is to define the continuous problem by setting z and u in Problem (10.5) to be equal to the optimal values $z^\star$ and $u^\star$ and then solve the resulting continuous problem:

$$\min_{x \in \mathbb{R}^{2n_P n_T}} \left\{ c^\dagger \begin{bmatrix} z^\star \\ x \\ u^\star \end{bmatrix} \middle| Ax = b, Cx \leq d, \forall k, \underline{\delta}_k \leq \Gamma_k \begin{bmatrix} z_k^\star \\ x_k \\ u_k^\star \end{bmatrix} \leq \overline{\delta}_k \right\}, \tag{10.7}$$

- which is in the same form as the ramp-constrained economic dispatch Problem (10.1), is convex, and has similar properties to Problem (10.1).

Implementing the results of unit commitment, continued

- Typically MILP implementations solve a continuous problem of the form of Problem (10.7) during the solution process, so that the Lagrange multipliers on the system constraints $Ax = b, Cx \leq d$ in Problem (10.7) are available as a *by-product* of the MILP algorithm.
- Note that the minimizer of Problem (10.7) is the same as the minimizer $x^\star$ of Problem (10.5):
  - key difference is that there are well-defined Lagrange multipliers on the system constraints $Ax = b, Cx \leq d$ in Problem (10.7), whereas Problem (10.5) does not have well-defined Lagrange multipliers because of the integer variables.

Implementing the results of unit commitment, continued

- Let $\lambda^\star$ and $\mu^\star$, respectively, be the Lagrange multipliers on the system constraints $Ax = b, Cx \leq d$ in Problem (10.7).
- As in discussion of offer-based economic dispatch and locational marginal pricing, we can define prices using the pricing rule:

$$\pi_{x_k}^{\text{LMP}} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star. \tag{10.8}$$

- We have labeled these prices with superscript LMP to emphasize that the prices are from the solution of essentially the same problem as the problem solved for LMPs and in ramp-constrained economic dispatch:
  - formulation so far has not represented transmission constraints, but these will be considered in Section 10.8,
  - as in the discussion of ramp constraints in Section 10.1.3.2, a sequence of LMPs for the intervals in the day are being calculated,
  - if ramp rates were included in the unit commitment formulation, they would also be represented in Problem (10.7).

Implementing the results of unit commitment, continued

- If the generators happen to be committed consistently with $z^\star$ then, by Theorem 8.1 in Section 8.12.4.4, the prices $\pi_{x_k}^{\text{LMP}}$ provide incentives for profit-maximizing generators to dispatch consistently with the solution $x^\star$.
- However, these prices $\pi_{x_k}^{\text{LMP}}$ will not always provide incentives for profit-maximizing generators to *commit* and dispatch consistently with the solution $z^\star$ and $x^\star$ (and $u^\star$):
  - revenue from energy payment may not cover the start-up, min-load, and incremental energy costs,
  - this issue was explored in Section 4.7.4 and specifically in Sections 4.8.3 and 4.8.4 in the context of a very simple unit commitment problem for which there was no choice of prices on energy that could provide incentives for a profit-maximizing generator to *commit* and dispatch consistently with the ISO solution, and
  - the same issue can occur in general in unit commitment problems because of the non-convexity.

10.4.2 Unit commitment example

- Consider the previous example in Sections 4.8.3 and 10.3.2 where a single generator was available to meet a demand of $\overline{D} = 3$ MW in the single interval $n_T = 1$.
- The unit commitment problem (10.6) is:

$$\min_{z \in \mathbb{Z}, x \in \mathbb{R}} \{4z + x | -x = -3, 0 \leq z \leq 1, 2z \leq x \leq 4\},$$

- The corresponding problem (10.7) (with simplifications since there are no start-up variables nor constraints and no system inequality constraints) is:

$$\min_{x \in \mathbb{R}} \{4z^{\star} + x | -x = -3, 0 \leq z^{\star} \leq 1, 2z^{\star} \leq x \leq 4z^{\star}\},$$

- which has solution $x^{\star} = 3$.
- The Lagrange multiplier on the supply-demand constraint is $\lambda^{\star} = 1$.

Unit commitment example, continued

- Recall that if the generator were paid π for its production then its profit maximizing behavior would be:

$$x = \begin{cases} 0, & \text{if } \pi < 2, \\ 0 \text{ or } 4, & \text{if } \pi = 2, \\ 4, & \text{if } \pi > 2. \end{cases}$$

- This meant that no price would equate supply to demand of 3 MW.
- In particular, if we set the price using (10.8), we have:

$$\pi_x^{\text{LMP}} = -[-1]\lambda^\star = 1.$$

- The revenue for generating $x^\star = 3$ at this price is $\pi_x^{\text{LMP}} \times x^\star = 3$, but the cost of generating at this level is 7.
- A profit-maximizing generator will not choose to commit and generate at the level $x^\star = 3$ if the compensation is only based on its energy production remunerated at the price of π_x^{LMP}.

10.4.3 Unit commitment example with two generators

- Consider the previous example in Section 10.3.3, with two generators available to meet a demand of $\overline{D}$ MW in the single interval $n_T = 1$.
- The unit commitment problem (10.6) is:

$$\min_{z \in \mathbb{Z}^2, x^2 \in \mathbb{R}} \left\{ 4z_1 + z_2 + x_1 + 2x_2 \;\middle|\; \begin{array}{l} -x_1 - x_2 = -\overline{D}, \mathbf{0} \leq z \leq \mathbf{1}, \\ 2z_1 \leq x_1 \leq 4z_1, 0.5z_2 \leq x_2 \leq 4z_2 \end{array} \right\}.$$

- The corresponding problem (10.7) (with simplifications since there are no start-up variables nor constraints and no system inequality constraints) is:

$$\min_{x^2 \in \mathbb{R}} \left\{ 4z_1^\star + z_2^\star + x_1 + 2x_2 \;\middle|\; \begin{array}{l} -x_1 - x_2 = -\overline{D}, \mathbf{0} \leq z^\star \leq \mathbf{1}, \\ 2z_1^\star \leq x_1 \leq 4z_1^\star, 0.5z_2^\star \leq x_2 \leq 4z_2^\star \end{array} \right\},$$

- which has solution $x^\star$ and Lagrange multiplier $\lambda^\star$ on the supply-demand balance constraint of:

$x_1^\star = 0, x_2^\star = \overline{D}, \lambda^\star = 2$, for $0.5 \leq \overline{D} < 3$,
$x_1^\star = \overline{D}, x_2^\star = 0, \lambda^\star = 1$, for $3 \leq \overline{D} < 4$,
$x_1^\star = 4, x_2^\star = \overline{D} - 4, \lambda^\star = 2$, for $4 \leq \overline{D} \leq 8$.

Unit commitment example with two generators, continued

- Using (10.8), we again set the price $\pi_{x_k}^{\mathrm{LMP}}$ equal to the Lagrange multiplier $\lambda^\star$ on the supply-demand balance constraint $-x_1 - x_2 = -\overline{D}$.
- Figure 10.1 shows the resulting price $\pi_{x_k}^{\mathrm{LMP}}$ versus demand $\overline{D}$:
  - we could also interpret this curve as showing the supply curve as in Section 6.5.
- We again find that profit-maximizing generators will not typically choose to commit and generate at the level $x^\star$ if the compensation is only based on its energy production remunerated at the price of $\pi_{x_k}^{\mathrm{LMP}}$.
- Moreover, note that the prices are not non-decreasing with demand:
  - in contrast, in Figure 6.6 the supply curve was non-decreasing.

Unit commitment example with two generators, continued

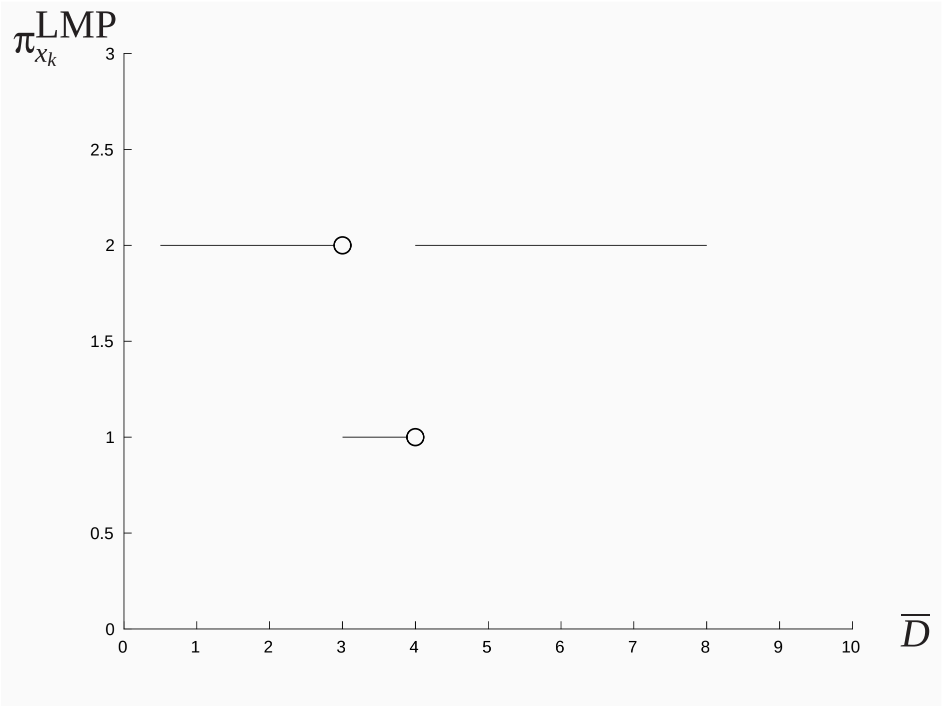

Fig. 10.1. Locational marginal price $\pi_{x_k}^{\text{LMP}}$ versus demand $\overline{D}$ for two generator system.

10.4.4 Aligning generator profit-maximization with ISO decisions

- The essential problem in both examples is that compensation based on prices for energy (and reserves in the more general case) do not compensate the generator for all of the costs of committing and dispatching at the levels $z^\star$ and $x^\star$ determined in the ISO unit commitment problem:
  - note that the ISO solicited the start-up, min-load, and incremental energy offers from the generators, and used this information to decide on the commitment and dispatch,
  - the ISO is asking the generators to incur operating costs based on the ISO's decisions,
  - from basic notions of property rights, the ISO must expect to at least compensate the generator for the generator offer costs incurred in committing and dispatching consistently with the ISO decisions.

Aligning generator profit-maximization with ISO decisions, continued

- We consider an approach to aligning generator profit-maximization with ISO unit commitment by compensating the generator for its offer costs at the ISO-determined commitment and dispatch levels.
- It involves an additional payment that is conditional on the generator committing according to the ISO solution.
- Suppose the ISO determines energy and reserve prices π_x:
  - for example, using (10.8), with resulting price for generator k:

$$\pi_{x_k}^{\text{LMP}} = -[A_k]^{\dagger}\lambda^{\star} - [C_k]^{\dagger}\mu^{\star}.$$

- We consider the profit maximizing response to these prices.

Aligning generator profit-maximization with ISO decisions, continued

- For some generators, their profit maximizing generation based on these energy and reserves prices will be consistent with the ISO decision:
  - these generators are paid based on these energy and reserves prices,
  - no further payment besides remuneration based on energy and reserves.
- For the rest of the generators, additional revenue is necessary to pay based on:
  - the energy and reserves prices, plus
  - an additional **make-whole** payment that is contingent on the generators committing consistently with the ISO decision.
- What would the make-whole payment be for the generator to induce it to produce 3 MW, given an energy price of $1/MWh?

Aligning generator profit-maximization with ISO decisions, continued

- We seek a general expression for the make-whole payment that would induce behavior consistent with optimal commitment and dispatch.
- Suppose the ISO specifies a vector of energy and reserve prices $\pi_{x_k} \in \mathbb{R}^{2n_T}$ for each generator k:
  - for example, LMPs as defined in (10.8),
  - will consider another choice of prices in Section 10.5.
- We consider two cases:
  - (i) generator k can choose its commitment $z_k^{\star\star}$ and dispatch and reserves $x_k^{\star\star}$ to maximize its operating profit given π_{x_k}, and
  - (ii) generator k commits and dispatches consistent with the solution of the ISO optimal commitment $z_k^{\star}$ and dispatch and reserves $x_k^{\star}$.

10.4.4.1 Generator profit maximization

- Generator k operating profit maximum, given prices π_{x_k}, is:

$$\Pi_k^{\star\star}(\pi_{x_k}) = \max_{\begin{bmatrix} z_k \\ x_k \end{bmatrix} \in \mathbb{S}_k} \left\{ [\pi_{x_k}]^\dagger x_k - f_k\left(\begin{bmatrix} z_k \\ x_k \end{bmatrix} \right) \right\},$$

- where, as previously, the double star refers to generator operating profit maximization.

10.4.4.2 Profit under optimal commitment and dispatch from ISO problem

- Given prices π_{x_k} and given that generator k operated according to the optimal commitment $z_k^\star$ and dispatch $x_k^\star$ determined by the ISO, the profit for generator k would be:

$$[\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right).$$

10.4.4.3 Comparison of profits

- Note that, by definition:

$$\Pi_k^{\star\star}(\pi_{x_k}) \geq [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right).$$

- Moreover, if:

$$\Pi_k^{\star\star}(\pi_{x_k}) = [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right).$$

- then the profit maximizing decision of generator k is consistent with the ISO optimal commitment and dispatch:
  - the vector of prices π_{x_k} **supports** the ISO optimal commitment and dispatch.

10.4.5 Make-whole payment

- We consider the two possible cases:
- If $\Pi_k^{\star\star}(\pi_{x_k}) = [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right)$:
  - then the profit maximizing behavior of generator k in response to π_{x_k} alone is consistent with optimal commitment and dispatch,
  - no make-whole payment is needed.
- If $\Pi_k^{\star\star}(\pi_{x_k}) > [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right)$:
  - then the profit maximizing behavior of generator k in response to π_{x_k} alone is inconsistent with optimal commitment and dispatch,
  - an additional make-whole payment of:

$$\Pi_k^{\star\star}(\pi_{x_k}) - \left([\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right)\right),$$

is necessary to induce behavior that is consistent with optimal commitment and dispatch.

Make-whole payment, continued

- We can combine both cases by observing that the payment is equal to

$$\Pi_k^{\star\star}(\pi_{x_k}) - \left([\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix} \right) \right), \tag{10.9}$$

  in both cases.
- Note that make-whole payment is only made to generator k if generator k commits according to $z_k^\star$.
- By design, the make-whole payment adjusts the profit for generator k so that $\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}$ is generator k's profit maximizing commitment and dispatch.
- In principle, no additional inducement is necessary for generator k to behave consistently with centralized optimal unit commitment and dispatch.

10.4.6 Simplified make-whole payment

- To develop a simplified make-whole payment, observe that there are three possibilities for profit-maximizing behavior $z_k^{\star\star}$ and $x_k^{\star\star}$ by generator k in response to the price π_{x_k}:
  (i) the generator would prefer not to commit, so that $z_k^{\star\star} = \mathbf{0}$ and $x_k^{\star\star} = \mathbf{0}$ and $0 = \Pi_k^{\star\star}(\pi_{x_k}) \geq [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right)$,
  (ii) the generator prefers to commit and dispatch consistently with ISO optimal commitment and dispatch, so that $z_k^{\star\star} = z_k^\star$ and $x_k^{\star\star} = x_k^\star$, and $\Pi_k^{\star\star}(\pi_{x_k}) = [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) > 0$, or
  (iii) the generator prefers to commit and dispatch, but inconsistently with ISO optimal commitment and dispatch, so that $z_k^{\star\star} \neq z_k^\star$ and/or $x_k^{\star\star} \neq x_k^\star$, and $\Pi_k^{\star\star}(\pi_{x_k}) > [\pi_{x_k}]^\dagger x_k^\star - f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right)$.

Simplified make-whole payment, continued

- Note that for the first alternative, a make-whole payment of $f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}]^\dagger x_k^\star$ would be required to make generator k indifferent between:
  - not committing, and
  - commitment and dispatching consistently with ISO optimal commitment and dispatch.
- In the second alternative, no make-whole payment is required since profit-maximization is consistent with ISO optimal commitment and dispatch.
- So, if we ignore the third alternative (or can otherwise prohibit the generator from committing and dispatching inconsistent with the ISO solution), then the make-whole payment can be simplified to:

$$\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}]^\dagger x_k^\star\right\}. \tag{10.10}$$

Simplified make-whole payment, continued

- The simplified make-whole payment of $\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}]^\dagger x_k^\star\right\}$ is used in ERCOT and other markets, even though it does not have the correct incentives in the case that both:
$$\Pi_k^{\star\star}(\pi_{x_k}) \neq 0, \text{ and } \Pi_k^{\star\star}(\pi_{x_k}) > f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}]^\dagger x_k^\star.$$
- The make-whole is paid only if the generator k actually commits "close enough" to $z_k^\star$ during the operating day.
- (There is also generally a requirement that the generator dispatches "close enough" to the dispatch signals in the real-time market. See Section 11.3.2.)

10.4.7 Simplified make-whole payment in unit commitment example

- For the previous example in Sections 4.8.3, 10.3.2, and 10.4.2 we have that the simplified make-whole payment is equal to:

$$\begin{aligned} \max\left\{0, f\left(\begin{bmatrix} z^{\star} \\ x^{\star} \end{bmatrix}\right) - [\pi_x]^{\dagger} x^{\star}\right\} &= \max\{0, 7-3\}, \\ &= 4, \end{aligned}$$

consistent with compensating the generator for the difference between its costs and the remuneration from energy.

10.4.8 Simplified make-whole payment in unit commitment example with two generators

- Consider the previous example in Sections 10.3.3 and 10.4.3, with two generators available to meet a demand of $\overline{D}$ MW in the single interval $n_T = 1$.
- Figure 10.2 shows the resulting simplified make-whole payment versus demand $\overline{D}$.

Unit commitment example with two generators, continued

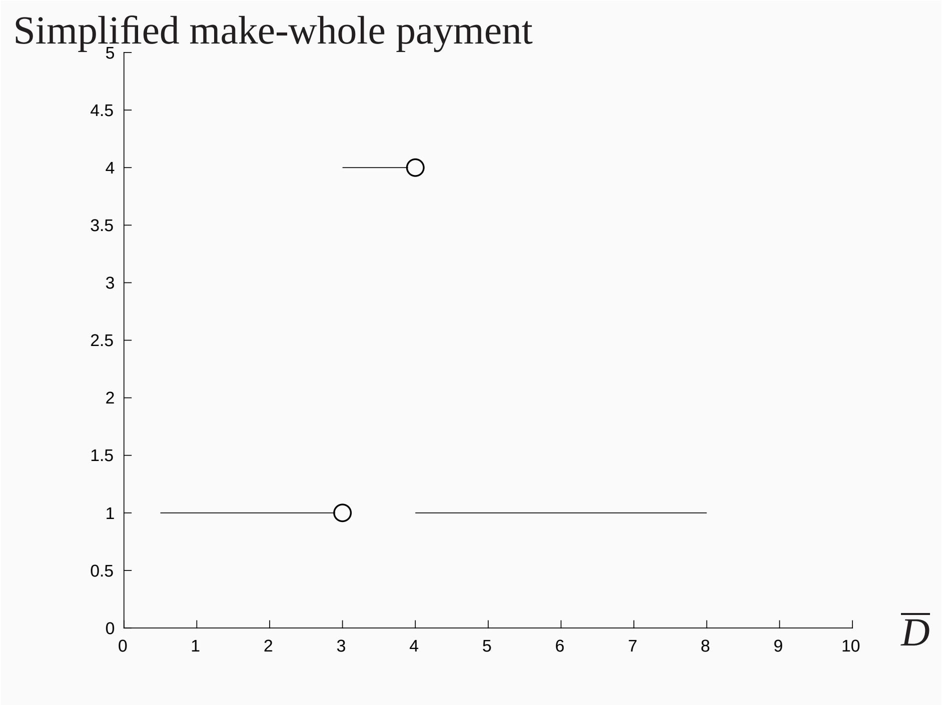

Fig. 10.2. Simplified make-whole payment with locational marginal prices versus demand $\overline{D}$ for two generator system.

10.4.9 Discussion of locational marginal prices and make-whole

- The combination of locational marginal prices and make-whole provides a straightforward approach to aligning profit maximization with the ISO commitment and dispatch:
  - the locational marginal prices are available as a by-product of the unit commitment optimization, and
  - the simplified make-whole can be conveniently calculated on a daily basis to ensure that the operating profits are non-negative day-by-day.
- Most ISOs in the US currently use a pricing rule based on locational marginal pricing for energy and reserves as specified in (10.8) and a simplified make-whole rule based on (10.10).

Discussion of locational marginal prices and make-whole, continued

- However, there are several drawbacks of locational marginal prices and make-whole payments:
  - the make-whole payments could potentially be large and, as will be discussed in more detail in Section 10.7, it is not as "visible" to the market as energy prices, which makes investment decisions about profitable new entry more difficult;
  - energy prices are not monotonically non-decreasing with increasing total production, so that an increase in demand can result in a decrease in price as in the two generator example when demand increases above 3, which can be misleading to market participants if supply is tight; and
  - the offers of generators that are at their maximum or minimum production do no play a role in setting price, although their costs are economically significant in determining unit commitment.
- In the next sections, we will consider an alternative pricing approach that reduces the severity of these issues.

10.5 Lagrangian relaxation

10.5.1 Alternative approaches to pricing

- The discussion of make-whole in the last section was not specific to locational marginal prices:
  - the make-whole payment could even be used with *arbitrary* prices.
- In this section, we will consider an alternative pricing rule based on Lagrangian relaxation of the unit commitment problem, instead of the rule in (10.8) that is based on setting the integer variables at their optimal values.
- Will consider possible advantages of such a rule.

10.5.2 Description

- As in Sections 4.8.3 and 4.8.4, we will apply Lagrangian relaxation to the unit commitment problem:
  - previously used by ISOs to approximately solve unit commitment before it was supplanted by MIP software.
- Recall from Section 4.7.4 that Lagrangian relaxation involves maximizing a dual problem.
- We dualize the system constraints to obtain the maximization problem:

$$\max_{\lambda,\mu\geq\mathbf{0}} \left\{ \min_{\forall k, \begin{bmatrix} z_k \\ x_k \end{bmatrix} \in \mathbb{S}_k} \left\{ f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) + \lambda^{\dagger}(Ax - b) + \mu^{\dagger}(Cx - d) \right\} \right\}. \tag{10.11}$$

- This problem is called the **Lagrangian dual problem**.
- Solving this problem is analogous to solving the economic dispatch problem by dualizing the system constraints:
  - in contrast to economic dispatch and the analysis in the last section, here we maintain the commitment decisions as discrete decision variables in the inner problem.

Description, continued

- Dualizing separates the unit commitment problem into:
  - a sub-problem for each generator equivalent to profit maximization for the generator given the value of dual variables, and
  - the problem of finding the values of the dual variables that maximize the dual.
- We will consider a pricing rule based on either the current value of the dual variables at a particular iteration or based on the maximizer of the dual.
- Since convenient calculation of the dual involves the convex hull as introduced in Section 4.8.3, we refer to these prices as **convex hull prices** (CHP).
- In particular, we define the **convex hull prices** to be:

$$\pi_{x_k}^{\text{CHP}} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star, \qquad (10.12)$$

- where $\lambda^\star$ and $\mu^\star$ maximize the dual problem, Problem (10.11).
- Each generator (conceptually) maximizes its operating profit for the given vector of prices, as specified by the current values of the dual variables.

Description, continued

- The dual variables are updated until a maximum of the dual function is obtained:
  - As suggested in Exercise 4.9, there are more efficient approaches to finding or approximating the dual maximizer (see Exercise 10.6).
- There may be a duality gap.
- Since the duality gap is typically non-zero, an *ad hoc* post-processing heuristics are required in order to produce a solution that satisfies the system constraints:
  - the heuristics to find a feasible solution from the results of Lagrangian relaxation are very detailed and “brittle,” particularly with transmission constraints.
  - the heuristics are problematic in a market setting, where a particular heuristic may have significant implications for profitability or be vulnerable to “strategic” offers, allowing market participants to influence outcomes through changes to offers that do not represent economic fundamentals.

Description, continued

- As mentioned in Section 10.3, all North American ISOs now use mixed-integer programming software to solve a linear formulation of the unit commitment problem, since it has several advantages over Lagrangian relaxation including that there is generally less post-processing required, even if the MIP is not solved to optimality.
- PJM estimates $60 million per year or more of savings (out of approximately $10 billion) with MILP formulation compared to previous Lagrangian relaxation and linear programming based algorithms.
- Other US ISOs report similar savings.

Description, continued

- Lagrangian relaxation solution is here only being used to define the convex hull prices, not to find the commitment and dispatch.
- The maximizer of the dual can provide important insights into prices even if it does not yield the optimal unit commitment.
- In Exercise 10.6, we will also consider a more efficient way to calculate the dual maximizer that builds on Exercise 4.9.
- In MISO, prices based on (an approximation of) the dual maximizer are used in conjunction with values of commitment and dispatch obtained by the ISO in a MILP solution.

10.5.3 Unit commitment example

- Consider again the previous one generator example in Sections 4.8.3 and 10.4.2 in the context of duality gaps where a single generator was available to meet a demand of $\overline{D} = 3$ MW in the single interval $n_T = 1$.
- Now consider the case of a generator with a parametrized cost function:

$$f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) = 4z + \beta x, z \in \{0,1\}, 2z \leq x \leq 4z,$$

- where $\beta \geq 0$ is a parameter.
- Suppose that the generator is paid π for its power production x and that it finds the value of production that maximizes profit specified by:

$$\pi x - f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right).$$

- We perform similar analysis to previously to find the profit maximizing x (and z).

Unit commitment example, continued

- To maximize profit $\pi x - f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) = (\pi - \beta)x - 4z$, we must compare:

  the profit for $z = 0$ and $x = 0$, (namely, a profit of 0), to
  the maximum profit over $2 \leq x \leq 4$ for $z = 1$.
- We consider various cases for π.

$\pi \leq \beta$

$$\begin{aligned} 0 &> -4, \\ &\geq (\pi - \beta)x - 4, \text{ for } 2 \leq x \leq 4. \end{aligned}$$

  - So, the profit is maximized for $z^{\star\star} = 0, x^{\star\star} = 0$.

$\beta < \pi < 1 + \beta$

  - Then $(\pi - \beta)x < 4$ for $2 \leq x \leq 4$.

$$0 > (\pi - \beta)x - 4, \text{ for } 2 \leq x \leq 4.$$

  - So, the profit is again maximized for $z^{\star\star} = 0, x^{\star\star} = 0$.

Unit commitment example, continued

$\pi = 1 + \beta$

- Then $0 > (\pi - \beta)x - 4$ for $2 \leq x < 4$.
- Also, $0 = (\pi - \beta)x - 4$ for $x = 4$.
- So, the profit has two maximizers:

  $z^{\star\star} = 0, x^{\star\star} = 0$, and
  $z^{\star\star} = 1, x^{\star\star} = 4$.

$\pi > 1 + \beta$

$$0 < (\pi - \beta)x - 4, \text{ for } x = 4.$$

- Moreover, the right-hand side increases with increasing x, so it is maximized over $2 \leq x \leq 4$ by $x = 4$.
- So, the profit is maximized for $z^{\star\star} = 1, x^{\star\star} = 4$.

Unit commitment example, continued

- Therefore, if the generator were paid π for its production then its profit maximizing behavior would be:

$$x = \begin{cases} 0, & \text{if } \pi < 1+\beta, \\ 0 \text{ or } 4, & \text{if } \pi = 1+\beta, \\ 4, & \text{if } \pi > 1+\beta. \end{cases}$$

- If we have just one generator having marginal cost β then there will still typically be no price that equates supply to demand, unless demand were changed to $\overline{D} = 0$ or 4.
- The price, $\pi = 1+\beta$, at which the generator starts to produce depends on β.
- We still typically have a duality gap since the minimum of Problem (10.4) is strictly greater than the maximum of Problem (10.11).

10.5.4 Larger example

- Suppose that we generalize the example problem from the last section to the case where there are multiple generators with different cost characteristics β and a larger demand.
- Suppose that demand was $\overline{D} = 303$ MW.
- Assume that there are no reserve requirements, so $x_k = P_k$ for generator k.
- Suppose that there are 100 generators, with generator $k = 1, \ldots, 100$ having cost function:

$$f_k\left(\begin{bmatrix} z_k \\ x_k \end{bmatrix}\right) = 4z_k + \beta_k x_k, z_k \in \{0,1\}, 2z_k \leq x_k \leq 4z_k,$$

- where:

$$\forall k = 1, \ldots, 100, \beta_k = 1 + k/100.$$

- The feasible operating set for each generator k is:

$$\mathbb{S}_k = \left\{ \begin{bmatrix} z_k \\ x_k \end{bmatrix} \middle| z_k \in \{0,1\}, 2z_k \leq x_k \leq 4z_k \right\}.$$

10.5.4.1 Solution

- Each generator has a slightly different operating cost function, with higher values of k associated with more expensive generators.
- The optimal commitment is for:
  - generators $1,\ldots,75$ to be committed and producing at full capacity of 4,
  - generator 76 to be committed and producing 3, and
  - generators $77,\ldots,100$ to be off.
- Minimum cost is therefore:

$$\sum_{k=1}^{75} [4 \times 1 + (1 + k/100) \times 4] + [4 \times 1 + (1 + 76/100) \times 3] = 723.28.$$

- This is the minimum of Problem (10.4), which we could find in this case by inspection because of the simple structure of the problem.
- We will investigate the maximizer of the dual problem, Problem (10.11), and see the insights it provides into the minimum and minimizer of Problem (10.4).

10.5.4.2 Maximizer of dual

- The dual problem, Problem (10.11), in this case is:

$$\max_{\lambda \in \mathbb{R}} \left\{ \min_{\forall k=1,\ldots,100, \begin{bmatrix} z_k \\ x_k \end{bmatrix} \in \mathbb{S}_k} \left\{ f\left(\begin{bmatrix} z \\ x \end{bmatrix}\right) + \lambda \left(\overline{D} - \sum_{k=1}^{100} x_k \right) \right\} \right\}.$$

- Suppose we set λ so that $2+75/100 < \lambda < 2+76/100$.
  - For example, suppose that we set the price to be $\lambda = 2.755$.
  - We have that $1+\beta_k < \lambda$ for $k = 1,\ldots,75$ and $1+\beta_k > \lambda$ for $k = 76,\ldots,100$.
  - Generators $k = 1,\ldots,75$ will produce 4 MW.
  - Generators $k = 76,\ldots,100$ will produce nothing.
  - Total production will be 300 MW.

Maximizer of dual, continued

- Summarizing, suppose we set λ so that $2+75/100 < \lambda < 2+76/100$.
  - In particular, suppose that we set the price to be $\lambda = 2.755$,
  - Then the dual function is:

$$\begin{aligned}\mathcal{D}(2.755) &= \sum_{k=1}^{75}[4\times 1+(1+k/100)\times 4]+2.755\times\left(\overline{D}-\sum_{k=1}^{75}4\right),\\ &= 722.265.\end{aligned}$$

- For values of $\lambda \leq 2+75/100$, the value of the dual will be less than or equal to 722.265.

Maximizer of dual, continued

- Now suppose that we set λ so that $2+76/100 < \lambda < 2+77/100$.
  - For example, suppose that we set the price to be $\lambda = 2.765$.
  - We have that $1+\beta_k < \lambda$ for $k = 1,\ldots,76$ and $1+\beta_k > \lambda$ for $k = 77,\ldots,100$.
  - Generators $k = 1,\ldots,76$ will produce 4 MW.
  - Generators $k = 77,\ldots,100$ will produce nothing.
  - Total production will be 304 MW.
  - The dual function is:

$$\begin{aligned}\mathcal{D}(2.765) &= \sum_{k=1}^{76}[4\times 1+(1+k/100)\times 4]+2.765\times\left(\overline{D}-\sum_{k=1}^{76}4\right),\\ &= 722.275.\end{aligned}$$

- For values of $\lambda \geq 2+77/100$, the value of the dual will be less than or equal to 722.275.
- The maximizer of the dual, Problem (10.11), is $\lambda^\star = 2.76$.

Maximizer of dual, continued

- Now suppose that we set the energy price to be:

$$\pi^{\text{CHP}} = \lambda^\star = 2.76.$$

- Profit-maximizing generators would choose to generate as follows:
  - Generators $k = 1, \ldots, 75$ will produce 4 MW.
  - Generator $k = 76$ is indifferent to either not producing or producing 4 MW.
  - Generators $k = 77, \ldots, 100$ will produce nothing.
  - Total production is either 300 or 304 MW.
  - The dual function is:

$$\begin{aligned}
\mathcal{D}(2.76) &= \sum_{k=1}^{75} [4 \times 1 + (1 + k/100) \times 4] + 2.76 \times \left(\overline{D} - \sum_{k=1}^{75} 4 \right), \\
&= \sum_{k=1}^{76} [4 \times 1 + (1 + k/100) \times 4] + 2.76 \times \left(\overline{D} - \sum_{k=1}^{76} 4 \right), \\
&= 722.28.
\end{aligned}$$

Maximizer of dual, continued

- There is no price were supply equals demand of 303 MW.
- However, the supply-demand constraint is violated by a *relatively* smaller amount than in the smaller examples in Sections 4.8.3 and 10.5.3.
- Moreover, the commitment and dispatch decisions for generators $k = 1, \ldots, 75$ and $77, \ldots, 100$ in the generator profit maximization problems are correct given that the price is $\pi^{\text{CHP}} = 2.76$.
- The duality gap is $723.28 - 722.28 = 1$.
- The duality gap is relatively smaller as a fraction of the minimum of the unit commitment problem.

10.6 Duality gaps

10.6.1 Discussion

- Typically there is a duality gap between the minimum of the unit commitment problem and the maximum of its dual:

  The maximum of the dual obtained by dualizing the system constraints is strictly less than the minimum of the primal problem.

  The commitment variables $z^{\star\star}$ and the dispatch variables $x^{\star\star}$ resulting from the generator profit maximization sub-problems do not satisfy the system constraints.

- However, the duality gap is *relatively* smaller in the larger example in Section 10.5.4 than in the single generator example in Section 10.5.3 and the system constraints are violated by a relatively smaller amount, so the commitment and dispatch values corresponding to the dual maximizer can provide a useful approximate guide to the optimum of the unit commitment Problem (10.4).

Discussion, continued

- If the generator cost characteristics are heterogeneous then the duality gap (and the violation of the system constraints) becomes relatively smaller as the number of generators grows large.
- This is the key to application of Lagrangian relaxation to large-scale systems since the post-processing step to create a feasible solution involves a smaller adjustment for larger systems.
- What are reasons for heterogeneity and homogeneity in the cost functions of generators?

10.6.2 Non-existence of dispatch-supporting prices

- Unfortunately, the non-zero duality gap means that prices on the system constraints alone cannot encourage profit-maximizing generators to commit and dispatch in a way that is (exactly) consistent with optimal commitment and dispatch.
- For each value of the price vector, some system constraint will fail to be satisfied by the resulting profit-maximizing decisions of the generators.

Non-existence of dispatch-supporting prices, continued

- As Stoft argues, by modifying demand slightly we can typically obtain dispatch supporting prices:
  - if the generation stock is heterogeneous then modification will be small,
  - in the larger example, the modification would be at most 2 MW,
  - since there are other uncertainties and errors in dispatch, it may be reasonable to ignore the duality gap in this case.
- This is the basis of a principled argument against centralized unit commitment:
  - might still utilize a centralized day-ahead economic dispatch process, but unit commitment decisions would be taken by individual market participants.

Non-existence of dispatch-supporting prices, continued

- In a centralized day-ahead economic dispatch market without centralized unit commitment and without start-up or min-load offers, market participants are faced with making “marked-up” energy and reserve offers that cover their start-up and min-load costs:
  - energy and reserve offers will be increased above marginal costs to cover the start-up and min-load costs,
  - ideally, dispatch decisions by ISO using marked-up energy offers alone will result in commitment and dispatch by market participants that roughly approximates optimal commitment and dispatch,
  - in practice, it is difficult for a market participant to estimate the “right” mark-up that would be consistent with optimal commitment and dispatch, unless it owns a large fraction of total generation capacity.

Non-existence of dispatch-supporting prices, continued

- We will continue to assume that the ISO performs centralized unit commitment:
  - ERCOT and other US ISOs optimize the commitment and dispatch in the day-ahead market, reflecting the complexity of the various constraints, particularly transmission constraints.
- In the next section, we will consider the convex hull prices in conjunction with a **make-whole** payment to align the incentives of profit-maximizing generators with the centralized ISO commitment and dispatch decision, as in Section 10.4.5.
- Using convex hull prices will result in a smaller make-whole payment than with LMPs.

10.6.3 Make-whole payment with convex hull prices

- As discussed above, the non-zero duality gap means that prices on the system constraints *alone* cannot encourage profit-maximizing generators to *all* commit and dispatch in a way that is exactly consistent with optimal unit commitment.
- A make-whole payment is necessary.
- For convex hull prices, the simplified make-whole payment (10.10) as defined in Section 10.4.6 is:

$$\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}^{\mathrm{CHP}}]^\dagger x_k^\star\right\},$$

where the convex hull prices were defined in (10.12):

$$\pi_{x_k}^{\mathrm{CHP}} = -[A_k]^\dagger \lambda^\star - [C_k]^\dagger \mu^\star,$$

with $\lambda^\star$ and $\mu^\star$ maximizing the Lagrangian dual problem (10.11).

10.6.4 Make-whole payment with convex hull prices in example

- In the example in Section 10.5.4 having $\overline{D} = 303$, all but one of the generators would be committed and dispatched correctly if the price were set equal to the maximizer of the dual $\pi^{\text{CHP}} = 2.76$:
  - generators $k = 1, \ldots, 75$ would produce 4 MW, while
  - generators $k = 77, \ldots, 100$ will produce nothing.
- Generators $k = 1, \ldots, 75$ and $77, \ldots, 100$ would collectively produce a total of 300 MW:
  - the CHP included sufficient compensation to cover both their min-load and incremental energy costs.

Make-whole payment with convex hull prices in example, continued

- To meet the total demand of $\overline{D} = 303$ MW, generator $k = 76$ should produce 3 MW:
  - the cost for generator $k = 76$ to produce 3 MW is:

$$f_{76}\left(\begin{bmatrix} z^{\star}_{76} \\ x^{\star}_{76} \end{bmatrix}\right) = 4z^{\star}_{76} + \beta_{76}x^{\star}_{76},$$
$$= 4 \times 1 + (1 + 76/100) \times 3,$$
$$= 9.28.$$

  - with an energy price of $\pi^{\text{CHP}} = \$2.76$/MWh, generator $k = 76$ would receive revenues of $\pi^{\text{CHP}} \times x^{\star}_{76} = 2.76 \times 3 = 8.28$ if it produced $x^{\star}_{76} = 3$.
  - generator $k = 76$ would need an additional payment of $9.28 - 8.28 = \$1$/h in order to have non-negative profit, based on an energy price of $\pi^{\text{CHP}} = \$2.76$/MWh,
  - this difference is equal to the duality gap.

Make-whole payment with convex hull prices in example, continued

- To achieve optimal unit commitment in the example:
  - price energy based on the dual maximizer, $\pi^{\text{CHP}} = \$2.76/\text{MWh}$,
  - profit-maximizing behavior of generators $1,\ldots,75$, and $77,\ldots,100$ in response to this price is to behave consistently with centralized optimal unit commitment and dispatch, but
  - an additional **make-whole payment** is paid to generator 76 of:

$$
\begin{aligned}
&\Pi_{76}^{\star\star}(\pi^{\text{CHP}}) - \left(\pi^{\text{CHP}} x_{76}^{\star} - f_{76}\left(\begin{bmatrix} z_{76}^{\star} \\ x_{76}^{\star} \end{bmatrix}\right)\right) \\
&= \; 0 - \left(\pi^{\text{CHP}} x_{76}^{\star} - f_{76}\left(\begin{bmatrix} z_{76}^{\star} \\ x_{76}^{\star} \end{bmatrix}\right)\right), \\
&= \; \max\left\{0, f_{76}\left(\begin{bmatrix} z_{76}^{\star} \\ x_{76}^{\star} \end{bmatrix}\right) - [\pi^{\text{CHP}}]^{\dagger} x_{76}^{\star}\right\}, \\
&= \; 4 z_{76}^{\star} + \beta_{76} x_{76}^{\star} - \pi^{\text{CHP}} x_{76}^{\star}, \\
&= \; 9.28 - 8.28 = 1.
\end{aligned}
$$

Make-whole payment with convex hull prices in example, continued

- To summarize, generator 76 requires an additional \$1/h to induce it to generate consistent with optimal commitment and dispatch.
- Demand pays for:
  - energy based on $\pi^{\text{CHP}} \times \overline{D} = 2.76 \times 303 =$ \$836.28/h, plus
  - the make-whole payment to generator 76 of \$1/h.
- The make-whole payment is charged as an uplift to demand.
- Note that the payment to generator 76 is qualitatively different to the payment to other generators since it involves a make-whole payment.

10.6.5 *Make-whole payment with locational marginal prices in example*

- If locational marginal prices were used, the price would be:

$$\pi^{\text{LMP}} = 1.76,$$

- since this is the marginal cost of the marginal generator.
- Note that at this price, the profit-maximizing response of all generators would be to *not* commit, since their min-load and incremental energy costs are not covered.
- To induce generator $k = 76$ to commit and to generate 3 MW, a make-whole payment of:

$$4 \times 1 + (1.76) \times 3 - 1.76 \times 3 = 4,$$

- would be required.
- To induce generators $k = 1, \ldots, 75$ to commit and to generate 4 MW, a make-whole payment of:

$$4 \times 1 + (1 + k/100) \times 4 - 1.76 \times 4 = (24 + k)/25,$$

- would be required.
- The total make-whole payment is \$80.48/h.

10.6.6 Comparison of convex hull and locational marginal prices

- The total make-whole payment is much higher under LMP in the example than under CHP.
- The energy price is lower under LMP in the example than under CHP.
- Total payment for energy and make-whole by demand is lower under LMP than under CHP.
- Although this example is extreme, make-whole payments under CHP are provably lower than make-whole payments under LMP, when the exact make-whole expression (10.9) is used:
  - the make-whole payments under (10.9) are equal to the difference between the minimum of the unit commitment problem and the value of the dual,
  - this difference is minimized by the dual maximizer,
  - so the convex hull prices minimize the make-whole payment as defined by (10.9).
- Convex hull prices may not minimize the make-whole payments under the simplified make-whole payment (10.10).

10.6.7 Demand response

- Demand response can reduce the duality gap (and therefore reduce the make-whole payment required to achieve optimality).
- Suppose that instead of fixed demand of 303 MW, the demand was the sum of:

  a fixed demand of $\overline{D} = 290$ MW, plus
  price-responsive demand ΔD with willingness-to-pay of

$$(2.755+10)\ \$/\text{MWh} - 1\ \$/\text{h} \times \Delta D,\ 0 \leq \Delta D \leq 20\ \text{MW}.$$

- Consider again convex hull prices for this unit commitment problem.
- At a price of $\pi^{\text{CHP}} = \$2.755/\text{MWh}$, the price-responsive demand would be $\Delta D = 10$ MW, so that total demand would be $290+10 = 300$ MW.
- At a price of $\pi^{\text{CHP}} = \$2.755/\text{MWh}$, the supply equals 300 MW.
- So, supply equals demand and there is no duality gap and no need for a make-whole payment.
- In general, price-responsive demand can reduce the duality gap and reduce the make-whole payments.

Demand response, continued

- This demand response example is somewhat unrealistic in that demand is generally not willing to voluntarily curtail at prices that are close to typical generation marginal costs:
  - we will assume fixed demand in subsequent examples.
- Such price responsiveness does, however, have an important effect in the presence of scarcity and/or market power where offer prices might otherwise rise to far above generation marginal costs.
- See market power course, `www.ece.utexas.edu/~baldick/classes/394V_market_power/`
- Moreover, as in Section 8.12.9.7, there may be representation of responsive demand for adequacy reserve.

10.6.8 Unit commitment example with two generators

- Consider again the previous example in Sections 10.3.3, 10.4.3, and 10.4.8 with two generators available to meet a demand of $\overline{D}$ MW in the single interval $n_T = 1$.
- Figure 10.3 shows the resulting convex hull prices:
  - non-decreasing in increasing supply, in contrast to the locational marginal prices as shown in Figure 10.1, and
  - reflect the no-load costs into the energy price.
- Figure 10.4 shows the resulting simplified make-whole payment versus demand $\overline{D}$:
  - convex hull prices typically result in smaller make-whole payments than under locational marginal pricing,
  - make-whole payments are smaller with convex hull prices than with locational marginal prices as shown in Figure 10.2.

Unit commitment example with two generators, continued

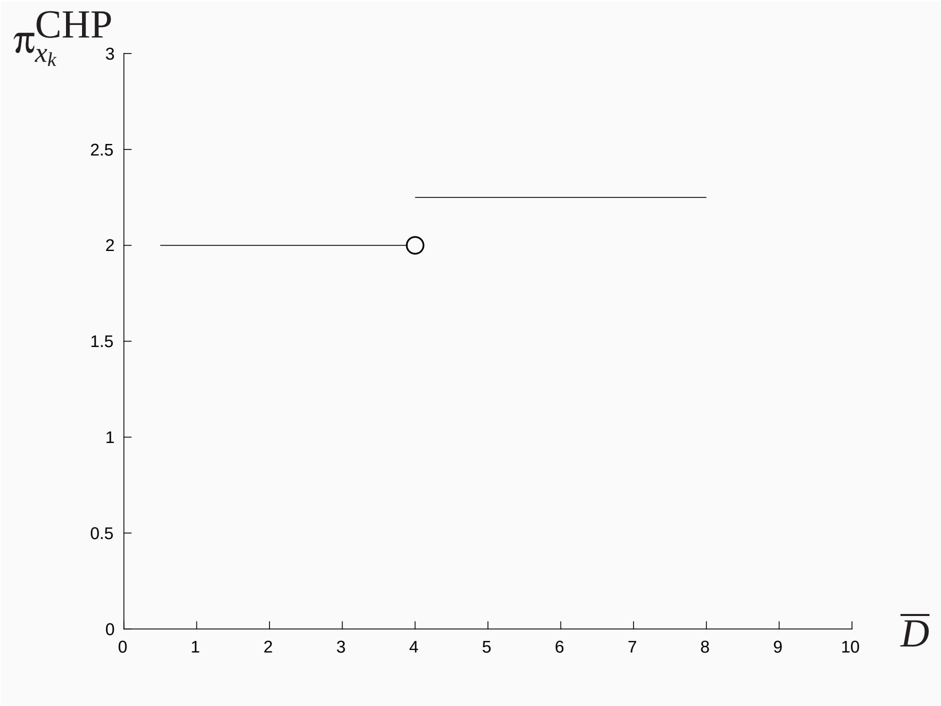

Fig. 10.3. Price $\pi_{x_k}^{\mathrm{CHP}}$ versus demand $\overline{D}$ for two generator system.

Unit commitment example with two generators, continued

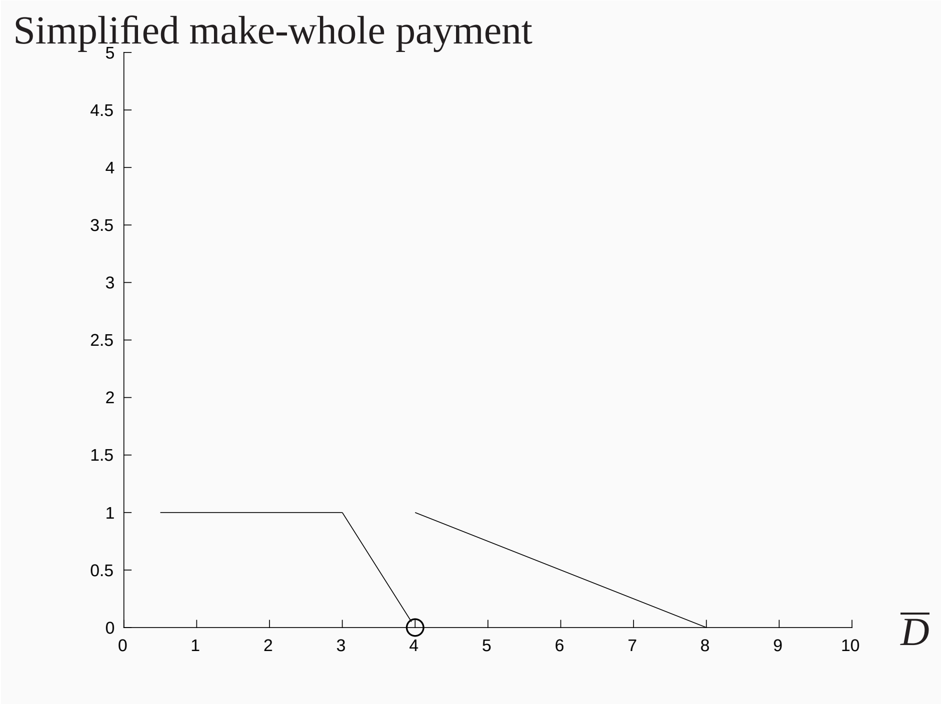

Fig. 10.4. Simplified make-whole payment with convex hull prices versus demand $\overline{D}$ for two generator system.

10.6.9 Summary of make-whole payments

- The goal of make-whole payments is to ensure that each generator is paid enough to cover its offer costs and so that it commits and dispatches consistently with the optimal commitment and dispatch as determined by the ISO:
  - all centralized unit commitment formulations require an uplift from demand.
- The simplified make-whole payment $\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - \pi^\dagger x_k^\star\right\}$ can be applied to any pricing rule on energy and reserves in order to induce a particular behavior:
  - make-whole payments are paid to a generator that commits according to (or close enough to) $z_k = z_k^\star$,
  - as previously mentioned, the simplified payment is used in practice even though it does not provide the exactly correct incentives.

Make-whole payments, continued

- In ISOs except MISO:
  - commitment $z^\star$ and dispatch $x^\star$ from solution of offer-based unit commitment Problem (10.4),
  - energy and reserves prices using LMPs based on Lagrange multipliers $\lambda^\star$ and $\mu^\star$ obtained from the solution of the convex problem, Problem (10.7), obtained by fixing the integer variables at their optimal values $z^\star$,
  - make-whole payment based on a daily calculation of make-whole payment using LMPs:

$$\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}^{\mathrm{LMP}}]^\dagger x_k^\star\right\},$$

  - where prices $\pi_{x_k}^{\mathrm{LMP}}$ are based on $\lambda^\star$ and $\mu^\star$ from Problem (10.7).

Make-whole payments, continued

- For its day-ahead and real-time market, MISO uses prices that approximate the convex hull prices:
  - commitment $z^\star$ and dispatch $x^\star$ from solution of unit commitment Problem (10.4),
  - energy and reserves payments based on the the maximizer $\lambda^\star$ and $\mu^\star$ of the dual problem, Problem (10.11), or an approximation to this problem, resulting in non-decreasing prices with increasing demand, and
  - make-whole payment based on a daily calculation of make-whole payment using CHPs:

$$\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}^{\mathrm{CHP}}]^\dagger x_k^\star\right\},$$

  - where prices $\pi_{x_k}^{\mathrm{CHP}}$ are based on $\lambda^\star$ and $\mu^\star$ from Problem (10.11).

10.7 Role of prices and implications for investment

- Two important roles for prices:
  - inform dispatch and consumption decisions, and
  - inform potential new entrants to the market about whether new entry would be profitable.
- Prices π_{x_k} on system equality and inequality constraints are paid for production of energy and provision of reserves independent of the producer, but possibly varying by location, and are said to be **uniform**.
- However, make-whole payments are not uniform since different market participants receive different payments, even if located at the same bus.
- Non-uniformity makes it harder for a potential entrant to determine if new entry would be profitable, particularly if the make-whole payments are not disclosed publicly.
  - It is difficult for a new entrant to understand if it would be profitable to enter at the current prices if the total remuneration from the market is not *transparent* to market participants.
- Moreover, larger make-whole payment implies that less of generation costs are reflected into energy prices.

Role of prices and implications for investment, continued

- Even if the make-whole payments are disclosed, make-whole payments can distort investment decisions.
- Make-whole payments contribute to the infra-marginal rents of some generators.
- These rents are not also available to everyone else.
- The incentives for building new capacity may be depressed compared to the remuneration to existing generation owners.
- Convex hull prices minimize the exact make-whole expression (10.9) over choices of uniform prices:
  - the prices are as "transparent" as possible and reflecting as much of the operational costs as possible for uniform prices,
  - minimize the distortion of investment decisions, and
  - since make-whole is charged to demand, arguably also minimize distortions of demand-side decisions.

10.8 Transmission constraints

- In the examples so far we have not explicitly considered transmission constraints.
- However, transmission constraints can limit the dispatch decisions.
- In practice, transmission-constrained unit commitment can be an extremely difficult problem to solve.
- See Exercise 10.6.

10.8.1 Transmission-constrained example

- We consider day-ahead unit commitment and dispatch across two hours, $n_T = 2$, with demands:

t	0	1	2
$\overline{D}_t$	90	110	125

- The $t = 0$ entry in the table is the demand for the last hour of today.
- The $t = 1,2$ entries are the demands for the first two hours of tomorrow.
- Also, $P_{10} = 90$ MW and $P_{20} = 0$ MW are the generations in the last hour of today, with generator 2 out-of-service at the end of today.
- We ignore reserves, min-load costs, and ramp-rate constraints.
- The offers are specified by:

$$\begin{aligned} \forall t = 1,2, s_{1t} = 1000, \forall P_{1t} \in [0,200], \nabla f_{1t}(P_{1t}) &= \$25/\text{MWh}, \\ \forall t = 1,2, s_{2t} = 1000, \forall P_{2t} \in [0,50], \nabla f_{2t}(P_{2t}) &= \$35/\text{MWh}. \end{aligned}$$

Transmission-constrained example, continued

- The generators are located in the following one-line two-bus system.
- We use the DC power flow approximation and the transmission line has transmission capacity of 100 MW.
- We solve the transmission constrained, offer-based unit commitment for this system.
- We will calculate and consider LMPs based on Problem (10.7).
- Make-whole payments will be based on

$$\max\left\{0, f_k\left(\begin{bmatrix} z_k^\star \\ x_k^\star \end{bmatrix}\right) - [\pi_{x_k}^{\text{LMP}}]^\dagger x_k^\star\right\}.$$

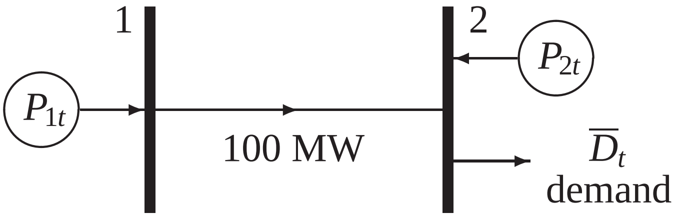

Fig. 10.5. One-line two-bus network.

Transmission-constrained example, continued

- Because of the transmission constraint, it will be necessary to commit generator 2 and run it during intervals 1 and 2.
- The optimal offer-based commitment and dispatch is:

t	0	1	2
$\overline{D}_t$	90	110	125
$z^{\star}_{1t}$	1	1	1
$P^{\star}_{1t}$	90	100	100
$z^{\star}_{2t}$	0	1	1
$P^{\star}_{2t}$	0	10	25

Transmission-constrained example, continued

- We calculate the locational marginal prices using commitment variables fixed at their optimal values, noting that $\pi_{P_{kt}}^{\text{LMP}} = \lambda_{kt}^{\star}$, where $\lambda_{kt}^{\star}$ is the Lagrange multiplier on supply-demand balance at the bus of generator k in interval t:

t	0	1	2
$\overline{D}_t$	90	110	125
$\pi_{P_{1t}}^{\text{LMP}}$	25	25	25
$\pi_{P_{2t}}^{\text{LMP}}$	25	35	35

- Since generator 1 is already committed at the start of the day, and since the revenue (just) covers its incremental energy costs, there is no make-whole payment for generator 1.
- Generator 2 must be started, but the revenue only just covers its incremental energy costs.
- Therefore, the make-whole payment to generator 2 is equal to its start-up cost of $s_{21} = \$1000$.

10.9 Robust, stochastic, and reliability unit commitment

10.9.1 Role of reserves

- Reserves provide capacity for *recourse* to cope with uncertain outcomes:
  - spinning reserve provides capacity to replace production if a generator trips out of service, while
  - regulation reserve provides capacity to compensate for variation of supply–demand balance and forecast error during a real-time dispatch interval.
- Quantities of procured reserves can be based on considerations such as:
  - assessment of largest credible loss of generation (for spinning reserve), and
  - historical and forecast variability of net load and forecast error (for regulation reserve).
  - reserves serve to make the commitment and dispatch robust to failures and forecast error, as discussed in Section 8.12.1.4.
- In simplest implementations, the choice of quantity of procured reserves is not directly incorporated into commitment and dispatch model:
  - procured quantity is exogenous decision.

10.9.2 Stochastic unit commitment

- Recall the discussion in Section 8.12.9.5 of operating reserve demand curve.
- Level of adequacy reserve was trade-off between expected value of unserved energy and the cost of procuring the reserve:
  - simple formulation involved off-line determination of parameters in operating reserve demand curve.
- In principle, consideration of random failures could be endogenous to unit commitment problem:
  - Minimize expected cost over probabilities of outage scenarios.
- In addition to random outages, could also consider random production of renewables:
  - Minimize expected cost of probabilities of production by renewables.
- **Stochastic unit commitment** formulations consider these issues, possibly including consideration of **risk**:
  - avoid downside of unfavorable outcomes.

10.9.3 Robust unit commitment

- If distributions of random variables are not available, or are uncertain, an alternative is to ensure feasibility even despite uncertainty within an **uncertainty set**.
- Solution is **robust** to uncertainty:
  - standard robust formulations optimize worst-case value of objective over uncertainties.

10.9.4 Interaction with market

- A concern with stochastic and robust solutions is that it assumes that the ISO can compile information about the various uncertainties in the market.
- A philosophical concern is that one of the functions of the market is to solicit this information implicitly in offer prices.

10.9.5 Comparison to reserve formulations

- Spinning reserve provides "robust" solution in that feasibility will be maintained for outages of size up to the amount of procured reserve:
  - historically chosen to be the largest "credible" contingency, analogous to uncertainty set in robust optimization,
  - makes decisions "robust" to credible contingencies, but
  - with objective given by base-case system.
- Spinning reserve formulation does not consider worst-case objective, so not literally consistent with standard robust optimization formulation.

10.9.6 Reliability unit commitment

- In addition to procurement of reserves, all ISOs perform an additional **reliability unit commitment** to ensure that there is enough committed generation capacity available to meet ISO forecast of demand:
  - uses ISO forecast of demand, instead of day-ahead bids or specifications of demand by load-serving entities,
  - uses information about physical commitments of generation, instead of financial commitments from day-ahead market,
  - represents transmission system more fully,
  - typically performed in day-ahead and hour-ahead timeframes,
  - additional costs of commitment are charged as an uplift to demand, targeted in ERCOT towards demand that occurs in real-time but was not bid or specified into the day-ahead market.
- Since commitment of additional capacity in such out-of-market processes will tend to increase supply and decrease prices, there are various mechanisms to offset or price this effect:
  - **reliability adder** in ERCOT.

10.10 Summary

- In this section we have considered temporal issues.
- We formulated the unit commitment problem.
- We considered make-whole rules.
- We investigated the duality gap in the problem and the implications for commitment-supporting prices.
- Transmission constraints and robust and stochastic unit commitment were briefly discussed.

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Homework exercises

10.1 Use GAMS or MATLAB to solve the ramp-constrained dispatch problem from Section 10.1.2.6. Verify that your solution is consistent with the values in Section 10.1.2.6.

10.2 Suppose that we have two generators, $n_P = 2$, with offers:

$$\forall t, \nabla f_{1t}(P_{1t}) = 2, 100 \leq P_{1t} \leq 400,$$
$$\forall t, \nabla f_{2t}(P_{2t}) = 5, 100 \leq P_{2t} \leq 300.$$

The generators have ramp-rate limits of $\Delta_1 = 50$ MW/h and $\Delta_2 = 100$ MW/h, respectively. We consider day-ahead dispatch across five hours, $n_T = 5$, with demands:

t	0	1	2	3	4	5
$\overline{D}_t$	250	350	400	425	450	475

The $t = 0$ entry in the table is the demand for the last hour of today. Also, $P_{10} = 150$ MW and $P_{20} = 100$ MW. We ignore reserves.

(i) Solve the ramp-constrained economic dispatch problem.
(ii) What price is paid for energy in each hour?
(iii) What do you notice about the relationship between demand and prices?

10.3 In this exercise, we explore a formulation of unit commitment that avoids the non-linear objective terms of (10.3) to represent min-load and incremental energy costs. Suppose that the minimum and maximum production capacities of generator k are $\underline{P}_k = 2$ and $\overline{P}_k = 4$, respectively, and that the marginal cost of a committed generator (in monetary units per MW per interval) is given by:

$$\forall P_{kt} \in [\underline{P}_k, \overline{P}_k] = [2,4], \nabla f_{ktP}(P_{kt}) = 1.$$

(i) For $P_{kt} \in [\underline{P}_k, \overline{P}_k]$, evaluate:

$$\int_{P'_{kt}=\underline{P}_k}^{P'_{kt}=P_{kt}} \nabla f_{ktP}(P'_{kt})\, dP'_{kt}.$$

(ii) Show that for all $\begin{bmatrix} z_{kt} \\ P_{kt} \end{bmatrix}$ satisfying the generator constraint $\underline{P}_k z_{kt} \leq P_{kt} \leq \overline{P}_k z_{kt}$ and such that $z_{kt} \in \{0,1\}$, we can express the incremental energy costs as a linear function of $\begin{bmatrix} z_{kt} \\ P_{kt} \end{bmatrix}$. In particular, show that for all such $\begin{bmatrix} z_{kt} \\ P_{kt} \end{bmatrix}$ we have that:

$$\left[\int_{P'_{kt}=\underline{P}_k}^{P'_{kt}=P_{kt}} \nabla f_{ktP}(P'_{kt})\, dP'_{kt}\right] z_{kt} = P_{kt} - 2z_{kt}.$$

(iii) Now suppose that there are min-load costs of 6 monetary units per interval. Show that the min-load and incremental energy costs can be expressed as a linear function of $\begin{bmatrix} z_{kt} \\ P_{kt} \end{bmatrix}$.

10.4 In this exercise, we explore a formulation of unit commitment that avoids the non-linear objective terms of (10.2) to represent start-up costs by defining additional variables to represent the coupling between intervals. Together with the linear expression for the incremental energy costs analyzed in Exercise 10.3, this formulation results in a mixed-integer linear program and also allows for convenient representation of minimum up- and down-time constraints. Consider a unit commitment formulation for tomorrow with intervals $t = 1, \dots, n_T$, We continue to assume that z_{kt} represents the commitment status of generator k in interval t, with generator k on in interval t if $z_{kt} = 1$ and off in interval t if $z_{kt} = 0$. As previously, we also assume that the commitment status z_{k0} for the interval $t = 0$ at the end of today is known and specified. We define additional "start-up" variables $u_{kt}, t = 1, \dots, n_T$ that will enable a linear representation of start-up costs, at the expense of creating a formulation with more variables. Collect the entries $z_{kt}, u_{kt}, t = 1, \dots, n_T$ together into vectors z_k and u_k.

(i) Consider the following (linear) "start-up" inequality constraints:

$$\begin{aligned} \forall t = 1, \dots, n_T, u_{kt} &\geq z_{kt} - z_{k,(t-1)}, \\ \forall t = 1, \dots, n_T, u_{kt} &\geq 0, \end{aligned}$$

and the following (linear) "start-up" expression to evaluate the start-up costs:

$$\sum_{t=1}^{n_T} s_{kt} u_{kt}.$$

Assume that $s_{kt} \geq 0, \forall t = 1, \ldots, n_T$. Show that, for every binary vector z_k, the minimum of this start-up expression over continuous u_k, subject to the start-up inequality constraints, is equal to (10.2). Moreover, show that if $s_{kt} > 0, \forall t = 1, \ldots, n_T$ then the minimizer $u_k^\star$ is unique and is a binary vector. That is, show that:

$$\forall z_k \in [0,1]^{n_T}, \sum_{t=1}^{n_T} s_{kt} z_{kt} (1 - z_{k,(t-1)})$$
$$= \min_{u_k \in \mathbb{R}^{n_T}} \left\{ \sum_{t=1}^{n_T} s_{kt} u_{kt} \;\middle|\; \begin{array}{rl} u_{kt} \geq z_{kt} - z_{k,(t-1)}, & \forall t = 1, \ldots, n_T; \\ u_{kt} \geq 0, & \forall t = 1, \ldots, n_T \end{array} \right\},$$

and show that, for any binary z_k, if $s_{kt} > 0, \forall t = 1, \ldots, n_T$ then the minimizer $u_k^\star$ is unique and binary. That is, the non-linear formulation of the objective (10.2) can be replaced by a formulation that has a linear objective and constraints and is therefore solvable as a mixed-integer linear program.

(ii) Many generators have **minimum up- and down-time** requirements. A minimum up-time requirement specifies that, once committed, generator k stays on for at least, say, L_k intervals, and once it is switched off, it must stay off for at least, say, ℓ_k intervals. Without loss of generality, we need only consider $1 \leq L_k, \ell_k \leq n_T$. Suppose that generator k is either:

- on in interval $t = 0$ and has been on for at least its minimum up time, or
- off in interval $t = 0$ and has been off for at least its minimum down time,

so that we can ignore minimum up- and down-time requirements relating to earlier commitment status. Then the minimum up- and down-time requirements can be expressed as follows:

- $\forall t' = 1, \ldots, n_T$, if $z_{k,(t'-1)} = 0$ and $z_{kt'} = 1$ then $z_{ki} = 1$ for $i = t'+1, \ldots, \min\{t'+L_k-1, n_T\}$, and
- $\forall t' = 1, \ldots, n_T$, if $z_{k,(t'-1)} = 1$ and $z_{kt'} = 0$ then $z_{ki} = 0$ for $i = t'+1, \ldots, \min\{t'+\ell_k-1, n_T\}$.

(The representation of minimum up- and down-time requirements relating to earlier commitment status is similar.)

Now consider the following minimum up- and down-time inequality constraints:

$$\begin{aligned}\sum_{i=t-L_k+1}^{t} u_{ki} &\leq z_{kt}, \forall t = L_k, \ldots, n_T, \\ \sum_{i=t-\ell_k+1}^{t} u_{ki} &\leq 1 - z_{k,(t-\ell_k)}, \forall t = \ell_k, \ldots, n_T.\end{aligned}$$

Suppose that z_k is binary. Show that z_k satisfies the minimum up- and down-time requirements if and only if there exists a u_k such that z_k and u_k satisfy the start-up inequality constraints from the last part and the minimum up- and down-time inequality constraints.

10.5 Suppose that we have two generators, $n_P = 2$, with offers:

$$\begin{aligned} \forall t, \nabla f_{1t}(P_{1t}) &= 2, 200 \leq P_{1t} \leq 400, \\ \forall t, \nabla f_{2t}(P_{2t}) &= 3, 50 \leq P_{2t} \leq 150. \end{aligned}$$

There are no ramp-rate limits nor min-load costs, but the start-up costs are:

$$\begin{aligned} s_{1t} &= 1000, t = 1, \ldots, n_T, \\ s_{2t} &= 200, t = 1, \ldots, n_T. \end{aligned}$$

We consider day-ahead commitment and dispatch across ten hours, $n_T = 10$, with demands:

t	0	1	2	3	4	5	6	7	8	9	10
$\overline{D}_t$	200	350	500	400	300	200	300	400	500	350	200

The $t = 0$ entry in the table is the demand for the last hour of today. Also, $P_{10} = 200$ MW, and $P_{20} = 0$ MW with generator 2 out-of-service at the end of today. We ignore both ramp-rates and reserves.

(i) Solve the unit commitment problem and evaluate the total cost of commitment and dispatch.

(ii) What are the energy prices $\pi_{P_k}^{\text{LMP}}$ obtained from the solution of the convex Problem (10.7) obtained by fixing the integer variables at their optimal values from the solution of unit commitment and optimizing P_{1t} and P_{2t}?

(iii) What is the make-whole payment for each generator based on prices from Part (ii)?

(iv) Find the maximizer of the dual Problem (10.11) obtained by dualizing the demand constraint in each hour. (Hint: What price will induce generator 2 to be indifferent between being off and being on at full capacity in intervals 2 and 8. What is the price in the other intervals?)

(v) What is the make-whole payment for each generator when prices are set equal to the convex hull prices $\pi_{P_k}^{\text{CHP}}$?

10.6 Consider the example four-line four-bus system from Section 9.6 and illustrated in Figure 10.6. Assume that the only limiting transmission element is the line from bus 2 to bus 3, with capacity $\overline{p}_{23} = 300$ MW.

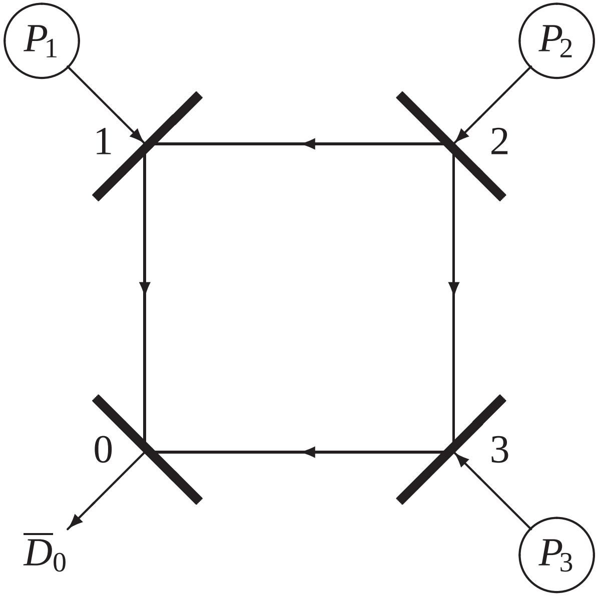

Fig. 10.6. Four-line four-bus network for homework exercise.

Recall that if we set $\sigma = 0$ to be the slack/price reference bus and bus $\rho = 0$ to be the angle reference bus then we can express power balance constraint and the flow constraint in each interval t as:

$$\begin{aligned} -P_{1t} - P_{2t} - P_{3t} &= -\overline{D}_{0t}, \\ 0.2P_{1t} + 0.4P_{2t} - 0.2P_{3t} &\leq \overline{p}_{23}, \end{aligned}$$

where P_{kt} is the (average) power production by generator k in interval t, and $\overline{D}_{0t}$ is the (average) demand at bus 0 in interval t.

We consider day-ahead dispatch across four hours, $n_T = 4$, with demands only at bus 0:

t	0	1	2	3	4
$\overline{D}_{0t}$	500	1200	3000	1200	500

- The $t = 0$ entry in the table is the demand for the last hour of today.
- The $t = 1, \ldots, 4$ entries are the demands for the first four hours of tomorrow.
- Also, $P_{20} = 500$ MW is the production of generator 2 in the last hour of today, and the other generators our off during the last hour of today.
- We ignore reserves, min-load costs and min-load limits, and ramp-rate constraints.

- The start-up and incremental energy offers are specified by:

$$\begin{aligned}
\forall t = 1,\ldots,4, s_{1t} = 10,000, \forall P_{1t} \in [0,1500], \nabla f_{1t}(P_{1t}) &= \$40/\text{MWh},\\
\forall t = 1,\ldots,4, s_{2t} = 10,000, \forall P_{2t} \in [0,1000], \nabla f_{2t}(P_{2t}) &= \$20/\text{MWh},\\
\forall t = 1,\ldots,4, s_{3t} = 10,000, \forall P_{3t} \in [0,1500], \nabla f_{3t}(P_{3t}) &= \$50/\text{MWh}.
\end{aligned}$$

(i) Using the formulation for start-up variables and inequality constraints from the first part of Exercise 10.4, use GAMS or MATLAB to solve the transmission constrained, offer-based unit commitment for this system for optimal values $z^\star$, $P^\star$, and $u^\star$,.

(ii) Calculate the LMPs, $\pi^{\text{LMP}}_{P_k}$, for the offer-based optimal power flow problems for each hour t obtained by fixing the variables z and u at their optimal values $z^\star$ and $u^\star$. That is, solve Problem (10.7).

(iii) Calculate the make-whole payments based on the LMPs. That is, for each k evaluate $\max\left\{0, f_k\left(\begin{bmatrix} z^\star_k \\ P^\star_k \end{bmatrix}\right) - [\pi^{\text{LMP}}_{P_k}]^\dagger P^\star_k\right\}$, where f_k is the total cost for generator k in the four hours and $\pi^{\text{LMP}}_{P_k}$ is the vector of LMPs at bus k for the four hours.

(iv) Solve the continuous problem obtained by relaxing the binary variables z and u to being continuous. Calculate the resulting Lagrange multipliers

$\lambda_{kt}^{\star}$ on supply–demand balance at each generator k in each interval t. As in Exercise 4.9, these Lagrange multipliers equal the dual maximizer of the Lagrangian relaxation problem obtained by dualizing the system constraints.

(v) Calculate the make-whole payments based on the dual maximizer obtained in the previous part and convex hull prices $\pi_{P_k}^{\text{CHP}}$. That is, for each k evaluate $\max\left\{0, f_k\left(\begin{bmatrix} z_k^{\star} \\ P_k^{\star} \end{bmatrix}\right) - [\pi_{P_k}^{\text{CHP}}]^{\dagger} P_k^{\star}\right\}$.

(vi) Compare the total make-whole payments based on the convex hull prices to the total make-whole payments based on the LMPs.