Course notes for EE394V Restructured Electricity Markets: Locational Marginal Pricing

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Power flow

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3.1 Review of power concepts

- **Power** is the rate of doing work, measured in W, kW, MW, or GW.
- **Energy** is the work accomplished over time, measured in Wh, kWh, MWh, or GWh.
- When power varies over time, the energy is the integral over time of the power.
- Figure 3.1 shows conceptually that generators inject power into the transmission (and in some cases the distribution) system, while demand withdraws power from the distribution (and in some cases the transmission) system.

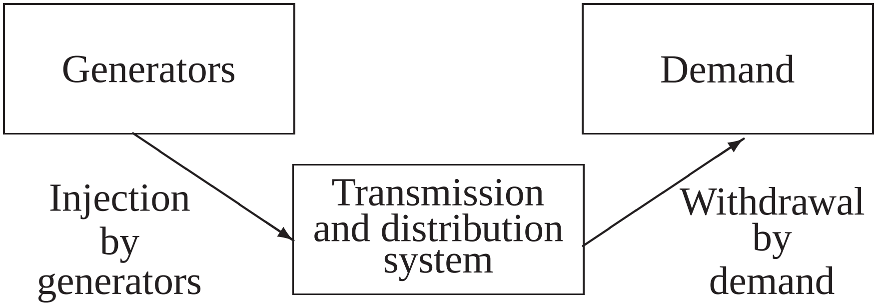

Fig. 3.1. Injection and withdrawal of power.

Review of power concepts, continued

- In principle, power could be generated, transmitted, and consumed using either **direct current** (DC) or **alternating current** (AC).
- Cost-effective and low loss transmission of bulk power relies on being able to create high voltages:
  - power is proportional to the product of current and voltage, so higher voltages allow for higher power levels at a given current,
  - for a given power, higher voltage means lower current, which implies lower losses for a given resistance of conductor.
- Generation and consumption is more convenient at lower voltages:
  - we will mostly focus on generation and transmission, modeling consumption through aggregated net loads at distribution substations,
  - step-down transformers at substations and on distribution feeders allow for consumption (and in some cases generation) at lower voltages.
- Other relevant issues include the probability of generation, transmission, and distribution failures, which affect the **quality of supply** to end users:
  - local effects on distribution system typically affect quality of supply more noticeably than generation and transmission failures.

Review of power concepts, continued

- Until the advent of power electronics, only AC power could easily be transformed from one voltage to another:
  - basic reason for ubiquity of AC power systems.
- AC transmission of power also involves the back-and-forth flow of power between electric and magnetic fields:
  - this back-and-forth flow is called **reactive power**,
  - to distinguish reactive power from the power that can actually be consumed by a load, the latter is called **real power**,
  - as we will see in Section 3.2.6, **complex power** is a complex number defined by $(\text{real power}) + (\text{reactive power})\sqrt{-1}$ and has convenient properties for computation.

Review of power concepts, continued

- The relationship between voltage and current in a circuit is determined by the characteristics of the circuit elements and **Kirchhoff's laws**.
- Kirchhoff's current law:
  - due to conservation of charge passing a bus or node of circuit,
  - analogous laws apply in other **transportation networks**,
  - implies that supply of *electric* power always equals demand of electric power plus losses,
  - mis-match between mechanical and electrical power is smoothed by inertia of system and results in frequency change,
  - enforcing supply–demand balance between mechanical power and electrical power is different to enforcing supply–demand balance in typical markets, such as a market for apartments to be described in Section 6.
- Kirchhoff's voltage law:
  - sum of voltages around loop is zero,
  - electric transmission network behaves differently to most other transportation networks because of Kirchhoff's voltage law.

Review of power concepts, continued

- Kirchhoff's laws implicitly determine the voltages and currents due to the real and reactive power injections at the generators and the withdrawals at the loads.
- Using Kirchhoff's laws to solve for the voltages and currents in a circuit consisting of generators, the transmission and/or distribution system, and loads is called the **power flow problem**:
  - assumes a particular operating condition,
  - **quasi-static** assumption that ignores dynamics and changes.
- The solution provides information about the flow of current and power on the transmission and distribution lines.
- The lines have limited capacities, so calculation of power flow enables us to decide whether or not a particular pattern of generation would result in acceptable flows on lines:
  - constraints on transmission operation implicitly determine limitations on the patterns of injections and withdrawals,
  - the locational marginal pricing market reflects these limitations into prices that vary by bus (or node).

3.2 Formulation of power flow

3.2.1 Variables

3.2.1.1 Phasors

- We can use complex numbers, called **phasors**, to represent the magnitude and angle of the AC voltages and currents at a fixed frequency.
- The **magnitude** of the complex number represents the root-mean-square magnitude of the voltage or current.
- The **angle** of the complex number represents the angular displacement between the sinusoidal voltage or current and a reference sinusoid.

3.2.1.2 Reference angle

- The angles of the voltages and currents in the system would all change if we changed the angle of our reference sinusoid, but this would have no effect on the physical system.
- We can therefore arbitrarily assign the angle at one of the buses to be zero and measure all the other angles with respect to this angle.
- We call this bus the **reference bus** or the **angle reference bus**, bus ρ, and typically number the buses so that $\rho = 0$ or $\rho = 1$.

3.2.1.3 *Representation of complex numbers*

- To represent a complex number $V \in \mathbb{C}$ with real numbers requires two real numbers, either:
  - the **magnitude** $|V|$ and the **angle** $\angle V$, so that $V = |V| \exp(\angle V \sqrt{-1})$, or
  - the **real** $\Re\{V\}$ and **imaginary** $\Im\{V\}$ parts, so that $V = \Re\{V\} + \Im\{V\}\sqrt{-1}$.
- Since we need to compare voltage magnitudes to limits to check satisfaction of voltage limit constraints, we will represent voltages as magnitudes and angles:
  - Some recent developments in power flow have used the real and imaginary parts representation.

3.2.1.4 *Scaling and "per unit"*

- There are voltage transformers throughout a typical power system:
  - "step-up" voltage at a generator to transmission voltages to enable transfer from generator to transmission system,
  - transform from one transmission voltage to another,
  - "step-down" voltage at a distribution substation and in distribution feeder for convenient use by load.
- The nominal voltage magnitude varies considerably across the system by several orders of magnitude.
- We scale the voltage magnitude so that an actual value of 121 kV in the 110 kV part of the system would be represented by a scaled value of:

$$\frac{121 \text{ kV}}{110 \text{ kV}} = 1.1,$$

- while an actual value of 688.5 kV in the 765 kV part of the system would be represented by a scaled value of:

$$\frac{688.5 \text{ kV}}{765 \text{ kV}} = 0.9.$$

3.2.2 Symmetry

3.2.2.1 Three-phase circuits

- Generation-transmission systems are usually operated as balanced **three-phase systems**, with generators, lines, and (roughly) distribution system loads arranged as symmetric triplets.

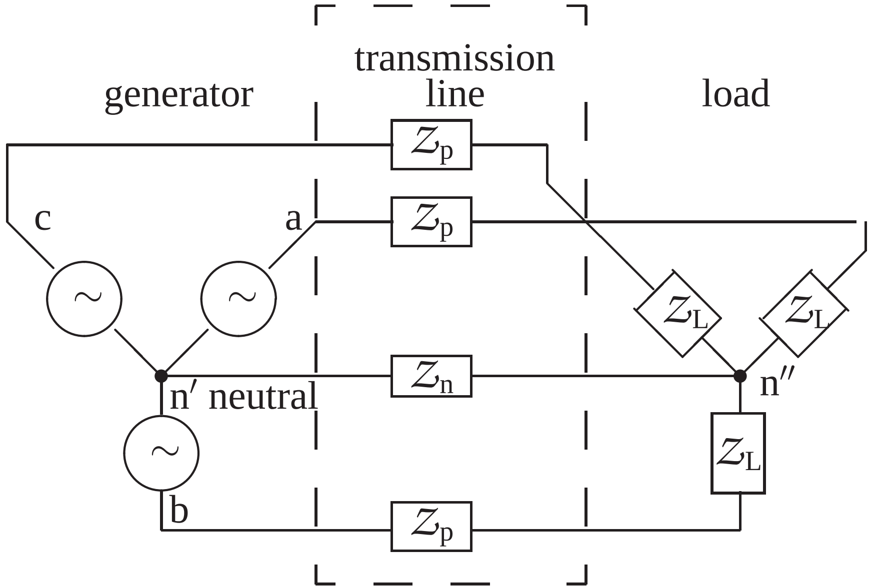

Fig. 3.2. An example balanced three-phase system.

3.2.2.2 Per-phase equivalent

- The quasi-static behavior of a balanced three-phase circuit can be completely determined from the behavior of a **per-phase equivalent circuit**.
- Figure 3.3 shows the a-phase equivalent circuit of the three-phase circuit of Figure 3.2.

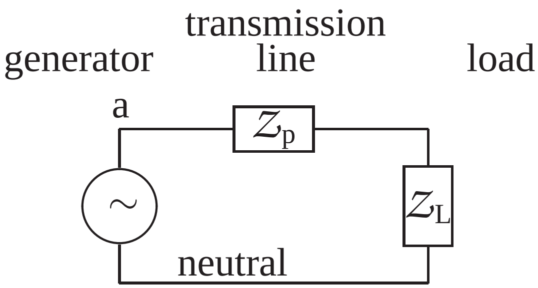

Fig. 3.3. Per-phase equivalent circuit for the three-phase circuit in Figure 3.2.

3.2.3 Transmission lines

- Transmission lines are physically extended objects, so the boxes in Figures 3.2 and 3.3 representing transmission lines are actually **distributed parameter circuits**,
- We can represent the terminal behavior of such distributed parameter circuits with a **π-equivalent** circuit.
- In the model, there are two **shunt components** connected from the terminals to neutral and a **series component** bridging the terminals.
- Each component of the π-equivalent has an impedance (or, equivalently, an **admittance**) determined by the characteristics of the line.

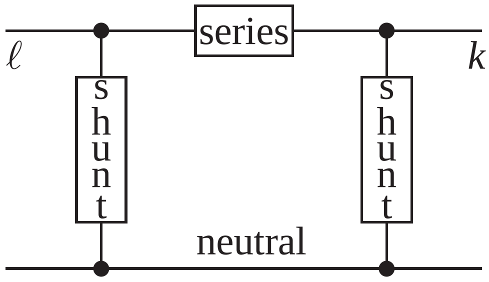

Fig. 3.4. Equivalent π circuit of per-phase equivalent of transmission line.

3.2.4 Bus admittance matrix

- Consider the per-phase equivalent of a three bus, three line transmission system as illustrated in Figure 3.5.
- For each bus $\ell = 1, 2, 3$, the pair of shunt π elements joining node ℓ to neutral can be combined together to form a single shunt element.

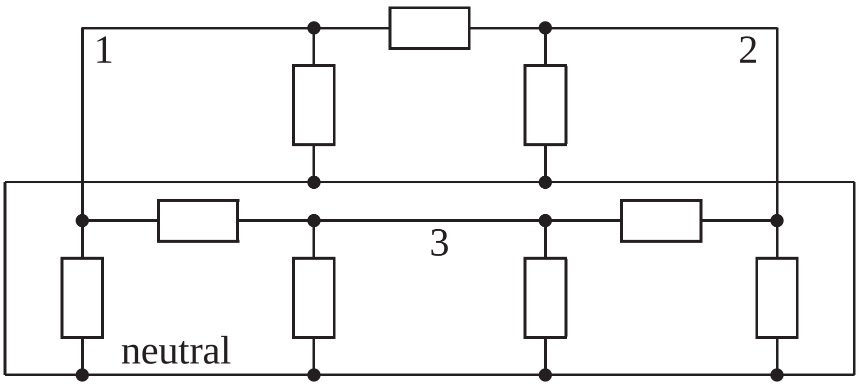

Fig. 3.5. Per-phase equivalent circuit model for three bus, three line system.

- This yields a circuit with:
  - one element corresponding to each of the buses $\ell = 1, 2, 3$, joining node ℓ to neutral, and
  - one element corresponding to each line,
- as illustrated in Figure 3.6.

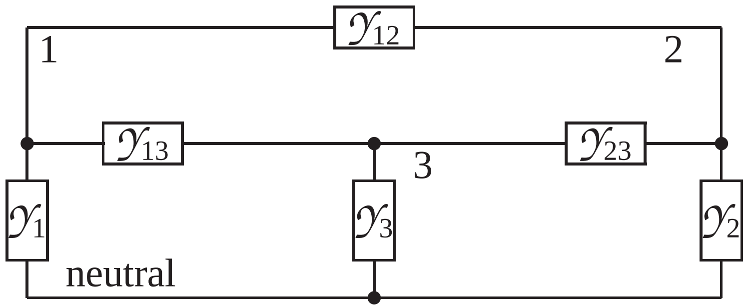

Fig. 3.6. Per-phase equivalent circuit model for three bus, three line system with parallel components combined.

Bus admittance matrix, continued

- As shown in Figure 3.6, let us write $\mathcal{Y}_\ell$ for the admittance of the element joining node ℓ to neutral, and
- $\mathcal{Y}_{\ell k}$ for the admittance of the series element corresponding to a line joining buses ℓ and k.
- The series element is most easily characterized in terms of its impedance.
- For a series impedance $\mathcal{Z}_{\ell k} = \mathcal{R}_{\ell k} + \mathcal{X}_{\ell k}\sqrt{-1}$ between buses ℓ and k, the corresponding admittance $\mathcal{Y}_{\ell k}$ is given by:

$$\begin{aligned} \mathcal{Y}_{\ell k} &= \frac{1}{\mathcal{Z}_{\ell k}}, \\ &= \frac{1}{\mathcal{R}_{\ell k} + \mathcal{X}_{\ell k}\sqrt{-1}} \\ &= \frac{1}{\mathcal{R}_{\ell k} + \mathcal{X}_{\ell k}\sqrt{-1}} \times \frac{\mathcal{R}_{\ell k} - \mathcal{X}_{\ell k}\sqrt{-1}}{\mathcal{R}_{\ell k} - \mathcal{X}_{\ell k}\sqrt{-1}} \\ &= \frac{\mathcal{R}_{\ell k} - \mathcal{X}_{\ell k}\sqrt{-1}}{(\mathcal{R}_{\ell k})^2 + (\mathcal{X}_{\ell k})^2}. \end{aligned} \quad (3.1)$$

- If $\mathcal{Z}_{\ell k} = 0.1 + \sqrt{-1}$, what is $\mathcal{Y}_{\ell k} = 1/\mathcal{Z}_{\ell k}$?

Bus admittance matrix, continued

- Let $V \in \mathbb{C}^{n_V}$ be the vector of phasor voltages at all the n_V buses in the system and let $I \in \mathbb{C}^{n_V}$ be the vector of phasor current injections into the transmission network at all of the buses in the system.
- Using Kirchhoff's laws, we can obtain a relationship of the form $AV = I$ between current and voltage, where $A \in \mathbb{C}^{n_V \times n_V}$:

$$\forall \ell, k, A_{\ell k} = \begin{cases} \mathcal{Y}_\ell + \sum_{k' \in \mathbb{J}(\ell)} \mathcal{Y}_{\ell k'}, & \text{if } \ell = k, \\ -\mathcal{Y}_{\ell k}, & \text{if } k \in \mathbb{J}(\ell) \text{ or } \ell \in \mathbb{J}(k), \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

- where $\mathbb{J}(\ell)$ is the set of buses joined directly by a transmission line to bus ℓ.
- The linear simultaneous equations $AV = I$ represent conservation of current at each of the buses.
- A is called the **bus admittance matrix**:
  - the ℓ-th diagonal entry is the sum of the admittances connected to bus ℓ,
  - the ℓk-th off-diagonal entry is minus the admittance connecting bus ℓ and k.

Bus admittance matrix, continued

- The bus admittance matrix can be thought of as being "built up" as the sum of matrices due to individual series and shunt elements.
- For example, if a line joins bus ℓ to bus k and its series element has admittance $\mathcal{Y}_{\ell k}$ then the contribution to the bus admittance matrix is $\mathcal{Y}_{\ell k} w^{\text{series}} (w^{\text{series}})^\dagger \in \mathbb{C}^{n_V \times n_V}$,
- where $w^{\text{series}} \in \mathbb{R}^{n_V}$ is a vector with a one in the ℓ-th entry, a minus one in the k-th entry, and zeros elsewhere, so:

$$\mathcal{Y}_{\ell k} w^{\text{series}} (w^{\text{series}})^\dagger$$

$$= \mathcal{Y}_{\ell k} \begin{bmatrix} & \overbrace{}^{\ell\text{-th column}} & \overbrace{}^{k\text{-th column}} & \\ & 1 & -1 & \} \ \ell\text{-th row} \\ & -1 & 1 & \} \ k\text{-th row} \\ & & & \end{bmatrix} = \begin{bmatrix} & \overbrace{}^{\ell\text{-th column}} & \overbrace{}^{k\text{-th column}} & \\ & \mathcal{Y}_{\ell k} & -\mathcal{Y}_{\ell k} & \} \ \ell\text{-th row} \\ & -\mathcal{Y}_{\ell k} & \mathcal{Y}_{\ell k} & \} \ k\text{-th row} \\ & & & \end{bmatrix}.$$

Bus admittance matrix, continued

- If the shunt element at bus ℓ has admittance $\mathcal{Y}_\ell$ then the contribution to bus admittance matrix is $\mathcal{Y}_\ell w^{\text{shunt}}(w^{\text{shunt}})^\dagger \in \mathbb{C}^{n_V \times n_V}$,
- where $w^{\text{shunt}} \in \mathbb{R}^{n_V}$ is a vector with a one in the ℓ-th entry and zeros elsewhere, so:

$$\mathcal{Y}_\ell w^{\text{shunt}}(w^{\text{shunt}})^\dagger = \mathcal{Y}_\ell \begin{bmatrix} & \overbrace{}^{\ell\text{-th column}} & & \\ & 1 & & \} \ \ell\text{-th row} \\ & & & \end{bmatrix} = \begin{bmatrix} & \overbrace{\phantom{\mathcal{Y}_\ell}}^{\ell\text{-th column}} & & \\ & \mathcal{Y}_\ell & & \} \ \ell\text{-th row} \\ & & & \end{bmatrix}.$$

- Summing these terms or **stamps** corresponding to each series and shunt element results in the bus admittance matrix A with entries as defined in (3.2).

Bus admittance matrix, continued

- With this characterization, we can write $A = W\mathcal{Y}W^\dagger$, where:
  - the matrix $\mathcal{Y}$ is a diagonal matrix with entries corresponding to the admittances of the series and shunt elements in the line models, and
  - the matrix W is the bus-to-element incidence matrix whose columns are of the form w^{series} or w^{shunt}.
- If $\mathcal{Y} = \mathcal{G} + \sqrt{-1}\mathcal{B}$ then $A = G + \sqrt{-1}B$, where

$$G = W\mathcal{G}W^\dagger, \quad (3.3)$$

$$B = W\mathcal{B}W^\dagger. \quad (3.4)$$

3.2.5 Changes in bus admittance matrix

- Using the characterization $A = W\mathcal{Y}W^{\dagger}$, we can consider the effect of adding or removing a line from the system.
- For example, if we incorporate an additional line into the system between buses ℓ and k having series element with admittance $\mathcal{G}_{\ell k} + \sqrt{-1}\mathcal{B}_{\ell k}$, then the admittance matrix is modified from A to $A + (\mathcal{G}_{\ell k} + \sqrt{-1}\mathcal{B}_{\ell k})w^{\text{series}}(w^{\text{series}})^{\dagger}$, with a similar observation for shunt elements.
- Conversely, if we remove a line joining buses ℓ and k from the system then the admittance matrix is modified from A to $A - (\mathcal{G}_{\ell k} + \sqrt{-1}\mathcal{B}_{\ell k})w^{\text{series}}(w^{\text{series}})^{\dagger}$, with a similar observation for shunt elements.
- Removing the line changes the real and imaginary parts of the admittance matrix to: $G - \mathcal{G}_{\ell k}w^{\text{series}}(w^{\text{series}})^{\dagger}$ and $B - \mathcal{B}_{\ell k}w^{\text{series}}(w^{\text{series}})^{\dagger}$, respectively.
- We will consider how such **outages** of lines affects the system in Section 3.12.

3.2.6 Generators and loads

- When electricity is bought and sold, the (real) power and energy are the quantities that are usually priced, not the voltage or current.
- However, real power does not completely describe the interaction between generators or loads and the system.
- We also have to characterize the injected reactive power.
- We can combine the real and reactive powers into the **complex power**, which is the sum of:

  the real power, and
  $\sqrt{-1}$ times the reactive power.

Generators and loads, continued

- The usefulness of this representation is that, for example, the net complex power S_ℓ injected at node ℓ into the network is given by:

$$S_\ell = V_\ell I_\ell^*,$$

- where the superscript $*$ indicates **complex conjugate**:
  - note difference between complex conjugate, denoted superscript $*$, and optimal or desired value, denoted superscript $\star$.
- The current I_ℓ equals the sum of:

  the current flowing into the shunt element $\mathcal{Y}_\ell$, and
  the sum of the currents flowing into each line connecting ℓ to a bus
  $k \in \mathbb{J}(\ell)$ through admittance $\mathcal{Y}_{\ell k}$.
- We can substitute for the currents to obtain:

$$\begin{aligned} S_\ell &= V_\ell [A_{\ell\ell} V_\ell + \sum_{k \in \mathbb{J}(\ell)} A_{\ell k} V_k]^*, \\ &= |V_\ell|^2 A_{\ell\ell}^* + \sum_{k \in \mathbb{J}(\ell)} A_{\ell k}^* V_\ell V_k^*. \end{aligned} \qquad (3.5)$$

Generators and loads, continued

- Let $A_{\ell k} = G_{\ell k} + B_{\ell k}\sqrt{-1}, \forall \ell, k$, where we note that by (3.1) and (3.2):
  - we have that $G_{\ell k} < 0$ and $B_{\ell k} > 0$ for $\ell \neq k$, and
  - we have that $G_{\ell\ell} > 0$ and the sign of $B_{\ell\ell}$ is indeterminate but typically less than zero;
- let $S_\ell = P_\ell + Q_\ell\sqrt{-1}, \forall \ell$, with:
  - for generator buses, $P_\ell > 0$ and Q_ℓ is typically positive,
  - for load buses, $P_\ell < 0$ and $Q_\ell < 0$;
- and let $V_\ell = v_\ell \exp(\theta_\ell\sqrt{-1}), \forall \ell$, with:
  - the voltage magnitude $v_\ell \approx 1$ in scaled units to satisfy voltage limits,
  - the voltage angle θ_ℓ typically between $-\pi/4$ and $\pi/4$ radians.
- Sometimes we will explicitly distinguish the real power injected by a generator from the real power consumed by a load, by writing D_ℓ for the real power load at bus ℓ:
  - the net real power injection at a bus with generation P_ℓ and load D_ℓ is then $P_\ell - D_\ell$, with both P_ℓ and D_ℓ typically positive.
  - Similarly, we will write E_ℓ for the reactive power load at bus ℓ, so that the net reactive power injection is $Q_\ell - E_\ell$.

Generators and loads, continued

- For notational convenience in the following development, we will write P_ℓ and Q_ℓ for the net real and reactive injections:
  - later cases where we explicitly distinguish generation from load will be clear from context.
- We can separate (3.5) into real and imaginary parts:

$$P_\ell = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \tag{3.6}$$

$$Q_\ell = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)]. \tag{3.7}$$

- The equations (3.6) and (3.7), which are called the **power flow equality constraints**, must be satisfied at each bus ℓ.
- That is, there are two constraints that must be satisfied at each bus.
- For a 5000 bus system, how many power flow equality constraints must be satisfied?

3.2.7 The power flow problem

- The power flow problem is to find values of voltage angles and magnitudes that satisfy the power flow equality constraints.

3.2.7.1 Real and reactive power balance

- For convenience, we will say *PQ* **bus** for a bus where the real and reactive power injection is specified.
- We specify:
  - the real and reactive generations at the *PQ* generator buses according to the generator control settings, and
  - the (typically negative) real and reactive net power injections at the *PQ* load buses according to supplied data.
- At each such bus, we have two specified parameters (the real and reactive power injection) and two unknowns that are entries in the decision vector (the voltage magnitude and angle).
- However, we cannot arbitrarily specify the real and reactive power at all the buses since this would typically violate the first law of thermodynamics!
  - Not all the buses can be *PQ* buses.

3.2.7.2 *Slack bus*

- A traditional, but *ad hoc* approach to finding a solution to the equations is to single out a **slack bus**, bus σ.
- At this slack bus, instead of specifying injected real and reactive power, there is assumed to be a generator that produces whatever is needed to "balance" the real and reactive power for the rest of the system, assuming that such a solution exists.
- Typically, the slack bus is the same as the reference bus:
  - in this case, we will typically number the buses so that $\sigma = \rho = 0$ or $\sigma = \rho = 1$,
  - however, the slack bus and reference bus can be different, and the slack can even be (conceptually) "distributed" across multiple buses.
- In optimal power flow (Section 9.1), we can in principle avoid this issue and not define a slack bus.
- For reasons that will become clear in the context of locational marginal pricing, we also call the slack bus the **price reference bus**.

Slack bus, continued

- We re-interpret the real and reactive power generation at the slack bus, P_σ and Q_σ, to be decision variables in our power flow formulation.
- We will find values of P_σ and Q_σ that satisfy the overall real and reactive power balance in the system as implied by the first law of thermodynamics:
  - for reasons that will become clear in the next section, we will not have to represent P_σ and Q_σ explicitly in the decision vector x when solving power flow,
  - we can simply evaluate P_σ and Q_σ at the end of the calculation by evaluating an expression,
  - in Section 5 in the context of **economic dispatch** where we are considering the choice of generation at all the buses, we will also consider the real and reactive generations to be decision variables and so P_σ and Q_σ together with all the other real and reactive generations will be explicitly in the decision vector x,
  - example of where the definition of x will depend on context.

Slack bus, continued

- The generator at the slack bus supplies whatever power is necessary for real and reactive power balance.
- To keep the number of unknowns equal to the number of equations, the voltage magnitude at the slack bus is specified as any particular value:
  - in Section 9 in the context of **optimal power flow** we will re-interpret the voltage magnitude at the slack bus to also be part of the decision vector.
- If the reference bus and the slack bus are the same bus, then we can call it a $V\theta$ bus, since both the voltage magnitude and angle are specified.
- At the $V\theta$ bus, we still have two specified parameters (the voltage magnitude and angle) and two unknowns (the real and reactive power injections).
- For most of the rest of the development of power flow, we will typically assume that the reference bus and the slack bus are the same and typically number the buses so that the reference/slack bus is bus 1:
  - we will sketch how to consider the case where the reference and slack buses are different.

3.2.8 Non-linear equations

- We have n_{PQ} PQ buses, including both the PQ generators and the loads.
- Let $n = 2n_{PQ}$ and define a decision vector $x \in \mathbb{R}^n$ consisting of the voltage magnitudes and angles at the PQ buses:
  - unknown real P_σ and reactive Q_σ generation at the slack bus will be evaluated in terms of x and so are not represented explicitly in the decision vector.
- For every bus ℓ (that is, including the slack bus as well as the PQ buses) define functions $p_\ell : \mathbb{R}^n \to \mathbb{R}$ and $q_\ell : \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall x \in \mathbb{R}^n, p_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], \quad (3.8)$$

$$\forall x \in \mathbb{R}^n, q_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)]. \quad (3.9)$$

- The functions p_ℓ and q_ℓ represent the real and reactive power flow, respectively, from bus ℓ into the lines in the rest of the system.
- Kirchhoff's laws require that the net real and reactive flow out of a bus must be zero, so that $p_\ell(x) - P_\ell = 0$ and $q_\ell(x) - Q_\ell = 0$ at every bus ℓ.

Non-linear equations, continued

- Consider the special case $\rho = \sigma = 1$ and define a vector function $g : \mathbb{R}^n \to \mathbb{R}^n$ with entries given by $p_\ell - P_\ell$ and $q_\ell - Q_\ell$ for all the PQ buses: $\forall x \in \mathbb{R}^n, g(x) = \begin{bmatrix} p_2(x) - P_2 \\ p_3(x) - P_3 \\ \vdots \\ q_2(x) - Q_2 \\ q_3(x) - Q_3 \\ \vdots \end{bmatrix}.$
- If we solve the non-linear simultaneous equations:

$$g(x) = \mathbf{0}, \tag{3.10}$$

- for $x^\star$ and then set:

$$\begin{aligned} P_1 &= p_1(x^\star), \\ Q_1 &= q_1(x^\star), \end{aligned}$$

- then we will have satisfied the power flow equality constraints at all buses including the slack bus $\sigma = 1$.

Non-linear equations, continued

- In the general case where the reference bus ρ may not be bus 1, and where the slack bus σ may not be bus 1:
  - the decision vector x consists of all of the voltage magnitudes and angles except at the reference bus ρ, and
  - the vector function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has entries given by $p_\ell - P_\ell$ and $q_\ell - Q_\ell$ for all $\ell \neq \sigma$.
- Again, if we solve the non-linear simultaneous equations (3.10), $g(x) = \mathbf{0}$, for $x^\star$ and then set:

$$\begin{aligned} P_\sigma &= p_\sigma(x^\star), \\ Q_\sigma &= q_\sigma(x^\star), \end{aligned}$$

- then we will have satisfied the power flow equality constraints at all buses including the slack bus σ.
- Note that we simply evaluated P_σ and Q_σ by evaluating the expressions $p_\sigma(x^\star)$ and $q_\sigma(x^\star)$.

Non-linear equations, continued

- We have formulated the power flow problem as the solution of non-linear simultaneous equations:

$$g(x) = \mathbf{0}.$$

- The vector g includes real and reactive power entries for each bus except the slack bus:
  - we will calculate the real and reactive power injection at the slack bus after we have solved $g(x) = \mathbf{0}$.
- The vector x includes voltage angles and magnitudes for each bus except the reference bus:
  - the voltage angle and magnitude for the reference/slack bus are specified.
- Note that as we develop other problems, we will re-define x and g as needed for the formulation.

Non-linear equations, continued

- For future notational convenience:
  - Let θ, v, p, q, P, and Q be vectors consisting, respectively, of the entries θ_ℓ, v_ℓ, p_ℓ, q_ℓ, P_ℓ, and Q_ℓ for *all* the buses.
- We will often need to refer to a sub-vector with a particular entry omitted:
  - let subscript $-k$ on a vector denote that vector with the entry k omitted,
  - so θ_{-k}, v_{-k}, p_{-k}, q_{-k}, P_{-k}, and Q_{-k} are, respectively, the sub-vectors of θ, v, p, q, P, and Q with the entries θ_k, v_k, p_k, q_k, P_k, and Q_k omitted.
- We will maintain these definitions of θ, v, p, q, P, and Q throughout the course:
  - recall that we will change the definition of x and g depending on the particular problem being formulated.
- Also, let subscript $-k$ on a matrix denote that matrix with row k omitted:
  - so A_{-k} is the admittance matrix A with the k-th row omitted.

Non-linear equations, continued

- With $\rho = \sigma = 1$ the reference/slack bus then:

$$x = \begin{bmatrix} \theta_{-1} \\ v_{-1} \end{bmatrix},$$
$$g = \begin{bmatrix} (p_{-1}) - P_{-1} \\ (q_{-1}) - Q_{-1} \end{bmatrix}.$$

- With this notation, the simultaneous equations $g(x) = \mathbf{0}$ can also be expressed in the equivalent form:

$$p_{-1}(x) = p_{-1}\left(\begin{bmatrix} \theta_{-1} \\ v_{-1} \end{bmatrix}\right) = P_{-1},$$
$$q_{-1}(x) = q_{-1}\left(\begin{bmatrix} \theta_{-1} \\ v_{-1} \end{bmatrix}\right) = Q_{-1}.$$

- For a 5000 bus system, how many entries are in θ, v, p and q? How about in θ_{-1}, v_{-1}, p_{-1}, q_{-1}, x, and g?

Non-linear equations, continued

- In the general case where the reference bus ρ may not be bus 1, and where the slack bus σ may not be bus 1:

$$
\begin{aligned}
x &= \begin{bmatrix} \theta_{-\rho} \\ v_{-\rho} \end{bmatrix}, \\
g &= \begin{bmatrix} (p_{-\sigma}) - P_{-\sigma} \\ (q_{-\sigma}) - Q_{-\sigma} \end{bmatrix}.
\end{aligned}
$$

- The simultaneous equations $g(x) = \mathbf{0}$ can also be expressed in the equivalent form:

$$
\begin{aligned}
p_{-\sigma}(x) = p_{-\sigma}\left(\begin{bmatrix} \theta_{-\rho} \\ v_{-\rho} \end{bmatrix}\right) &= P_{-\sigma}, \\
q_{-\sigma}(x) = q_{-\sigma}\left(\begin{bmatrix} \theta_{-\rho} \\ v_{-\rho} \end{bmatrix}\right) &= Q_{-\sigma}.
\end{aligned}
$$

Non-linear equations, continued

- In summary, to solve Kirchhoff's equations for the electric power network, we:
  - (i) solve (3.10), $g(x) = \mathbf{0}$, which is a system of non-linear simultaneous equations, and
  - (ii) substitute the solution $x^\star$ into (3.6) and (3.7) for the slack bus $\ell = \sigma$ to find the real and reactive power generated at the slack bus.
- The real power generation at the slack bus is $P_\sigma = p_\sigma(x^\star)$, so $x^\star$ also satisfies $p(x^\star) = P$ and, moreover:

$$\mathbf{1}^\dagger P = \mathbf{1}^\dagger p(x^\star).$$

- This expression evaluates the total losses in the system, since it sums the total net real power injected into the transmission lines.
- Line currents and real and reactive power flows can also be calculated once x is known:
  - see in Section 3.7.

3.2.9 Example

- Consider the three bus system shown in Figure 3.7 with buses $\ell = 1, 2, 3$ and with bus $\rho = \sigma = 1$ the reference/slack bus.
- Net generation of $P_\ell, \ell = 1, 2, 3$, is shown at each bus and transmission lines are represented by the lines joining the buses.

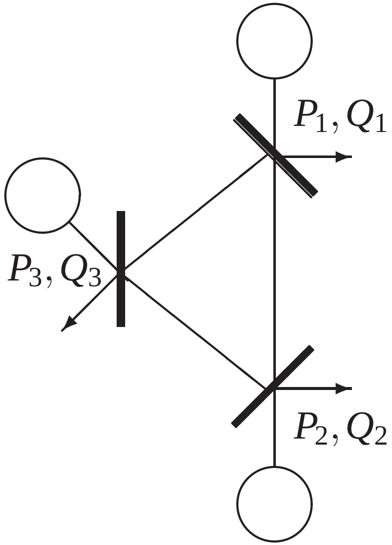

Fig. 3.7. Three bus, three line network.

Example, continued

- The entries of $x \in \mathbb{R}^4$ and $g : \mathbb{R}^4 \to \mathbb{R}^4$ are:

$$x = \begin{bmatrix} \theta_{-1} \\ v_{-1} \end{bmatrix} = \begin{bmatrix} \theta_2 \\ \theta_3 \\ v_2 \\ v_3 \end{bmatrix},$$

$$\forall x \in \mathbb{R}^4, g(x) = \begin{bmatrix} p_{-1}(x) - P_{-1} \\ q_{-1}(x) - Q_{-1} \end{bmatrix} = \begin{bmatrix} p_2(x) - P_2 \\ p_3(x) - P_3 \\ q_2(x) - Q_2 \\ q_3(x) - Q_3 \end{bmatrix},$$

- where P_2 is the net generation (generation minus demand) at bus 2, and similarly for other buses and for the reactive power at the buses.

Example, continued

- If we solve $g(x) = \mathbf{0}$, we can then use the resulting solution $x^\star$ to evaluate the real power and reactive power that must be produced at the reference/slack bus to satisfy real and reactive power balance at every bus:

$$\begin{aligned} P_1 &= p_1(x^\star), \\ Q_1 &= q_1(x^\star). \end{aligned}$$

- Losses in the system are given by:

$$\mathbf{1}^\dagger p(x^\star) = P_1 + P_2 + P_3.$$

- If the reference bus is $\rho = 2$ and the slack bus is $\sigma = 3$, what are the entries in x and g?
- If the slack bus is $\sigma = 3$, what are the expressions to evaluate the real and reactive power injection at the slack bus?

3.3 Problem characteristics and solution

3.3.1 Number of variables and equations

- There are the same number of variables as equations in (3.10).
- For a 5000 bus system, with one $V\theta$ bus and the rest PQ buses, how many variables and equations are there?

3.3.2 Non-existence of direct algorithms

- Because the equations are non-linear, there is no direct algorithm, such as factorization, to solve for the solution $x^{\star}$ for arbitrary systems.
- The Newton–Raphson algorithm from Section 2.5 can be applied to this problem, requiring:
  - an initial guess $x^{(0)}$,
  - evaluation of partial derivative terms in the Jacobian, $\frac{\partial g}{\partial x}$, and
  - solution of the Newton–Raphson update (2.10)–(2.11) at each iteration.

3.3.3 Number of solutions

- There may be no solutions, one solution, or even multiple solutions to (3.10).
- However, power systems are usually designed and operated so that the voltage magnitudes are near to nominal and the voltage angles are relatively close to 0°.
- If we restrict our attention to solutions such that voltage magnitudes are all close to 1 (and make some other assumptions) then we can find conditions for the there to be at most one solution.
- How many solutions are there to $2+\sin(\theta)=0$?
- How many solutions are there to $0.1+\sin(\theta)=0$?
- How many solutions are there to $0.1+\sin(\theta)=0$ with $-\pi/4\leq\theta\leq\pi/4$?

3.3.4 Admittance matrix

3.3.4.1 Symmetry

- The admittance matrix A is symmetric.

3.3.4.2 Sparsity

- The matrix A is only sparsely populated with non-zero entries and each component of g depends on only a few components of x.
- Sparsity is the key to practical solution of problems with large numbers of buses.
- For a 5000 bus system having 5000 lines, how many non-zero entries are there in the admittance matrix A?

3.3.4.3 Values

- A typical line impedance has positive real and imaginary parts.
- The corresponding line admittance $\mathcal{Y}_{\ell k}$ therefore has positive real part and negative imaginary part.
- If there is a line between buses ℓ and k then the entries $A_{\ell k} = G_{\ell k} + \sqrt{-1}B_{\ell k}$ in the admittance matrix satisfy $G_{\ell k} < 0, B_{\ell k} > 0$.
- The diagonal entries $A_{\ell\ell} = G_{\ell\ell} + \sqrt{-1}B_{\ell\ell}$ in the admittance satisfy $G_{\ell\ell} > 0$ and, typically, $B_{\ell\ell} < 0$.
- The resistance $\mathcal{R}_{\ell k}$ of transmission lines is relatively small compared to the inductive reactance $\mathcal{X}_{\ell k}$.
- Furthermore, the shunt elements are often also negligible compared to the the inductive reactance.
- This means that for transmission lines:

$$\forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, |G_{\ell k}| \ll |B_{\ell k}|.$$

- Note that distribution lines may have relatively higher resistance to inductive reactance ratios than transmission:
  - approximations described in following sections are less accurate for distribution systems.

3.4 Linearized power flow

3.4.1 Base-case

- Suppose that we are given values of real and reactive generation $P^{(0)} \in \mathbb{R}^{n_{PQ}+1}$ and $Q^{(0)} \in \mathbb{R}^{n_{PQ}+1}$ that specify a **base-case**.
  - For example, the base-case real and reactive generations could be the current operating conditions.
  - As another example, $P^{(0)} = \mathbf{0}$ is the (unrealistic) condition of zero net real power injection.
- Also suppose that we have a solution $x^{(0)} = \begin{bmatrix} \theta_{-1}^{(0)} \\ v_{-1}^{(0)} \end{bmatrix}$ to the base-case equations, so that $g(x^{(0)}) = \mathbf{0}$, or equivalently:

$$\begin{aligned} p_{-1}(x^{(0)}) &= P_{-1}^{(0)}, \\ q_{-1}(x^{(0)}) &= Q_{-1}^{(0)}, \end{aligned}$$

- where $P_{-1}^{(0)}$ and $Q_{-1}^{(0)}$ are the sub-vectors of $P^{(0)}$ and $Q^{(0)}$, respectively, that omit the reference/slack bus, where $\rho = \sigma = 1$.

3.4.2 Change-case

- Now suppose that the real and reactive power generations change:
  - from $P^{(0)}$ and $Q^{(0)}$,
  - to $P = P^{(0)} + \Delta P$ and $Q = Q^{(0)} + \Delta Q$, respectively.
- Similarly, we suppose that the value of x changes from $x^{(0)}$ to $x^{(0)} + \Delta x$ to re-establish satisfaction of the power flow equations $g(x) = \mathbf{0}$.
- That is, the **change-case** power flow equations are given by:

$$\begin{aligned} p_{-1}(x^{(0)} + \Delta x) &= P^{(0)}_{-1} + \Delta P_{-1}, \\ q_{-1}(x^{(0)} + \Delta x) &= Q^{(0)}_{-1} + \Delta Q_{-1}, \end{aligned}$$

- where ΔP_{-1} and ΔQ_{-1} are the sub-vectors of ΔP and ΔQ, respectively, that omit the reference/slack bus.
- The equations are non-linear equations in Δx.

Change-case, continued

- Note the change in net generation at the reference/slack bus is required to be consistent with the change Δx.
- So, we also have that:

$$\begin{aligned} p_1(x^{(0)} + \Delta x) &= P_1^{(0)} + \Delta P_1, \\ q_1(x^{(0)} + \Delta x) &= Q_1^{(0)} + \Delta Q_1. \end{aligned}$$

- That is, the change in generation at the reference/slack bus can be calculated (or estimated) once Δx is known or estimated.

3.4.3 First-order Taylor approximation

- To find an approximate solution to the change-case equations, we form **first-order Taylor approximations** to p_{-1} and q_{-1}:

$$\begin{aligned} p_{-1}(x^{(0)}+\Delta x) &\approx p_{-1}(x^{(0)})+\frac{\partial p_{-1}}{\partial x}(x^{(0)})\Delta x, \\ q_{-1}(x^{(0)}+\Delta x) &\approx q_{-1}(x^{(0)})+\frac{\partial q_{-1}}{\partial x}(x^{(0)})\Delta x. \end{aligned}$$

- For future reference, note that the matrices $\frac{\partial p_{-1}}{\partial x}(x)$ and $\frac{\partial q_{-1}}{\partial x}(x)$ form the Jacobian of the system of equations $p_{-1}(x) = P_{-1}, q_{-1}(x) = Q_{-1}$:

$$\begin{bmatrix} \frac{\partial p_{-1}}{\partial x}(x) \\ \frac{\partial q_{-1}}{\partial x}(x) \end{bmatrix}.$$

3.4.4 Linearization of change-case equations

- Substituting the first-order Taylor approximations into the change-case equations, we obtain:

$$p_{-1}(x^{(0)}) + \frac{\partial p_{-1}}{\partial x}(x^{(0)})\Delta x \approx P_{-1}^{(0)} + \Delta P_{-1},$$
$$q_{-1}(x^{(0)}) + \frac{\partial q_{-1}}{\partial x}(x^{(0)})\Delta x \approx Q_{-1}^{(0)} + \Delta Q_{-1}.$$

- From the base-case solution, we have $p_{-1}(x^{(0)}) = P_{-1}^{(0)}$ and $q_{-1}(x^{(0)}) = Q_{-1}^{(0)}$.
- Ignoring the error in the first-order Taylor approximation, we have:

$$\frac{\partial p_{-1}}{\partial x}(x^{(0)})\Delta x = \Delta P_{-1},$$
$$\frac{\partial q_{-1}}{\partial x}(x^{(0)})\Delta x = \Delta Q_{-1}.$$

Linearization of change-case equations, continued

- Typically, the Jacobian $\begin{bmatrix} \frac{\partial p_{-1}}{\partial x}(x^{(0)}) \\ \frac{\partial q_{-1}}{\partial x}(x^{(0)}) \end{bmatrix}$ is non-singular.
- That is, we can solve:

$$\begin{bmatrix} \frac{\partial p_{-1}}{\partial x}(x^{(0)}) \\ \frac{\partial q_{-1}}{\partial x}(x^{(0)}) \end{bmatrix} \Delta x = \begin{bmatrix} \Delta P_{-1} \\ \Delta Q_{-1} \end{bmatrix},$$

- for Δx.
- These are **sparse** linear equations, which can be solved efficiently for Δx.
- This approximation to the solution of the change-case power flow equations is equivalent to performing one iteration of the Newton–Raphson method, starting at the base-case specified by $x^{(0)}$.
- For a 5000 bus system, what is the size of the coefficient matrix of the linear equations?

Linearization of change-case equations, continued

- Moreover, the change in real and reactive power at the slack bus will approximately satisfy:

$$\Delta P_1 = \frac{\partial p_1}{\partial x}(x^{(0)})\Delta x,$$

$$\Delta Q_1 = \frac{\partial q_1}{\partial x}(x^{(0)})\Delta x.$$

3.4.5 Jacobian

3.4.5.1 Terms

- Recall that the entries in $p : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$ are defined by:

$$\forall x \in \mathbb{R}^n, p_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)].$$

- The entries in $q : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{PQ}+1}$ are defined by: $q_\ell : \mathbb{R}^n \rightarrow \mathbb{R}$:

$$\forall x \in \mathbb{R}^n, q_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)].$$

- The entries in the vector x are either of the form θ_k or of the form v_k.
- To examine the terms in the Jacobian, partition x so that all the voltage angles appear first in a sub-vector θ_{-1} followed by all the voltage magnitudes in a sub-vector v_{-1}, so that:

$$x = \begin{bmatrix} \theta_{-1} \\ v_{-1} \end{bmatrix}.$$

Terms, continued

- There are four qualitative types of partial derivative terms corresponding to each combination:

$$\forall x \in \mathbb{R}^n, \frac{\partial p_\ell}{\partial \theta_k}(x)$$
$$= \begin{cases} \sum_{j \in \mathbb{J}(\ell)} v_\ell v_j [-G_{\ell j} \sin(\theta_\ell - \theta_j) + B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ v_\ell v_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall x \in \mathbb{R}^n, \frac{\partial p_\ell}{\partial v_k}(x)$$
$$= \begin{cases} 2v_\ell G_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} v_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ v_\ell [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

Terms, continued

$$\forall x \in \mathbb{R}^n, \frac{\partial q_\ell}{\partial \theta_k}(x)$$

$$= \begin{cases} \sum_{j \in \mathbb{J}(\ell)} v_\ell v_j [G_{\ell j} \cos(\theta_\ell - \theta_j) + B_{\ell j} \sin(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ v_\ell v_k [-G_{\ell k} \cos(\theta_\ell - \theta_k) - B_{\ell k} \sin(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise,} \end{cases}$$

$$\forall x \in \mathbb{R}^n, \frac{\partial q_\ell}{\partial v_k}(x)$$

$$= \begin{cases} -2 v_\ell B_{\ell\ell} + \sum_{j \in \mathbb{J}(\ell)} v_j [G_{\ell j} \sin(\theta_\ell - \theta_j) - B_{\ell j} \cos(\theta_\ell - \theta_j)], & \text{if } k = \ell, \\ v_\ell [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)], & \text{if } k \in \mathbb{J}(\ell), \\ 0, & \text{otherwise.} \end{cases}$$

3.4.5.2 Partitioning by types of terms

- Based on the partitioning of x, we can partition the Jacobian into four blocks:

$$\begin{bmatrix} \frac{\partial p_{-1}}{\partial \theta_{-1}}(x) & \frac{\partial p_{-1}}{\partial v_{-1}}(x) \\ \frac{\partial q_{-1}}{\partial \theta_{-1}}(x) & \frac{\partial q_{-1}}{\partial v_{-1}}(x) \end{bmatrix}.$$

- For a 5000 bus system, with 5000 lines, how many non-zero entries are there in each of these blocks? Assume that there are exactly two lines connected to the slack bus.

3.4.6 Decoupled equations

- Recall that for typical lines $\forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, |G_{\ell k}| \ll |B_{\ell k}|$.
- Also note that for typical lines ℓk, $|\theta_\ell - \theta_k| \ll \pi/2$.
- This implies that the terms in the matrices $\frac{\partial p_{-1}}{\partial v_{-1}}$ and $\frac{\partial q_{-1}}{\partial \theta_{-1}}$ are small compared to the terms in the matrices $\frac{\partial p_{-1}}{\partial \theta_{-1}}$ and $\frac{\partial q_{-1}}{\partial v_{-1}}$.
- If we neglect all the terms in $\frac{\partial p_{-1}}{\partial v_{-1}}$ and $\frac{\partial q_{-1}}{\partial \theta_{-1}}$, then we can then approximate the Jacobian by $\begin{bmatrix} \frac{\partial p_{-1}}{\partial \theta_{-1}}(x) & \mathbf{0} \\ \mathbf{0} & \frac{\partial q_{-1}}{\partial v_{-1}}(x) \end{bmatrix}$.

Decoupled equations, continued

- Letting $\Delta x = \begin{bmatrix} \Delta\theta_{-1} \\ \Delta v_{-1} \end{bmatrix}$, this allows decoupling of the linearized equations into:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1} = \Delta P_{-1},$$
$$\frac{\partial q_{-1}}{\partial v_{-1}}(x^{(0)})\Delta v_{-1} = \Delta Q_{-1},$$

- The first set of equations relate real power and angles, while the second set of equations relate reactive power and voltage magnitudes.
- These decoupled equations require less computation than solving the full system.
- For a 5000 bus system, what is the size of the coefficient matrix of each of the the decoupled linear equations?

3.5 Fixed voltage schedule

- If **real power** generations and flows are our main concern and there is adequate **voltage support** in the form of controllable reactive sources then we may be justified in assuming that the voltage magnitudes can be held fixed by controlling reactive power:
  - instead of assuming that each bus except the reference/slack bus is a PQ bus, we assume that each such bus has a specified real power and voltage magnitude.
  - These are called PV buses.
  - A typical assumption is that all voltage magnitudes are 1 per unit, $v = \mathbf{1}$.
  - More generally, any fixed voltage schedule $v^{(0)}$ can be used.
  - We can evaluate the reactive power injections at PV buses at the end of the calculation by evaluating an expression:
    - This is similar to the situation for real and reactive power at the slack bus.
    - At a PV bus, we do not need to solve for the voltage magnitude, since it is specified.

Fixed voltage schedule, continued

- Assuming all buses, except the reference/slack bus, are PV buses:
  - The unknowns are: the voltage angles at all the buses except the reference/slack bus; the real power generation at the reference/slack bus; and the reactive power generations at all buses.
  - We first solve $p_{-1}\left(\begin{bmatrix}\theta_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right) = P_{-1}$ for θ_{-1}, given the fixed voltage schedule $v^{(0)}$, and call the solution $\theta^{\star}_{-1}$.
  - To complete the solution:
    - real power at the reference/slack bus, $P_1^{(0)}$ is chosen to satisfy $P_1^{(0)} = p_1\left(\begin{bmatrix}\theta^{\star}_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right)$, and
    - reactive generations at all buses, including the reference/slack bus, $Q^{(0)}$ are chosen to satisfy $Q^{(0)} = q\left(\begin{bmatrix}\theta^{\star}_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right)$, in order to achieve the voltage schedule $v^{(0)}$.

3.6 DC power flow

- We combine the ideas of fixed voltage profile and linearization.

3.6.1 Fixed voltage schedule

- We again assume that there are controllable voltage sources available to provide a fixed voltage schedule $v^{(0)}$.
- Based on the analysis in the previous section, we could first solve $p_{-1}\left(\begin{bmatrix}\theta_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right) = P_{-1}$ for θ_{-1} to obtain the solution $\theta^{\star}_{-1}$ and then evaluate $P_1^{(0)} = p_1\left(\begin{bmatrix}\theta^{\star}_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right)$.
- This again enables us to focus on real power generation and angles.
- However, instead of solving $p_{-1}\left(\begin{bmatrix}\theta_{-1}\\ v_{-1}^{(0)}\end{bmatrix}\right) = P_{-1}$ exactly for $\theta^{\star}_{-1}$, we solve a linearized version of the equations that is linearized about a base-case in order to estimate a change-case solution.

3.6.2 Linearization

- We linearize about a *fixed* base-case solution, $x^{(0)} = \begin{bmatrix} \theta_{-1}^{(0)} \\ v_{-1}^{(0)} \end{bmatrix}$.
- The change-case power flow equations are given by:

$$p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ v_{-1}^{(0)} \end{bmatrix}\right) = P_{-1}^{(0)} + \Delta P_{-1}.$$

- To find an approximate solution to the change-case equations, we form a first-order Taylor approximation to p_{-1}:

$$\begin{aligned} p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ v_{-1}^{(0)} \end{bmatrix}\right) &\approx p_{-1}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ v_{-1}^{(0)} \end{bmatrix}\right) + \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix} \theta_{-1}^{(0)} \\ v_{-1}^{(0)} \end{bmatrix}\right)\Delta\theta_{-1}, \\ &= p_{-1}(x^{(0)}) + \frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1}. \end{aligned}$$

Linearization, continued

- Substituting the first-order Taylor approximations into the change-case equations, we obtain:

$$p_{-1}(x^{(0)}) + \frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1} \approx P_{-1}^{(0)} + \Delta P_{-1}.$$

- From the base-case solution, we have $p_{-1}(x^{(0)}) = P_{-1}^{(0)}$.
- Ignoring the error in the first-order Taylor approximation, we have:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1} = \Delta P_{-1},$$

- which can be solved for $\Delta\theta_{-1}$.
- The change in the real power generation at the slack bus is then approximately:

$$\Delta P_{1} = \frac{\partial p_{1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1}.$$

Linearization, continued

- The base-case power generations $P^{(0)}$ that determine the base-case solution are chosen to be convenient for calculations.
- A typical base-case involves:
  - zero net real power generation at all buses, so that $P^{(0)} = \mathbf{0}$, and
  - all voltage magnitudes 1 per unit, so that $v^{(0)} = \mathbf{1}$.
- If we make the realistic assumption that the transmission lines have zero real values for their shunt elements then $\theta^{(0)} = \mathbf{0}$ solves the base-case.
- $x^{(0)} = \begin{bmatrix} \theta^{(0)}_{-1} \\ v^{(0)}_{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ is called a **flat start**.
- At the flat start, the linearization yields the following equations:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1} = \Delta P_{-1}.$$

Linearization, continued

- To summarize, we have linearized about the flat start condition to approximate the change-case solution $\theta = \theta^{(0)} + \Delta\theta = \mathbf{0} + \Delta\theta = \Delta\theta$ corresponding to injections $P = P^{(0)} + \Delta P = \mathbf{0} + \Delta P = \Delta P$.
- We now interpret:

  $P^{(0)} + \Delta P = \Delta P = P$ to be the power generation for the change-case we are trying to solve, and

  $\theta^{(0)} + \Delta\theta = \Delta\theta = \theta$ to be the solution for the angles for the change-case we are trying to solve.
- That is, we solve the linearized power flow equations for θ_{-1}:

$$\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-1} = P_{-1},$$

- where $\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is a constant matrix,
- θ_{-1} is the vector of unknown angles at the change-case solution, and
- P_{-1} is the sub-vector of P that omits the slack bus.

Linearization, continued

- These equations are in the form $J_{-1}\theta_{-1} = P_{-1}$, where the coefficient matrix is:

$$J_{-1} = \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right).$$

- The coefficient matrix J_{-1} relates real power and angles and is the sub-matrix of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ obtained by deleting row 1.
- The subscript -1 on J_{-1} is referring to bus 1 as the slack bus:
  - for the general case of bus σ as the slack bus, we will consider $J_{-\sigma} = \frac{\partial p_{-\sigma}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$,
  - where the reference bus is still bus 1, but $J_{-\sigma}$ has been obtained from J by deleting row σ.
  - We will discuss changing both the slack bus and the reference bus in Section 3.6.9.
- The equations $J_{-1}\theta_{-1} = P_{-1}$ (or $J_{-\sigma}\theta_{-1} = P_{-\sigma}$) are sparse linear equations, which can be solved efficiently for θ_{-1}.

Linearization, continued

- Paralleling the earlier observation, this approximation to the solution of the power flow equations for power generation P_{-1} is equivalent to performing one iteration of the Newton–Raphson method, starting at a flat start.
- Moreover, the real power at the slack bus for the change-case can be estimated by:

$$P_1 = \frac{\partial p_1}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta_{-1}, \tag{3.11}$$

- or $P_\sigma = \frac{\partial p_\sigma}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)\theta_{-1}$ if bus σ is the slack bus.
- We will see in Section 3.6.6 that the estimation of the real power generation at the slack bus can be written more directly in terms of the generations $P_{-\sigma}$ at the other buses.

3.6.3 DC power flow equations

- Combining $J_{-1}\theta_{-1} = P_{-1}$ and $P_1 = \frac{\partial p_1}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)\theta_{-1}$, we obtain:

$$J\theta_{-1} = P, \tag{3.12}$$

  where $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$.
- Equations (3.12) are called the **DC power flow equations**.
- Values of θ_{-1} and P that satisfy the DC power flow equations (3.12) then approximately satisfy the power flow equality constraints (3.6) for all buses ℓ.
- Recall that we have assumed that Q is chosen to satisfy the power flow equality constraints (3.7) for all buses ℓ for the assumed voltage magnitudes $v^{(0)} = \mathbf{1}$.

3.6.4 Interpretation

- We have interpreted the DC power flow approximation as equivalent to performing one iteration of the Newton–Raphson method, starting at a base-case specified by a flat start, or equivalently the power flow equations linearized about a flat start.
- This differs from the "traditional" interpretation that emphasizes:
  - the small angle approximations for cos and sin,
  - voltage magnitudes assumed to equal one per unit, and
  - the solution of DC power flow being the same as the solution of an analogous DC circuit with current sources specified by the power injections and voltages specified by the angles.
- The traditional interpretation in terms of a DC circuit is useful for solving small systems by hand:
  - See in Section 3.10 and Exercise 3.5.
- The traditional interpretation also allows for a more straightforward derivation in the case of no shunt elements:
  - See in Exercise 3.6.

Interpretation, continued

- Our interpretation in terms of linearization provides a clearer and more general perspective on the conditions when the DC power flow provides a good approximation:
  - See in Exercise 3.1.
- It also provides a connection to **decomposition** algorithms:
  - iteration between solution and linearization of the power flow equations, and calculation of a desired generation operating point.

3.6.5 Terms in Jacobian

- The entries of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$ are:

$$\frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right) = \begin{cases} \sum_{j\in\mathbb{J}(\ell)} B_{\ell j}, & \text{if } k=\ell,\\ -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell),\\ 0, & \text{otherwise,}\end{cases} \tag{3.13}$$

$$= \begin{cases} \sum_{j\in\mathbb{J}(\ell)} \left(\begin{array}{c}\text{minus the susceptance}\\ \text{joining buses } \ell \text{ and } j\end{array}\right), & \text{if } k=\ell,\\ -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell),\\ 0, & \text{otherwise,}\end{cases}$$

$$= \begin{cases} -B_{\ell k}, & \text{if } k\in\mathbb{J}(\ell)\cup\{\ell\},\\ 0, & \text{otherwise,}\end{cases} \left\{ \begin{array}{r}\text{if the shunt susceptances}\\ \text{are all equal to zero.}\end{array}\right.$$

- Note that these entries correspond to the imaginary part of the admittance matrix, B, where $A = G + B\sqrt{-1}$, *not* to the inverse of the line inductive reactances, as is often stated in derivations of the DC power flow:
  - these are different if the resistance is non-zero.

Terms in Jacobian, continued

- In the next slides, we will consider entries of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$, which includes the row consisting of the derivatives of p_1 corresponding to the slack bus.
- If the shunt admittances are all equal to zero then:

  J is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted,

  J_{-1} is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted and the row corresponding to the slack bus deleted, and

  if the slack bus is bus σ, then $J_{-\sigma}$ is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted and the row corresponding to the slack bus σ deleted.
- If the shunt admittances are non-zero then the entries of J corresponding to diagonal entries $B_{\ell\ell}$ of B will differ from the entries of $-B$ by the shunt admittance connected to bus ℓ; J omits the shunt terms.

Terms in Jacobian, continued

- Summing the entries in the ℓ-th column of $\frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$, we obtain:

$$\begin{aligned}
\forall k, \sum_{\ell} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right) &= \frac{\partial p_k}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right) + \sum_{\ell \neq k} \frac{\partial p_\ell}{\partial \theta_k}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right), \\
&\quad \text{where the summation over } \ell \text{ includes} \\
&\quad \text{the slack bus,} \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{\ell k}, \text{general case from (3.13),} \\
&\quad \text{considering possibly non-zero shunt admittances,} \\
&= \sum_{j \in \mathbb{J}(k)} B_{kj} - \sum_{\ell \in \mathbb{J}(k)} B_{k\ell}, \text{ since } B_{\ell k} = B_{k\ell}, \\
&= 0.
\end{aligned}$$

- That is, each column of $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)$ sums to zero, so $\mathbf{1}^\dagger J = \mathbf{0}$.

3.6.6 *Slack injection and losses*

- Equivalently,

$$\mathbf{1}^{\dagger}\frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) = \mathbf{0}. \tag{3.14}$$

- From (3.11), we can estimate the net injection at the slack bus as:

$$\begin{aligned} P_1 &= \frac{\partial p_1}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-1}, \\ &= -\mathbf{1}^{\dagger}\frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)\theta_{-1}, \\ &\quad \text{since } \mathbf{1}^{\dagger}\frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right) = \mathbf{0} \text{ from (3.14)}, \\ &= -\mathbf{1}^{\dagger}P_{-1}, \end{aligned}$$

- given the DC power flow approximation.
- That is, $\mathbf{1}^{\dagger}P = \mathbf{0}$ and there are zero losses, given the DC power flow assumption of the flat start as base-case, and assuming that the real part of the shunt admittances are zero.

3.6.7 Solving for the angles

- The matrix $J_{-1} = \frac{\partial p_{-1}}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is non-singular, so we can write $\theta_{-1} = [J_{-1}]^{-1} P_{-1}$, allowing us to solve for the angles θ_{-1}.
- As mentioned in Section 2.2, in practice for large systems we would not invert the matrix J_{-1}, but instead factorize it and use forwards and backwards substitution.
- Given the lossless assumption and a specification of P_{-1}, we have already evaluated the net generation at the reference/slack bus: $P_1 = -\mathbf{1}^\dagger P_{-1}$,
- so that the slack bus exactly compensates for the net demand or withdrawal summed across all other buses, since the approximation is lossless.

Solving for the angles, continued

- Summarizing, the DC power flow equations $J\theta_{-1} = P$ are equivalent to:

$$\begin{aligned} P_1 &= -\mathbf{1}^{\dagger}P_{-1}, \\ \theta_{-1} &= [J_{-1}]^{-1}P_{-1}. \end{aligned}$$

- See Exercise 3.4 for a formal demonstration of the equivalence of the DC power flow equations to this representation, which we will also refer to as the DC power flow equations.
- As mentioned in Section 2.2, in practice for large systems we would not invert the matrix J_{-1}, but instead factorize it and use forwards and backwards substitution.

3.6.8 Demand

- So far, the vector P has represented the vector of *net* injections at the buses.
- In some formulations, we want to consider demand and generation separately.
- For example, if the net injection is $P - D$, where:

  P is now the vector of generations, and
  D is the vector of demands,

- then the DC power flow equations are equivalent to:

$$
\begin{aligned}
-\mathbf{1}^{\dagger} P &= -\mathbf{1}^{\dagger} D, \\
\theta_{-1} &= [J_{-1}]^{-1}(P_{-1} - D_{-1}),
\end{aligned}
$$

- where P_{-1} and D_{-1} are the sub-vectors of P and D, respectively, that omit the reference/slack bus.

3.6.9 Slack bus and reference bus choices

- If the slack bus is bus σ then the DC power flow equations are equivalent to:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta_{-1} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where $P_{-\sigma}$ and $D_{-\sigma}$ are the sub-vectors of P and D, respectively, that omit the slack bus, and
- where the reference bus is still assumed to be bus 1.
- If the slack bus is bus σ and the reference bus is bus ρ then the DC power flow equations are equivalent to:

$$-\mathbf{1}^{\dagger}P = -\mathbf{1}^{\dagger}D, \quad (3.15)$$

$$\theta_{-\rho} = \left[J'_{-\sigma}\right]^{-1}(P_{-\sigma} - D_{-\sigma}), \quad (3.16)$$

- where $J' = \dfrac{\partial p}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$ is the matrix of partial derivatives with the reference bus assumed to be bus ρ and where the matrix J' differs from J in one column.

3.7 Line flow

- We typically use the results of power flow to evaluate whether the flow along a line is within limits:
  - this is most straightforward for flow limits expressed in terms of real power flow,
  - we can also consider flow limits expressed in terms of current magnitude or the magnitude of complex power.
- There is typically a flow limit in each direction on the line.
- For a line joining bus ℓ to bus k we can consider:
  - real and reactive flows $p_{\ell k}$ and $q_{\ell k}$ along the line from bus ℓ in the direction of bus k, and
  - real and reactive flows $p_{k\ell}$ and $q_{k\ell}$ along the line from bus k in the direction of bus ℓ.
- Without loss of generality, we explicitly consider only $p_{\ell k}$ and $q_{\ell k}$.

Line flow, continued

- Ignoring shunt elements in the models, we have that the real and reactive flows are given by:

$$\begin{aligned}
\forall x \in \mathbb{R}^n, p_{\ell k}(x) &= v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - (v_\ell)^2 G_{\ell k}, \\
\forall x \in \mathbb{R}^n, q_{\ell k}(x) &= v_\ell v_k [G_{\ell k} \sin(\theta_\ell - \theta_k) - B_{\ell k} \cos(\theta_\ell - \theta_k)] + (v_\ell)^2 B_{\ell k}.
\end{aligned}$$

- (The linearization analysis including shunt elements has the same result that we will present, but is notationally inconvenient since we need to define parameters for the shunt elements in each line.)
- We will approximate these expressions by again linearizing about a base-case:
  - for convenience, we will again assume $\rho = \sigma = 1$ in the derivation and then sketch the extensions to the general case.
- We linearize the expressions for $p_{\ell k}$ and $q_{\ell k}$ about $\theta_{-1}^{(0)}$.
- We continue to assume that the voltage magnitudes are fixed at $v^{(0)}$.

3.7.1 Linearized line flow

$$
\begin{aligned}
p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ v_{-1}^{(0)} \end{bmatrix}\right) & \\
\approx \; & p_{\ell k}(x^{(0)}) + \frac{\partial p_{\ell k}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1}, \\
= \; & p_{\ell k}(x^{(0)}) + v_{\ell}^{(0)} v_{k}^{(0)} \begin{bmatrix} -G_{\ell k}\sin(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \\ +B_{\ell k}\cos(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \end{bmatrix} (\Delta\theta_{\ell} - \Delta\theta_{k}), \\
q_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ v_{-1}^{(0)} \end{bmatrix}\right) & \\
\approx \; & q_{\ell k}(x^{(0)}) + \frac{\partial q_{\ell k}}{\partial \theta_{-1}}(x^{(0)})\Delta\theta_{-1}, \\
= \; & q_{\ell k}(x^{(0)}) + v_{\ell}^{(0)} v_{k}^{(0)} \begin{bmatrix} G_{\ell k}\cos(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \\ +B_{\ell k}\sin(\theta_{\ell}^{(0)} - \theta_{k}^{(0)}) \end{bmatrix} (\Delta\theta_{\ell} - \Delta\theta_{k}).
\end{aligned}
$$

Linearized Line flow, continued

- We focus on the real power $p_{\ell k}$ flowing along the line from bus ℓ in the direction of bus k.
- Define the row vector $K_{(\ell k)}$ of partial derivatives by:

$$\forall j \neq 1, K_{(\ell k)j} = \begin{cases} v_\ell^{(0)} v_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], & \text{if } j = \ell, \\ -v_\ell^{(0)} v_k^{(0)} [-G_{\ell k} \sin(\theta_\ell^{(0)} - \theta_k^{(0)}) + B_{\ell k} \cos(\theta_\ell^{(0)} - \theta_k^{(0)})], & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

- That is, $K_{(\ell k)j}$ is the j-th entry in the row vector $K_{(\ell k)}$, which has entries for every bus except the reference bus.
- Then the linear approximation to $p_{\ell k}$ is given by:

$$p_{\ell k}\left(\begin{bmatrix} \theta_{-1}^{(0)} + \Delta\theta_{-1} \\ v_{-1}^{(0)} \end{bmatrix}\right) \approx p_{\ell k}(x^{(0)}) + K_{(\ell k)} \Delta\theta_{-1}.$$

3.7.2 Line flow constraints

- Suppose that we have line flow constraints of the form $p_{\ell k}(x) \leq \overline{p}_{\ell k}$.
- Using the linear approximation, we obtain:

$$p_{\ell k}(x^{(0)}) + K_{(\ell k)} \Delta\theta_{-1} \leq \overline{p}_{\ell k}.$$

- By defining a matrix K with rows $K_{(\ell k)}$ and a vector d with entries $d_{(\ell k)}$ of the form:

$$d_{(\ell k)} = \overline{p}_{\ell k} - p_{\ell k}(x^{(0)}),$$

- we can approximate the collection of line flow constraints in the form:

$$K \Delta\theta_{-1} \leq d.$$

- This form of the line flow constraints includes the angles explicitly.

3.7.3 DC power flow approximation to line flow constraints

- Using a flat start $x^{(0)} = \begin{bmatrix} \theta_{-1}^{(0)} \\ v_{-1}^{(0)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ as the base-case for the linearization, we find:

$$
\begin{aligned}
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ v_{-1}^{(0)} \end{bmatrix}\right) &= v_\ell^{(0)}(v_k^{(0)} - v_\ell^{(0)})G_{\ell k}, \\
p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= 0, \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ v_{-1}^{(0)} \end{bmatrix}\right) &= \begin{cases} v_\ell v_k B_{\ell k}, & \text{if } j = \ell, \\ -v_\ell v_k B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \\
K_{(\ell k)j} = \frac{\partial p_{\ell k}}{\partial \theta_j}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) &= \begin{cases} B_{\ell k}, & \text{if } j = \ell, \\ -B_{\ell k}, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \\
d_{(\ell k)} &= \overline{p}_{\ell k} - p_{\ell k}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right), \\
&= \overline{p}_{\ell k}.
\end{aligned}
$$

DC power flow approximation to line flow constraints, continued

- Summarizing, we can approximate the flows at the angle $\theta_{-1} = \theta_{-1}^{(0)} + \Delta\theta_{-1} = \Delta\theta_{-1}$ using the linearized equations $K\theta_{-1} \leq d$, where:

$$\forall(\ell k), \forall j \neq 1, K_{(\ell k)j} = \begin{cases} B_{\ell k}, \text{ if } j = \ell, \\ -B_{\ell k}, \text{ if } j = k, \\ 0, \text{ otherwise,} \end{cases}$$
$$\forall(\ell k), d_{(\ell k)} = \overline{p}_{\ell k}.$$

- Again, this form of the constraints includes the angles explicitly and approximates the power flow on a line joining buses ℓ and k as being proportional to the angle difference $(\theta_\ell - \theta_k)$ across the line.
- The (ℓk) row of K has exactly two non-zero entries of equal magnitude and opposite sign, unless ℓ or k is the reference bus, in which case the row has one non-zero entry.

DC power flow approximation to line flow constraints, continued

- If the reference bus changes to bus ρ then the linearized line flow constraints would be $K'\theta_{-\rho} \leq d$, where:

$$\forall(\ell k), \forall j \neq \rho, K'_{(\ell k)j} = \begin{cases} B_{\ell k}, \text{ if } j = \ell, \\ -B_{\ell k}, \text{ if } j = k, \\ 0, \text{ otherwise}, \end{cases}$$

$$\forall(\ell k), d_{(\ell k)} = \overline{p}_{\ell k}.$$

- The matrix K' differs from the matrix K in one column.

3.7.4 Eliminating the angles

- In some cases, it can be convenient to eliminate the angle variables by expressing them in terms of the net power injections.
- We previously found that the DC power flow equations could be expressed as:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \theta_{-1} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where σ is the slack bus and $\rho = 1$ is the reference bus.
- We use the second equation to substitute into $K\theta_{-1} \leq d$ to obtain the equality and inequality constraints with the angles eliminated:

$$-\mathbf{1}^{\dagger}P = -\mathbf{1}^{\dagger}D, \tag{3.17}$$

$$K[J_{-\sigma}]^{-1}P_{-\sigma} \leq K[J_{-\sigma}]^{-1}D_{-\sigma} + d. \tag{3.18}$$

- This approximation to the flows is not always good:
  – it is used to represent transmission constraints in most day-ahead electricity markets,
  – will explore accuracy in homework exercise.

3.7.5 Shift factor matrix

- The matrix $K[J_{-\sigma}]^{-1}$ is the matrix of DC **shift factors** or **power transfer distribution factors**.
- That is, entries in the matrix represent the fraction of flow along each line for:

  injection at the buses represented in the vector $P_{-\sigma}$, and
  withdrawal at the slack bus σ.

- We occasionally want to express line flows in terms of the vector P of all net injections.
- First consider the case that $\sigma = 1$.
- For $\sigma = 1$, define the augmented shift factor matrix $\hat{C} = \left[\mathbf{0} \quad K[J_{-1}]^{-1}\right]$.
- That is, $\hat{C}$ consists of the columns of $K[J_{-1}]^{-1}$ augmented by an additional zero column corresponding to P_1.
- Each entry of $\hat{C}_k$ represents the fraction of the generation from generator at bus k that flows on the corresponding line.
- The flows are given by $\hat{C}(P - D)$.

Shift factor matrix, continued

- The equality and inequality constraints with the angles eliminated are then:

$$-\mathbf{1}^{\dagger}P = -\mathbf{1}^{\dagger}D, \tag{3.19}$$

$$\hat{C}P \leq \hat{C}D + d. \tag{3.20}$$

3.7.6 Shift factor matrix with other slack bus and reference bus choices

- Similarly, if the reference bus is some other bus ρ and the slack bus is some other bus σ, we can again define a corresponding augmented shift factor matrix $\hat{C}'$ such that the flows are given by $\hat{C}'(P-D)$, and the equality and inequality constraints are then still of the form:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \hat{C}'P &\leq \hat{C}'D+d. \end{aligned}$$

- For any particular example, we will typically maintain a given choice of reference bus ρ and slack bus σ.
- Slightly abusing notation, we will henceforth typically refer to the Jacobian as J and the shift matrix as C or $\hat{C}$, irrespective of the choice of reference and slack bus.
- We will make clear the choice of reference and slack bus in each example and occasionally use notation such as J and J' when when we are considering different choices of reference and/or slack bus.

3.7.7 Line flow constraints at other operating points

- The derivation so far used the flat start condition as the base-case for evaluating the shift factors and the line flow constraints.
- Other base-cases could be used, such as:
  - another assumed operating point, or
  - a measured or estimated operating point from a **state estimator**.
- The lossless assumption will typically not hold at other base-cases nor for the estimated change-case.

3.8 Example

- Consider the following one-line two-bus system with MW capacity and per unit impedance (on a 1 MVA base) as shown.
- Let bus $\rho = 1$ be the angle reference bus, so the unknown angle is θ_2.
- Let bus $\sigma = 2$ be the slack bus.
- There are generators and demand at both buses 1 and 2.

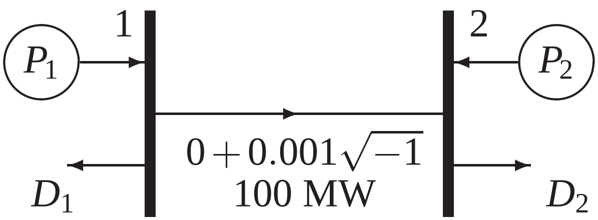

Fig. 3.8. One-line two-bus network.

3.8.1 Admittance matrix

- The line admittance is:

$$\begin{aligned} \mathcal{Y}_{12} &= \frac{1}{0+0.001\sqrt{-1}}, \\ &= -1000\sqrt{-1}. \end{aligned}$$

- The bus admittance matrix is:

$$\begin{aligned} \begin{bmatrix} \mathcal{Y}_{12} & -\mathcal{Y}_{12} \\ -\mathcal{Y}_{12} & \mathcal{Y}_{12} \end{bmatrix} &= \begin{bmatrix} -1000\sqrt{-1} & 1000\sqrt{-1} \\ 1000\sqrt{-1} & -1000\sqrt{-1} \end{bmatrix}, \\ &= \begin{bmatrix} B_{11}\sqrt{-1} & B_{12}\sqrt{-1} \\ B_{21}\sqrt{-1} & B_{22}\sqrt{-1} \end{bmatrix}. \end{aligned}$$

3.8.2 Jacobian

- Evaluating the sub-matrix of the Jacobian corresponding to real power and angles at the condition of flat start:

$$
\begin{aligned}
J &= \frac{\partial p}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right), \\
&= \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right), \\
&= \frac{\partial p}{\partial \theta_{2}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right), \\
&= \begin{bmatrix} -B_{12} \\ B_{12} \end{bmatrix}, \\
&= \begin{bmatrix} -1000 \\ 1000 \end{bmatrix}.
\end{aligned}
$$

3.8.3 DC power flow

- The DC power flow constraints are:

$$
\begin{aligned}
J\theta_{-\rho} &= P - D, \\
&= \begin{bmatrix} P_1 - D_1 \\ P_2 - D_2 \end{bmatrix}.
\end{aligned}
$$

- Substituting, we obtain:

$$
\begin{bmatrix} -1000 \\ 1000 \end{bmatrix} [\theta_2] = \begin{bmatrix} P_1 - D_1 \\ P_2 - D_2 \end{bmatrix}.
$$

3.8.4 Eliminating angles

- We eliminate θ_2 to obtain the following form:

$$\begin{aligned} -P_1 - P_2 &= -D_1 - D_2, \\ [\theta_2] &= [J_{-\sigma}]^{-1}[P_1 - D_1], \end{aligned}$$

- where, to form $J_{-\sigma}$, we have deleted the second row of J corresponding to the slack bus $\sigma = 2$:

$$\begin{aligned} J_{-\sigma} &= [-1000], \\ [J_{-\sigma}]^{-1} &= [-0.001]. \end{aligned}$$

- Note that the *angle* reference bus is bus $\rho = 1$, whereas the slack bus is bus $\sigma = 2$!

  Example shows that the angle and slack buses can be different buses!

Eliminating angles, continued

- The DC power flow equations are then:

$$\begin{aligned} -P_1 - P_2 &= -D_1 - D_2, \\ \theta_2 &= [-0.001][P_1 - D_1]. \end{aligned}$$

- For positive values of P_1, we have that $\theta_2 < 0 = \theta_1$. Power flows from "higher" to "lower" angles.

3.8.5 Line flow constraints

- Assume that the real power line flow limit of 100 MW applies only in the direction of the arrow in Figure 3.8.
- Ignore the constraint on flow in the direction opposite to the arrow.
- The line flow constraint is then specified by $K\theta_{-\rho} \leq d$, where:

$$\begin{aligned} d &= [\overline{p}_{(12)}], \\ &= [100], \\ K &= [-B_{12}], \\ &= [-1000]. \end{aligned}$$

- Therefore:

$$\begin{aligned} (K[\theta_2] \leq d) &\Leftrightarrow ([-1000][\theta_2] \leq [100]), \\ &\Leftrightarrow (\theta_2 \geq -0.1). \end{aligned}$$

- For $|\theta_2| \leq 0.1$ we have that $\sin(\theta_1 - \theta_2) = \sin(-\theta_2) \approx -\theta_2$, so that the DC power flow approximation is reasonable.

3.8.6 Shift factors

- The matrix of shift factors is:

$$\begin{aligned} K[J_{-\sigma}]^{-1} &= [-1000][-0.001], \\ &= [1]. \end{aligned}$$

- That is, if P_1 is the net injection at bus 1 and an equal power is withdrawn at bus 2 then $[1][P_1] = P_1$ will flow on the line between bus 1 and bus 2.
- If P_2 is the net injection at bus 2 and an equal power is withdrawn at bus 2 then no power will flow on the line between bus 1 and bus 2.
- That is, the augmented shift factor matrix is:

$$\begin{aligned} \hat{C} &= \left[K[J_{-\sigma}]^{-1} \quad 0 \right], \\ &= \left[1 \quad 0 \right]. \end{aligned}$$

3.8.7 Line flow constraints with angles eliminated

- The system equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ \hat{C}P &\leq \hat{C}D + d. \end{aligned}$$

- Since $\hat{C} = [1 \quad 0]$, and $d = [100]$, these constraints become:

$$\begin{aligned} -P_1 - P_2 &= -D_1 - D_2, \\ P_1 &\leq D_1 + 100. \end{aligned}$$

- We could see this from Figure 3.8 directly:
  - Generation at buses 1 and 2 must meet demand at buses 1 and 2.
  - For withdrawal at the price reference bus $\sigma = 2$, all net injection $(P_1 - D_1)$ at bus 1 flows on the line; therefore generation at bus 1 must be within the capacity of the line plus the demand at bus 1.
  - For withdrawal at the price reference bus $\sigma = 2$, no net injection at bus 2 flows on the line.
- What would the constraints be if $\sigma = 1$ were the price reference bus?

3.9 Larger example

- Consider the following four-line four-bus system with MW capacities and per unit impedances (on a 1 MVA base) as shown.
- Bus $\sigma = 0$ is the slack bus and there are no shunt admittances.
- Bus $\rho = 1$ is reference bus, so the unknown angles are $\theta_{-1} = \begin{bmatrix} \theta_0 \\ \theta_2 \\ \theta_3 \end{bmatrix}$.

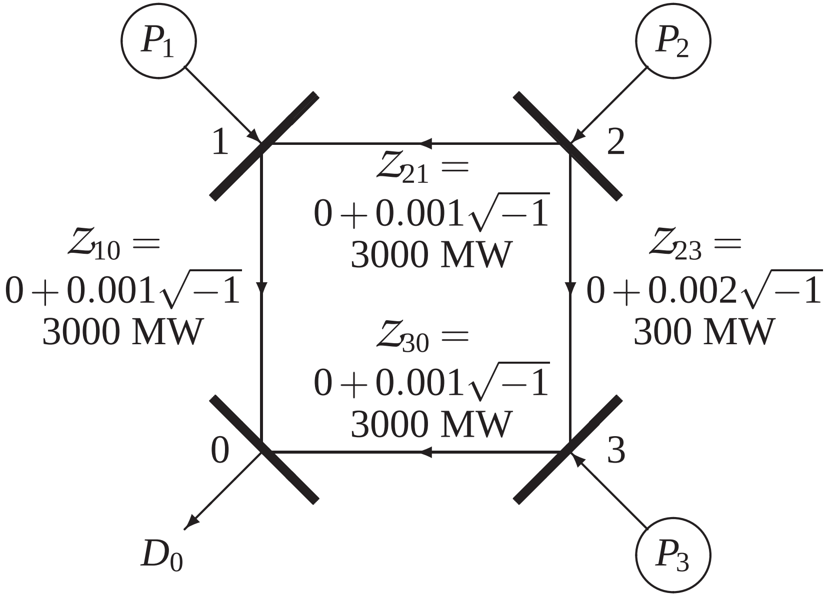

Fig. 3.9. Four-line four-bus network with generators at buses 1, 2, and 3, and demand at bus 0.

3.9.1 Admittance matrix

- The line admittances are:

$$\mathcal{Y}_{01} = \mathcal{Y}_{10} = \mathcal{Y}_{12} = \mathcal{Y}_{21} = \mathcal{Y}_{03} = \mathcal{Y}_{30} = \frac{1}{0+0.001\sqrt{-1}} = -1000\sqrt{-1},$$

$$\mathcal{Y}_{23} = \mathcal{Y}_{32} = \frac{1}{0+0.002\sqrt{-1}} = -500\sqrt{-1}.$$

Admittance matrix, continued

- The bus admittance matrix is:

$$\begin{bmatrix} \mathcal{Y}_{01}+\mathcal{Y}_{03} & -\mathcal{Y}_{01} & 0 & -\mathcal{Y}_{03} \\ -\mathcal{Y}_{10} & \mathcal{Y}_{10}+\mathcal{Y}_{12} & -\mathcal{Y}_{12} & 0 \\ 0 & -\mathcal{Y}_{21} & \mathcal{Y}_{21}+\mathcal{Y}_{23} & -\mathcal{Y}_{23} \\ -\mathcal{Y}_{30} & 0 & -\mathcal{Y}_{32} & \mathcal{Y}_{30}+\mathcal{Y}_{32} \end{bmatrix}$$

$$= \begin{bmatrix} -2000\sqrt{-1} & 1000\sqrt{-1} & 0 & 1000\sqrt{-1} \\ 1000\sqrt{-1} & -2000\sqrt{-1} & 1000\sqrt{-1} & 0 \\ 0 & 1000\sqrt{-1} & -1500\sqrt{-1} & 500\sqrt{-1} \\ 1000\sqrt{-1} & 0 & 500\sqrt{-1} & -1500\sqrt{-1} \end{bmatrix},$$

$$= \begin{bmatrix} B_{00}\sqrt{-1} & B_{01}\sqrt{-1} & 0 & B_{03}\sqrt{-1} \\ B_{10}\sqrt{-1} & B_{11}\sqrt{-1} & B_{12}\sqrt{-1} & 0 \\ 0 & B_{21}\sqrt{-1} & B_{22}\sqrt{-1} & B_{23}\sqrt{-1} \\ B_{30}\sqrt{-1} & 0 & B_{32}\sqrt{-1} & B_{33}\sqrt{-1} \end{bmatrix}.$$

3.9.2 Jacobian

- Since there are no shunts, $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$ is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus deleted:

$$J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right) = \begin{bmatrix} -B_{00} & 0 & -B_{03} \\ -B_{10} & -B_{12} & 0 \\ 0 & -B_{22} & -B_{23} \\ -B_{30} & -B_{32} & -B_{33} \end{bmatrix},$$

$$= \begin{bmatrix} 2000 & 0 & -1000 \\ -1000 & -1000 & 0 \\ 0 & 1500 & -500 \\ -1000 & -500 & 1500 \end{bmatrix}.$$

- Note that the rows of J are indexed by 0, 1, 2, 3, while the columns are indexed by 0, 2, 3.

3.9.3 DC power flow

- We can solve for θ_{-1} to obtain the following form for the DC power flow equations:

$$
\begin{aligned}
-\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\
\theta_{-1} &= [J_{-0}]^{-1}(P_{-0} - D_{-0}).
\end{aligned}
$$

- where J_{-0} is J with the row corresponding to $\sigma = 0$ deleted, so that:

$$
\begin{aligned}
J_{-0} &= \begin{bmatrix} -1000 & -1000 & -0 \\ 0 & 1500 & -500 \\ -1000 & -500 & 1500 \end{bmatrix}, \\
[J_{-0}]^{-1} &= \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix}.
\end{aligned}
$$

- Note that the subscript -1 on θ_{-1} is referring to $\rho = 1$, the reference bus, with the entry θ_1 omitted, (and columns of J also omit terms for θ_1),
- whereas the subscript -0 on J_{-0}, P_{-0}, and D_{-0} is referring to $\sigma = 0$, the slack bus, with row $\frac{\partial p_0}{\partial \theta_{-1}}$, and terms P_0 and D_0, respectively, omitted.

DC power flow, continued

- So far, the development considered generation and demand at all buses.
- The example only has demand at bus 0 and has generation at buses 1, 2, and 3.
- Since there is only demand at bus 0 then the DC power flow equations are:

$$
\begin{aligned}
-P_1 - P_2 - P_3 &= -D_0, \\
\theta_{-1} &= \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}.
\end{aligned}
$$

3.9.4 DC power flow approximation to line flow constraints

- In principle, there are limits on flow in both directions on each line.
- We will assume that the only binding limits are in the directions from buses 1 to 0, 2 to 1, 2 to 3, and 3 to 0, respectively, as suggested by the arrows in Figure 3.9.
- These four line flow inequality constraints are then specified by $K\theta_{-1} \leq d$, where:

$$d = \begin{bmatrix} \overline{p}_{10} \\ \overline{p}_{21} \\ \overline{p}_{23} \\ \overline{p}_{30} \end{bmatrix} = \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix},$$

$$K = \begin{bmatrix} -B_{10} & 0 & 0 \\ 0 & B_{21} & 0 \\ 0 & B_{23} & -B_{23} \\ -B_{30} & 0 & B_{30} \end{bmatrix} = \begin{bmatrix} -1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 500 & -500 \\ -1000 & 0 & 1000 \end{bmatrix}.$$

- Note that the rows of K are indexed by $(10), (21), (23), (30)$, while the columns are indexed by 0, 2, 3.

3.9.5 DC shift factors

- The matrix of DC shift factors is:

$$K[J_{-0}]^{-1} = \begin{bmatrix} -1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 500 & -500 \\ -1000 & 0 & 1000 \end{bmatrix} \begin{bmatrix} -0.0008 & -0.0006 & -0.0002 \\ -0.0002 & 0.0006 & 0.0002 \\ -0.0006 & -0.0002 & 0.0006 \end{bmatrix},$$

$$= \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix}.$$

- The augmented shift factor matrix is:

$$\hat{C} = \begin{bmatrix} \mathbf{0} & K[J_{-0}]^{-1} \end{bmatrix},$$

$$= \begin{bmatrix} 0.0 & 0.8 & 0.6 & 0.2 \\ 0.0 & -0.2 & 0.6 & 0.2 \\ 0.0 & 0.2 & 0.4 & -0.2 \\ 0.0 & 0.2 & 0.4 & 0.8 \end{bmatrix}.$$

DC shift factors, continued

- For example, for power injected at bus 1 and withdrawn at bus $\sigma = 0$, the shift factors to the lines 1 to 0, 2 to 1, 2 to 3, and 3 to 0 are, respectively $0.8, -0.2, 0.2, 0.2$.
- Moreover, the flow on any particular line is the sum of the flows due to individual injections at particular buses.
- For power injected at bus 0 and withdrawn at bus $\sigma = 0$, what are the shift factors to the lines from buses: 1 to 0; 2 to 1; 2 to 3; and 3 to 0?
- If 1 MW is injected at bus 1, 10 MW is injected at bus 2, and 100 MW is injected at bus 3, with 111 MW withdrawn at bus $\sigma = 0$, what is the flow on the line from bus 1 to bus 0?

3.9.6 Line flow constraints in terms of shift factors

- The flows on the lines are given by $\hat{C}(P - D)$ or, equivalently, $K[J_{-1}]^{-1}(P_{-0} - D_{-0})$.
- The equality and inequality constraints with angles eliminated are:

$$\begin{aligned} -\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\ K[J_{-0}]^{-1}P_{-0} &\leq K[J_{-0}]^{-1}D_{-0} + d. \end{aligned}$$

- Again note that $D_{-0} = \mathbf{0}$ for this particular example.
- Also, $d = \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}$, so these constraints become:

$$\begin{aligned} -P_1 - P_2 - P_3 &= -D_0, \\ \begin{bmatrix} 0.8 & 0.6 & 0.2 \\ -0.2 & 0.6 & 0.2 \\ 0.2 & 0.4 & -0.2 \\ 0.2 & 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} &\leq \begin{bmatrix} 3000 \\ 3000 \\ 300 \\ 3000 \end{bmatrix}. \end{aligned}$$

Line flow constraints in terms of shift factors, continued

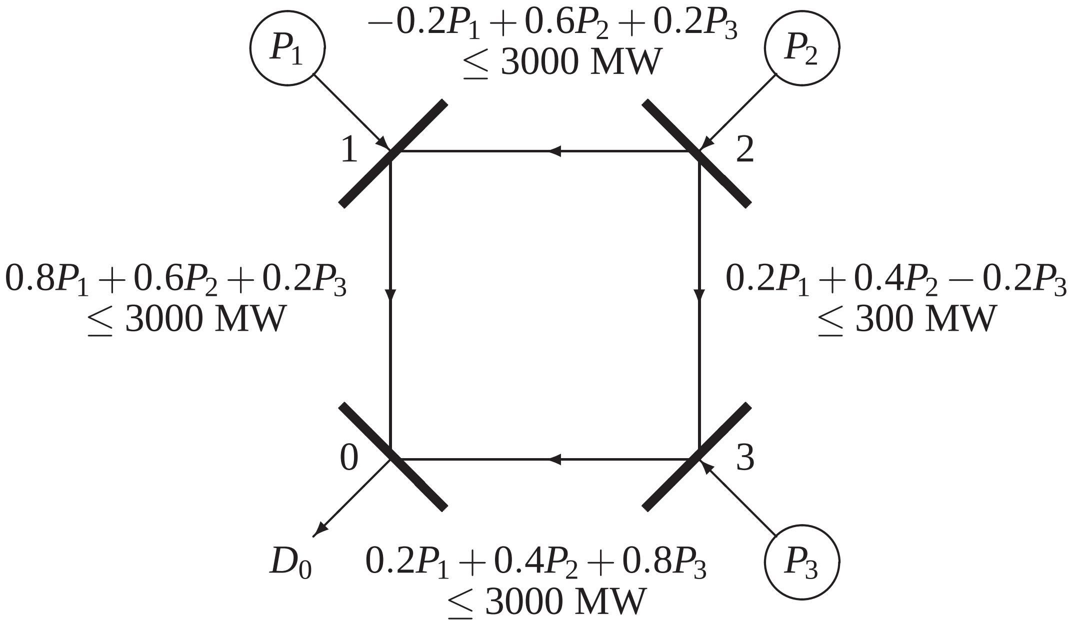

Fig. 3.10. DC power flow approximation to line flow constraints for four-line four-bus network.

3.10 DC power flow circuit interpretation

- As mentioned in Section 3.6.4, we can interpret the DC power flow approximation in terms of an analogous linear DC circuit:
  - the DC circuit interpretation is useful to solve small systems by hand.
- Recall that the power flow equations are in the form $J\theta_{-\rho} = P$, where ρ is the reference bus.
- If the shunt admittances are zero then J is given by $-B$ with the column corresponding to the reference bus deleted.
- Consider the following analogy with a DC circuit:
  - Bus ρ is the datum node in the circuit with DC voltage defined to be 0,
  - Real power injections P are analogous to DC current injections i at all buses,
  - Angles $\theta_{-\rho}$ are analogous to DC voltages $v_{-\rho}$ at all nodes except the datum node,
  - Entries in J are analogous to the admittance matrix of a circuit having resistors joining nodes ℓ and k with "conductance" $g_{\ell k} = |B_{\ell k}|$.
- The analogous linear DC circuit satisfies $Jv_{-\rho} = i$:
  - applies whether or not there are non-zero shunt admittances.

DC power flow circuit interpretation, continued

- *Current* injections and flows in the DC circuit correspond to *power* injections and flows in the power system.
- Recall that DC current is shared along parallel paths in proportion to the conductance of each path:
  - the current flowing in each parallel path due to a single injection and withdrawal is proportional to the conductance along each path.
- Recall that currents in a linear DC circuit can be superposed:
  - the current flowing in a branch due to multiple current injections and withdrawals is equal to the sum of the currents flowing in that branch due to each current injection and withdrawal considered separately.
- Therefore, we can:
  - evaluate sharing of *power* flow in the DC power flow approximation in the same way as we evaluate sharing of *current* in a DC circuit, and
  - superpose *power* flow in the DC power flow approximation in the same way as we superpose *current* flow in a DC circuit.

DC power flow circuit interpretation, continued

- For example, suppose that the DC circuit has two nodes, 1 and 2, joined by two conductances as shown in Figure 3.11, which we view as two "paths" between these nodes (possibly consisting of several branches in series):
  - From circuit theory, recall that if current is injected at one node and withdrawn at another node then current is shared on the branches in these paths in proportion to the path conductances.
  - If the two conductances are equal then each will have one-half of the total flow of current.
- If the two lines have equal admittance then power injected at bus 1 and withdrawn at bus 2 will be shared equally between the two lines.

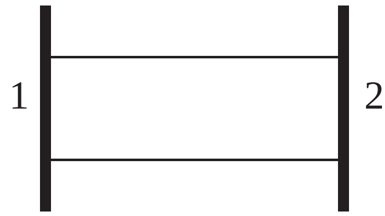

Fig. 3.11. Two-line two-bus network.

DC power flow circuit interpretation, continued

- Moreover, if there is current injected at multiple nodes and withdrawn at another node or nodes then the resulting total current in any branch is equal to the superposition of the currents in that branch due to the individual current injections and withdrawals.

DC power flow circuit interpretation, continued

- Recall the example from Figure 3.9, repeated in Figure 3.12.
- We will illustrate the DC circuit using this example.

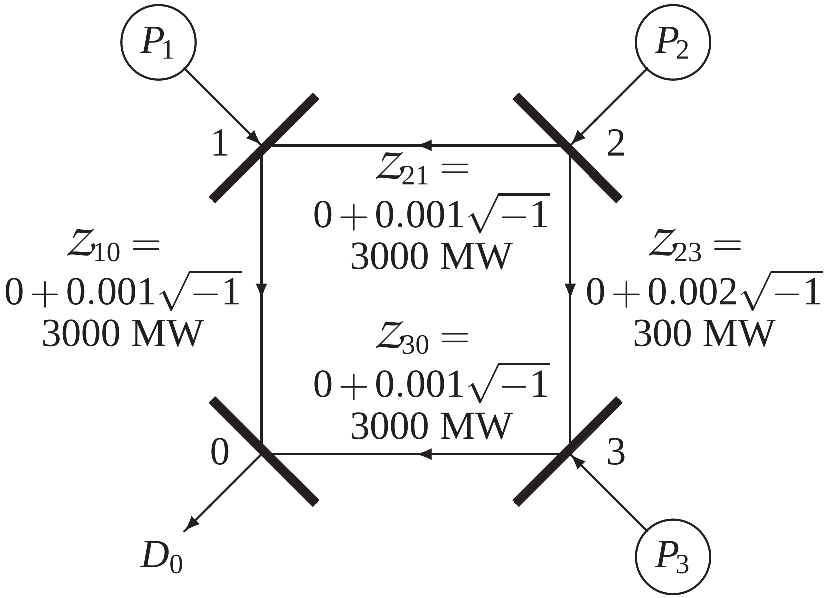

Fig. 3.12. Four-line four-bus network repeated from Figure 3.9.

DC power flow circuit interpretation, continued

- Suppose current is injected at node 1 in the analogous DC circuit and withdrawn at the node $\sigma = 0$.
- Note that the actual impedance directly joining buses 1 and 0 in the power system is $0 + 0.001\sqrt{-1}$:
  - analogous "conductance" of this path is $g_{10} = |B_{10}| = 1000$,
  - current on this analogous conductance is proportional to 1000.
- We can also think of the lines from buses 1 to 2, 2 to 3, and 3 to 0 as another impedance joining buses 1 and 0 in the power system:
  - total impedance in this path is:

$$0 + 0.001\sqrt{-1} + 0 + 0.002\sqrt{-1} + 0 + 0.001\sqrt{-1} = 0.004\sqrt{-1},$$

  - analogous "conductance" of path is $g_{1230} = 250$,
  - current on this analogous conductance due to the lines 1 to 2, 2 to 3, and 3 to 0 is proportional to 250.
- Current injected at node 1 and withdrawn at node 0 is shared between the analogous conductances in the paths in the proportion $1000 : 250 = 0.8 : 0.2$.

DC power flow circuit interpretation, continued

- In the power system, the DC power flow approximation means that power injected at bus 1 and withdrawn at the slack bus $\sigma = 0$ will be shared in the ratio 0.8:0.2 between:
  - the path consisting of the line directly joining buses 1 and 0, and
  - the lines forming the path from buses 1 to 2, 2 to 3, 3 to 0.
- That is, the shift factors, for injection at bus 1 and withdrawal at bus 0, to the lines from buses 1 to 0, 2 to 1, 2 to 3, and 3 to 0 are, respectively $0.8, -0.2, 0.2, 0.2$, exactly as calculated in the previous section.
- By superposition, we can approximate the total power flow on a line as the sum of the power flows due to individual power injections.

3.11 Losses

3.11.1 Losses under DC power flow approximation

- In Section 3.6.6, we found that losses under the DC power flow approximation are zero.
- This can be understood in terms of the linearization interpretation of DC power flow:
  - DC power flow involves a first-order Taylor approximation of injections as a function of angles, evaluated at a flat start, corresponding to zero net real power injections,
  - the condition of zero net real power injection corresponds to zero losses, while
  - linearizing about the flat start results in the derivative of losses with respect to injections being zero,
  - so a first-order Taylor expansion of losses has zero constant term and zero linear term!
- In the following sections we find a more accurate approximation by considering a second-order Taylor expansion of the exact expression for losses.

3.11.2 Exact loss expression

- Recall from Section 3.2.8 that the function $p_\ell : \mathbb{R}^n \to \mathbb{R}$ that evaluates the real power flow out of bus ℓ into the lines connected to it is given by (3.8), which we repeat here:

$$\forall x \in \mathbb{R}^n, p_\ell(x) = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)].$$

- Focusing on just the real power flow $p_{\ell k} : \mathbb{R}^n \to \mathbb{R}$ from bus ℓ into the line joining bus ℓ to a bus $k \in \mathbb{J}(\ell)$, ignoring shunts, but including the terms in the power flow for $k = \ell$ that involve $G_{\ell k}$, we obtain:

$$\forall x \in \mathbb{R}^n, p_{\ell k}(x) = v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) + B_{\ell k} \sin(\theta_\ell - \theta_k)] - (v_\ell)^2 G_{\ell k}.$$

- Similarly, the real power flow $p_{k\ell} : \mathbb{R}^n \to \mathbb{R}$ from bus k into this same line from its other end is:

$$\forall x \in \mathbb{R}^n, p_{k\ell}(x) = v_\ell v_k [G_{\ell k} \cos(\theta_\ell - \theta_k) - B_{\ell k} \sin(\theta_\ell - \theta_k)] - (v_k)^2 G_{\ell k}.$$

Exact loss expression, continued

- The losses $L_{\ell k} : \mathbb{R}^n \to \mathbb{R}$ on the line is the sum of the power injected from bus k and bus ℓ:

$$\begin{aligned} \forall x \in \mathbb{R}^n, L_{\ell k}(x) &= p_{\ell k}(x) + p_{k\ell}(x), \\ &= 2v_\ell v_k G_{\ell k} \cos(\theta_\ell - \theta_k) - [(v_\ell)^2 + (v_k)^2] G_{\ell k}. \end{aligned} \tag{3.21}$$

- The total losses in the system, $L : \mathbb{R}^n \to \mathbb{R}$ is the sum of the losses over the lines:

$$\forall x \in \mathbb{R}^n, L(x) = \sum_{\ell, k \in \mathbb{J}(\ell), k > \ell} L_{\ell k}(x).$$

- The total losses can also be evaluated as $L = \mathbf{1}^\dagger p$, where $p : \mathbb{R}^n \to \mathbb{R}^{n_{PQ}+1}$ is the vector of flows from the buses into the rest of the system.

3.11.3 Quadratic loss expression

- We approximate $L_{\ell k}$ and L through a second-order Taylor approximation about a flat start, again assuming a fixed voltage schedule, so that we only explicitly consider the dependence on angle:

$$\begin{aligned} L_{\ell k}(x) &\approx L_{\ell k}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right) + +\frac{\partial L_{\ell k}}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)\theta_{-\rho} + \frac{1}{2}[\theta_{-\rho}]^{\dagger}\frac{\partial^2 L_{\ell k}}{\partial {\theta_{-\rho}}^2}\left(\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}\right)\theta_{-\rho}, \\ &= 0 + \mathbf{0}^{\dagger}\theta_{-\rho} + (-G_{\ell k})(\theta_{\ell} - \theta_{k})^2, \end{aligned} \tag{3.22}$$

- on evaluating the terms.
- As mentioned above, there is a zero constant term and a zero linear term in the Taylor expansion of losses.
- The quadratic term is non-zero and has coefficient $(-G_{\ell k})$.

Quadratic loss approximation, continued

- Note that the loss approximation is proportional to the square of the angle difference across the line.
- Recall from Section 3.7.3 that, under the DC power flow approximation, the angle difference is approximately proportional to the real power flow.
- That is, losses are approximately proportional to the square of the power flow.
- Note that $G_{\ell k} \leq 0$, so that the loss function is therefore approximately a convex quadratic function of the power flow.
- Using the DC power flow approximation, the power flows are approximately linear in the real power injections.
- Therefore, the losses in each line are approximately a convex quadratic function of the real power injections.
- Summing losses over lines, the total losses in the system, $L = \sum_{\ell,k\in\mathbb{J}(\ell),k>\ell} L_{\ell k}$, is approximately a convex quadratic function of the real power injections.
- If the losses are small then they can be estimated by substituting from the flows given by the DC power flow approximation.

Quadratic loss approximation, continued

- We can write approximate the total losses by:

$$\begin{aligned}
\forall x \in \mathbb{R}^n, L(x) &= \sum_{\ell,k \in \mathbb{J}(\ell), k>\ell} L_{\ell k}(x), \\
&\approx \sum_{\ell,k \in \mathbb{J}(\ell), k>\ell} (-G_{\ell k})(\theta_\ell - \theta_k)^2, \\
&= \theta^\dagger W \mathcal{G} W^\dagger \theta, \\
&= [\theta_{-\rho}]^\dagger W_{-\rho} \mathcal{G} (W_{-\rho})^\dagger \theta_{-\rho}.
\end{aligned}$$

- by (3.3), where:
  - the matrix $\mathcal{G}$ is a diagonal matrix with entries corresponding to the conductances of the series elements of the lines,
  - the matrix W is the bus-to-series element incidence matrix; and,
  - the matrix $W_{-\rho}$ is the bus-to-series element incidence matrix with row ρ removed.

Quadratic loss approximation, continued

- By the DC power flow approximation in Section 3.6.9, using J for the Jacobian with reference bus ρ,

$$\theta_{-\rho} = [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}),$$

- by (3.16) where $J = \frac{\partial p}{\partial \theta_{-\rho}}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{1}\end{bmatrix}\right)$.
- Defining $\hat{L} : \mathbb{R}^{n_{PQ}-1} \to \mathbb{R}$ to be the losses expressed in terms of the generation $P_{-\sigma}$, we have that the losses are:

$$\begin{aligned} & \forall P_{-\sigma} \in \mathbb{R}^{n_{PQ}-1}, \hat{L}(P_{-\sigma}) \\ \approx \quad & (P_{-\sigma} - D_{-\sigma})^{\dagger}[[J_{-\sigma}]^{-1}]^{\dagger} W_{-\rho} \mathcal{G} (W_{-\rho})^{\dagger}[J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}). \end{aligned}$$

- That is, losses are a convex function of $P_{-\sigma}$ and losses are strictly convex if the conductances of all series elements are non-zero.

3.11.4 Example

- Consider a modified version of the one-line two-bus system from Section 3.8 as shown in in Figure 3.13.
- Bus $\rho = 1$ is the angle reference bus, so the unknown angle is θ_2.
- Bus $\sigma = 2$ is the slack bus.
- There is generation and demand at both buses 1 and 2.
- The admittance of the series element is modified to:

$$\mathcal{Y}_{12} = 100 - 1000\sqrt{-1}.$$

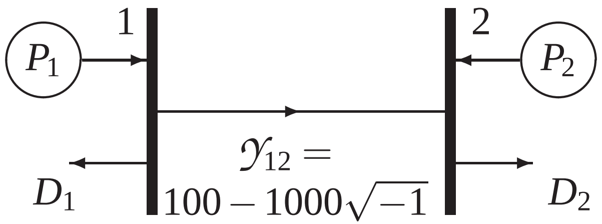

Fig. 3.13. One-line two-bus network.

3.11.4.1 Admittance matrix

- The bus admittance matrix is:

$$\begin{aligned}
W\mathcal{Y}W^{\dagger} &= \begin{bmatrix} \mathcal{Y}_{12} & -\mathcal{Y}_{12} \\ -\mathcal{Y}_{12} & \mathcal{Y}_{12} \end{bmatrix}, \\
&= \begin{bmatrix} 100-1000\sqrt{-1} & -100+1000\sqrt{-1} \\ -100+1000\sqrt{-1} & 100-1000\sqrt{-1} \end{bmatrix}, \\
&= \begin{bmatrix} G_{11}+B_{11}\sqrt{-1} & G_{12}+B_{12}\sqrt{-1} \\ G_{21}+B_{21}\sqrt{-1} & G_{22}+B_{22}\sqrt{-1} \end{bmatrix}.
\end{aligned}$$

- That is, we have:

$$\begin{aligned}
W\mathcal{G}W^{\dagger} &= \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}, \\
&= \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}, \\
W_{-\rho}\mathcal{G}(W_{-\rho})^{\dagger} &= [100].
\end{aligned}$$

3.11.4.2 Power flow equations and Jacobian

- We assume that the voltage magnitudes are maintained equal to one per unit, so that $v^{(0)} = \mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Noting that the imaginary part of the admittance matrix has not changed compared to the example in Section 3.8, we have that the power flow equations are:

$$\begin{aligned} -P_1 - P_2 &= -D_1 - D_2, \\ [\theta_2] &= [J_{-\sigma}]^{-1}[P_1 - D_1], \end{aligned}$$

- where:

$$\begin{aligned} J_{-\sigma} &= [-1000], \\ [J_{-\sigma}]^{-1} &= [-0.001]. \end{aligned}$$

3.11.5 Quadratic loss approximation

- The exact loss expression (3.21) for this system is:

$$\begin{aligned} \forall x \in \mathbb{R}^n, L_{12}(x) &= 2v_1 v_2 G_{12} \cos(\theta_1 - \theta_2) - [(v_\ell)^2 + (v_k)^2] G_{\ell k}, \\ &= 200(1 - \cos(\theta_2)), \end{aligned} \tag{3.23}$$

- since $v_1 = v_2 = 1$, $G_{12} = -100$, and assuming that the angle reference is $\theta_1 = 0$.
- A quadratic approximation to this function yields:

$$\forall x \in \mathbb{R}^n, L_{12}(x) \approx 100(\theta_2)^2. \tag{3.24}$$

Quadratic loss approximation, continued

- The quadratic loss approximation in terms of power injection is $\hat{L} : \mathbb{R} \to \mathbb{R}$:

$$\forall P_1 \in \mathbb{R}, \hat{L}(P_1)$$
$$\approx (P_{-\sigma} - D_{-\sigma})^{\dagger} [[J_{-\sigma}]^{-1}]^{\dagger} W_{-\rho} \mathcal{G} (W_{-\rho})^{\dagger} [J_{-\sigma}]^{-1} (P_{-\sigma} - D_{-\sigma}),$$
$$= (P_1 - D_1)^{\dagger} [-0.001][100][-0.001](P_1 - D_1),$$
$$= (0.0001)(P_1 - D_1)^2.$$

- For example, if $(P_1 - D_1) = 100$ MW, so that line is at capacity, then losses are $(0.0001)(100)^2 = 1$ MW, and there are 1% losses in the line under these conditions.

3.11.6 Errors with quadratic loss approximation

- If the losses are large then the approach can be erroneous.
- Consider the simple three bus, three line system shown in Figure 3.14.
- Assume that the slack and reference bus is bus one, so that $\rho = \sigma = 1$.
- Note that the DC shift factors for injection at bus 3 and flow to each of the lines are both equal to one. (See Exercise 3.3.)
  - the DC approximation models all power injected at bus 3 flowing on both of the lines,
  - does not model any losses occurring "on" these lines,
  - the loss approximation effectively models all of the losses as occurring "at" the slack bus.

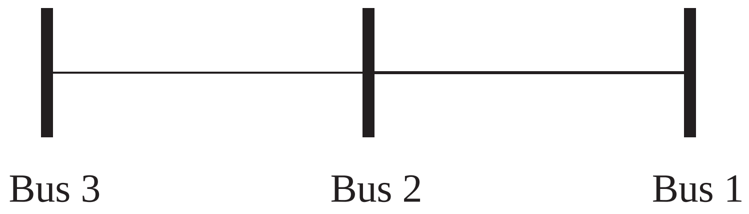

Fig. 3.14. Three bus, two line radial network.

Errors with quadratic loss approximation, continued

- If the losses on the line joining buses 3 and 2 are significant, then a significant amount of the injection at bus 3 will by lost on this line and will not flow on the line from bus 2 to bus 1:
  - the DC power flow approximation will ignore this issue, over-estimating the contribution of injection at bus 3 to flow on the line joining bus 2 to bus 1,
  - substituting from the DC power flow into the loss expression (3.22) for the line joining bus 2 to bus 1 will over-estimate the effect of injection at bus 3 on losses on this line.
- Will tend to over-estimate the contribution of remote generation to losses.
- This issue is treated in different ways in different market implementations.
- Despite this issue, losses are still approximately a convex quadratic function of power injections.

3.12 Contingency analysis

- Power flow analysis evaluates the angles, voltage magnitudes, and line power flows on a given system.
- We will see in Section 9 that we must dispatch generation so that, among other things, the flow on each line in the system does not exceed its capacity:
  - we will consider optimization formulations that seek dispatch to keep flows within normal or **long-term** ratings of transmission lines.
- In addition, we must consider the fact that lines may be outaged occasionally:
  - typical cause is due to lightning strike causing short-circuit.

Contingency analysis, continued

- Standard practice is to operate power systems so that even in the event of any given single outage, the resulting flows on the lines remain within limits:
  - because most outages are temporary, we generally use **short-term** or **emergency** ratings to evaluate whether flows in each contingency are acceptable,
  - we will also include these **contingency constraints** in the optimization formulation.
- In this section, we will utilize the DC power flow approximation to approximately evaluate the flow under a contingency.
- For a system having 5000 lines, how many line contingencies are possible?

3.12.1 Pre-contingency flows

- We first consider the DC power flow approximations for the original **pre-contingency** system.
- Suppose that the reference bus is bus ρ, while the slack bus is bus σ.
- From Section 3.6.9 the DC power flow equations are equivalent to:

$$
\begin{aligned}
-\mathbf{1}^{\dagger}P &= -\mathbf{1}^{\dagger}D, \\
\theta_{-\rho} &= [J_{-\sigma}]^{-1}(P_{-\sigma} - D_{-\sigma}),
\end{aligned}
$$

- where $P_{-\sigma}$ and $D_{-\sigma}$ are the sub-vectors of P and D, respectively, that omit the slack bus, and $J_{-\sigma}$ is minus the imaginary part of the admittance matrix, that is, $-B$, with the column corresponding to the reference bus ρ deleted and the row corresponding to the slack bus σ deleted.
- As in Section 3.7.3, we can evaluate the pre-contingency line flows as $K\theta_{-\rho}$, where each row of K corresponds to a line and has exactly two non-zero elements, with the non-zero values given by plus and minus the susceptance of the series element of the corresponding line.
- The matrix $K[J_{-\sigma}]^{-1}$ is the pre-contingency DC **shift factors**.

3.12.2 Post-contingency system

- As discussed in Section 3.2.5, we can consider removing a line from the system and we will also use the DC power flow approximation for this **post-contingency** system:
  - in principle, we need to consider the effect of removing each line in the system on the flows in the remaining system,
  - in practice, we may only select some contingencies for analysis,
  - even selecting only some contingencies, there will still typically be many to be considered, so computational effort is still significant.
- Consider the effect of removing a line joining bus ℓ to bus k, where the series element of the line model has imaginary part $\mathcal{B}_{\ell k}$,
- Let $w \in \mathbb{R}^{n_V}$ be a vector with a one in the ℓ-th entry, a minus one in the k-th entry, and zeros elsewhere.
- If the line is removed, the imaginary part of the admittance matrix changes from B to $(B - \mathcal{B}_{\ell k} w w^{\dagger})$.

3.12.3 Post-contingency flows

- Removing the line means that the DC power flow equations for the system will change to:

$$\begin{aligned} -\mathbf{1}^\dagger P &= -\mathbf{1}^\dagger D, \\ \theta'_{-\rho} &= \left[J'_{-\sigma}\right]^{-1}(P_{-\sigma} - D_{-\sigma}), \end{aligned}$$

- where $J'_{-\sigma}$ is minus the imaginary part of the changed admittance matrix, that is, $(-B + \mathcal{B}_{\ell k} w w^\dagger)$, with the column corresponding to the reference bus ρ deleted and the row corresponding to the slack bus σ deleted.
- That is, $J'_{-\sigma} = J_{-\sigma} + \mathcal{B}_{\ell k} w_{-\sigma}(w_{-\rho})^\dagger$, where $w_{-\sigma}$ and $w_{-\rho}$ are, respectively, the vector w with the σ-th and ρ-th entries, respectively, deleted.
- As mentioned in Section 2.2, in practice we will not invert the matrix $J'_{-\sigma}$.
- Instead of inverting $J'_{-\sigma}$, we could factorize it and use forwards and backwards substitution to solve for $\theta'_{-\rho}$.

Post-contingency flows, continued

- If we have already factorized $J_{-\sigma}$, which corresponds to the pre-contingency system, we can reduce the effort to factorize $J'_{-\sigma}$ or use the Sherman-Morrison formula.
- In particular, we can evaluate:

$$\theta'_{-\rho} = (J'_{-\sigma})^{-1}(P_{-\sigma} - D_{-\sigma}) = \left(J_{-\sigma} + \mathcal{B}_{\ell k} w_{-\sigma} (w_{-\rho})^{\dagger}\right)^{-1}(P_{-\sigma} - D_{-\sigma}),$$

using the following:

(i) solve $J_{-\sigma}\theta^{\star}_{-\rho} = (P_{-\sigma} - D_{-\sigma})$ and $J_{-\sigma}\theta^{\star\star}_{-\rho} = w_{-\sigma}$, so that $\theta^{\star}_{-\rho}$ are the angles in the pre-contingency or base-case system and $\theta^{\star\star}_{-\rho}$ are the angles that would occur in a system with unit injection of power at bus ℓ and unit withdrawal at bus k (and no other injections or withdrawals),

(ii) define $\Delta P_{-\sigma} = -\frac{\mathcal{B}_{\ell k} w_{-\sigma} (w_{-\rho})^{\dagger}}{1 + \mathcal{B}_{\ell k} (w_{-\rho})^{\dagger} \theta^{\star\star}_{-\rho}} \theta^{\star}_{-\rho}$, and

(iii) solve $J_{-\sigma}\Delta\theta'_{-\rho} = \Delta P_{-\sigma}$ and set $\theta'_{-\rho} = \theta^{\star}_{-\rho} + \Delta\theta'_{-\rho}$.

Post-contingency flows, continued

- Note that all of these calculations involve forward and backwards substitution using the factors of $J_{-\sigma}$, which corresponds to the pre-contingency system.
- We do not need to factorize $J'_{-\sigma}$, which is desirable since we must evaluate the effect of outages of multiple lines in the system.

Post-contingency flows, continued

- We can interpret the solution of the post-contingency system as being a perturbation on the solution $\theta^{\star}_{-\rho}$ of the pre-contingency system that is due to the flow on the line joining buses ℓ and k being redistributed to all of the other lines in the system.
- The vector of injections $\Delta P_{-\sigma} = -\frac{\mathcal{B}_{\ell k} w_{-\sigma} (w_{-\rho})^{\dagger}}{1+\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}\theta^{\star\star}_{-\rho}} \theta^{\star}_{-\rho}$ is used as an intermediate step to evaluate how the flow is distributed into the rest of the system:
  - since $\Delta P_{-\sigma} = \alpha w_{-\sigma}$, where $\alpha = -\frac{\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}}{1+\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}\theta^{\star\star}_{-\rho}} \theta^{\star}_{-\rho} \in \mathbb{R}$, we note that $\Delta P_{-\sigma}$ has exactly two non-zero entries, $\Delta P_{\ell} = \alpha$ and $\Delta P_k = -\alpha$,
  - that is, $\Delta P_{-\sigma}$ defines a matched injection and withdrawal at buses ℓ and k of power α.
  - These injections are equal to the pre-contingency flow on the line $\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}\theta^{\star}_{-\rho}$, scaled by the factor $-\frac{1}{1+\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}\theta^{\star\star}_{-\rho}}$,
  - The injections have the effect of creating a flow on the line that makes the flows in the rest of the system the same as an outage of the line.

Post-contingency flows, continued

- To understand the choice of α, consider Figure 3.15, which shows buses ℓ and k, the line joining them, and lines from these two buses to other buses in the system that are denoted by arrows.
- Let γ be the base-case flow on the line from bus ℓ to bus k:

$$\gamma = (-\mathcal{B}_{\ell k})(w_{-\rho})^{\dagger}\theta^{\star}_{-\rho}.$$

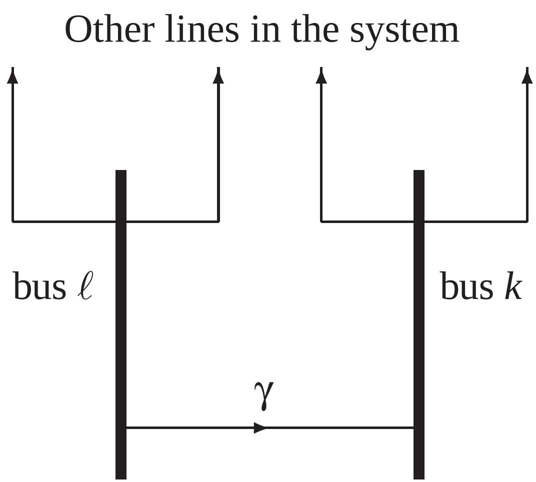

Fig. 3.15. Two buses, ℓ and k, in network. Source: This figure is adapted from figure 11.16 of Wood and Wollenberg (1996).

Post-contingency flows, continued

- Consider a matched injection and withdrawal at buses ℓ and k, assuming that there are no other injections and withdrawals in the system:
  - Let β be the flow on the line from bus ℓ to bus k if there were a unit injection of power at bus ℓ and unit withdrawal at bus k (and no other injections or withdrawals):

$$\beta = (-\mathcal{B}_{\ell k})(w_{-\rho})^{\dagger}\theta^{\star\star}_{-\rho}.$$

  - Now suppose that, instead of a unit injection, there is an injection of α at bus ℓ and withdrawal at bus k.
  - Then the flow from bus ℓ to bus k due to this injection would be scaled to $\alpha\beta$.

Post-contingency flows, continued

- Now consider the case that there is both the base-case flow and the flows due to the injection of α at bus ℓ and withdrawal of α at bus k:
  - Summing both the base-case flow and the injection of α, we would obtain a net flow on the line of $\alpha\beta + \gamma$.
- Consider the conditions on α so that the flows from buses ℓ and k into the rest of the system are zero:
  - at bus ℓ, this would require that the injection α equals the net flow $\alpha\beta + \gamma$ into the line ℓ,
  - at bus k, this would require that the withdrawal α equals the net flow $\alpha\beta + \gamma$ out of the line at bus k.
- If we conceptually "split" the bus, we can think of the flows into the rest of the system as being flows "across" the bus.
- If this flow across the bus is made equal to zero, as shown in Figure 3.16, then the effect of the injections and withdrawals at buses ℓ and k and the effect of the base-case flow on the line are cancelled.

Post-contingency flows, continued

- Summarizing, to achieve flow of zero across the bus, we require that the net injection α balances the flows $\alpha\beta+\gamma$.
- Re-arranging, we obtain $\alpha = -\frac{\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}}{1+\mathcal{B}_{\ell k}(w_{-\rho})^{\dagger}\theta^{\star\star}_{-\rho}}\theta^{\star}_{-\rho}$.

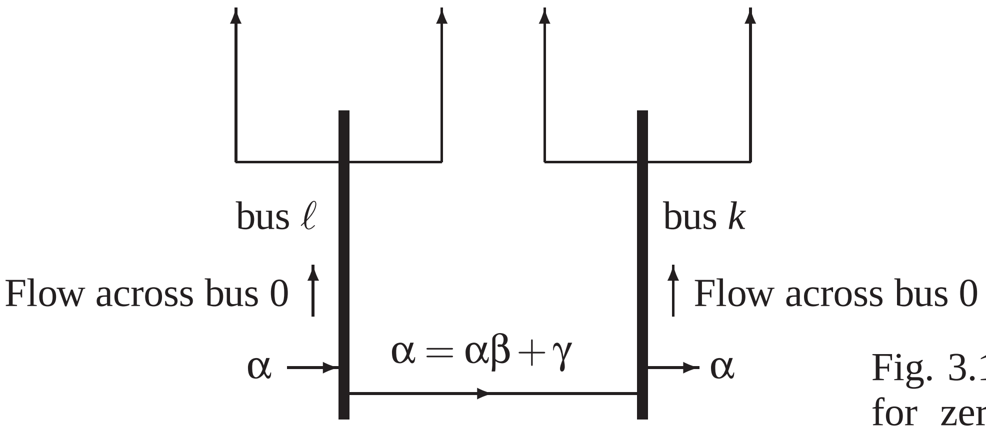

Fig. 3.16. Conditions for zero net flow into rest of network.

Post-contingency flows, continued

- The line has been effectively removed from the system through the superposition of the original flows together with the matched injection at bus ℓ and withdrawal at bus k.
- The zero flow across the buses is equivalent to the buses being split, and the line removed from the rest of the system, as shown in Figure 3.17.

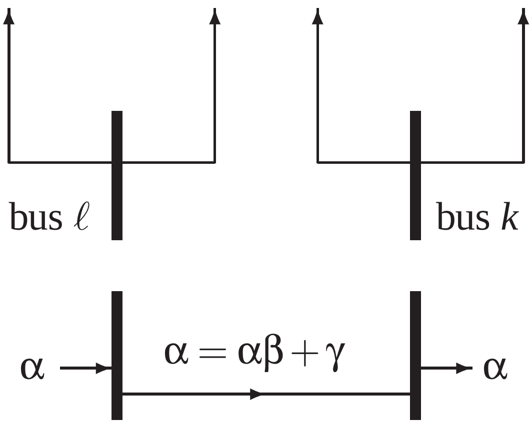

Fig. 3.17. Splitting buses.

3.12.4 Outage shift factors

- The flows on the lines in the rest of the system can be evaluated as $K'\theta'_{-\rho}$, where K' is obtained from K by deleting the row corresponding to the line joining buses ℓ and k.
- Similarly, post-contingency or **outage** shift factors can be evaluated as $K'(J'_{-\sigma})^{-1}$.
- It is sometimes more convenient to think of the outage shift factors as being the sum of the pre-contingency shift factors plus terms that represent:
  - the effect of injections on the pre-contingency flow on the line joining bus ℓ to bus k, and
  - the effect of redistributing the flow on this line to the other lines due to the contingency.
- This is useful for large-scale implementations.

3.13 Summary

- In this chapter we formulated the power flow problem, and considered:
  - linearization of power flow,
  - fixed voltage profiles,
  - DC power flow,
  - losses, and
  - contingency analysis.

This chapter is based on:

- Sections 8.2 and 9.2 of *Applied Optimization: Formulation and Algorithms for Engineering Systems*, Cambridge University Press 2006.
- Ross Baldick, "Variation of Distribution Factors with Loading," *IEEE Transactions on Power Systems*, 18(4):1316–1323, November 2003.
- Brian Stott, Jorge Jardim, and Ongun Alsaç, "DC Power Flow Revisited," *IEEE Transactions on Power Systems*, 24(3):1290–1300, August 2009.
- Allen J. Wood and Bruce F. Wollenberg, *Power Generation*, *Operation*, *and Control*, Second Edition, Wiley, New York, 1996.

Homework exercises

3.1 Consider a power system consisting of two buses and one transmission line:

- bus 1 (the reference/slack bus), where there is a generator, and
- bus 2, where there is load.

Suppose that the reference/slack bus voltage is specified to be $V_1 = 1\angle 0^\circ$ and that real power flow from bus 2 into the line is given by:

$$\forall v_2 \in \mathbb{R}_+, \forall \theta_2 \in \mathbb{R}, p_2(\theta_2, v_2) = v_2 \sin\theta_2.$$

(That is, we assume that $G_{22} = G_{12} = 0$ and $B_{12} = 1$.) Suppose $v_2 = 1.0$.

(i) What is the largest value of demand D_2 at bus 2 for which there is a solution to the equation $p_2(\theta_2, 1.0) + D_2 = 0$? What is the corresponding value of θ_2 with $\pi \geq \theta_2 \geq -\pi$? We will write $\underline{\theta}_2$ for this value of θ_2.

(ii) What happens if θ_2 becomes smaller than $\underline{\theta}_2$?

(iii) Show that there are two solutions to the equation $p_2(\theta_2, 1.0) + D_2 = 0$ with $0 \geq \theta_2 > -2\pi$ if $D_2 = 0.5$. What are the corresponding values of θ_2?

(iv) Use DC power flow to approximate the relationship between θ_2 and D_2.

(v) When do you expect the DC power flow to be a poor approximation to the exact solution?

3.2 Consider the example in Section 3.9, but suppose that bus 0 is both the reference and the slack bus, so that $\rho = \sigma = 0$.

(i) What is vector of unknown angles θ_{-0}?

(ii) Evaluate $J' = \frac{\partial p}{\partial \theta_{-0}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$.

(iii) Evaluate $[J'_{-0}]^{-1}$.

(iv) Write down the DC power flow equations in terms of generation at buses 1, 2, and 3, and demand at bus 0.

(v) Evaluate the matrix K' in the linearized representation of line flow inequality constraints $K'\Delta\theta_{-0} \leq d$.

(vi) Evaluate the shift factor matrix $K'[J'_{-0}]^{-1}$.

(vii) Write down the line flow inequality constraints in terms of the shift factors.

(viii) What do you notice about the line flow inequality constraints? Did the choice of reference bus change the form of the line flow constraints?

(ix) Repeat the previous parts, but with bus 1 both the reference and the slack bus, so that $\rho = \sigma = 1$.

3.3 Consider the three bus, two line system shown in Figure 3.18 and suppose that bus 1 is both the reference and the slack bus, so that $\rho = \sigma = 1$. The line capacities are shown. Assume that the susceptance joining bus 2 to bus 1 and the susceptance joining bus 3 to bus 2 are both non-zero.

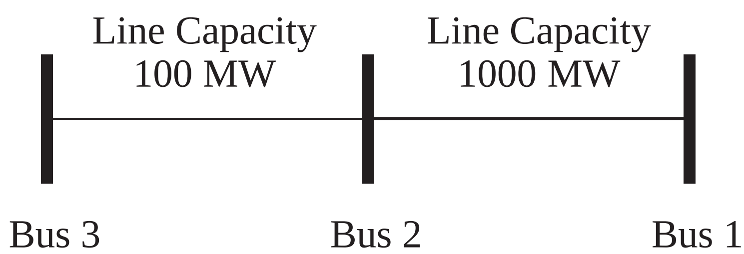

Fig. 3.18. Three bus, two line radial network.

(i) What is vector of unknown angles θ_{-1}?

(ii) Evaluate $J = \frac{\partial p}{\partial \theta_{-1}}\left(\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}\right)$. (Hint: See (3.13).)

(iii) Write down the DC power flow equations in terms of generation and demand at buses 2 and 3.

(iv) Evaluate $[J_{-1}]^{-1}$.

(v) Assume that the only flow constraints are from bus 2 to bus 1, and from bus 3 to bus 2. Evaluate the matrix K in the linearized representation of real power line flow limit inequality constraints $K\Delta\theta_{-1} \leq d$.

(vi) Evaluate the shift factor matrix $K[J_{-1}]^{-1}$.

(vii) Write down the real power line flow limit inequality constraints in terms of the shift factors as in (3.18).

(viii) Interpret these inequality constraints in terms of the figure.

3.4 Assuming that J_{-1} is invertible, show that the DC power flow equations $J\theta_{-1} = P$ are equivalent to:

$$\begin{aligned} P_1 &= -\mathbf{1}^\dagger P_{-1}, \\ \theta_{-1} &= [J_{-1}]^{-1} P_{-1}. \end{aligned}$$

That is, show that θ_{-1} satisfies $J\theta_{-1} = P$ if and only if θ_{-1} satisfies $\theta_{-1} = [J_{-1}]^{-1} P_{-1}$ and $P_1 = -\mathbf{1}^\dagger P_{-1}$. (Hint: See discussion in Section 3.6.5 or consider the invertible matrix $\mathcal{M} = \begin{bmatrix} 1 & \mathbf{1}^\dagger \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$.)

3.5 Consider the three bus system shown in Figure 3.19 with buses $\ell = 1, 2, 3$ and with bus $\rho = \sigma = 1$ the reference/slack bus. Assume all three lines have the same admittance. Using the analysis in Section 3.10, calculate the matrix of shift factors for this system to flows on the lines from buses 2 to 1, 3 to 1, and 2 to 3.

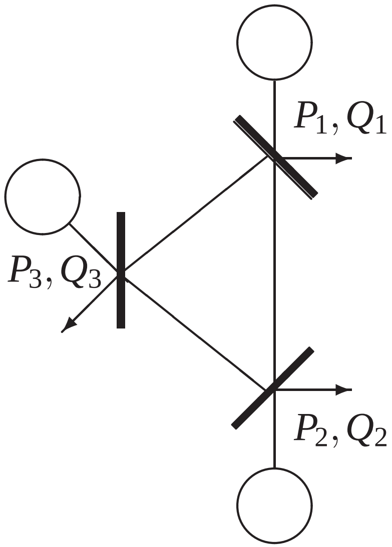

Fig. 3.19. Three-bus, three-line network.

3.6 As mentioned in Section 3.6.4, when there are no shunts, we can follow the "traditional" approach to analyzing DC power flow assuming:

- the small angle approximations for cos and sin, and
- voltage magnitudes assumed to equal one per unit,

in order to derive the DC power equations in a more straightforward manner than the more general sensitivity analysis used in Sections 3.6 and 3.7.
In particular, again adopting the fixed voltage schedule assumption of Section 3.5 with all buses *PV* buses, we assume that:

$$v_\ell \approx 1, \forall \ell,$$

and can therefore ignore the reactive power equality constraints (3.7) since they will be satisfied by controlled reactive injection at each bus. Moreover, assuming that angle differences are small we obtain that:

$$\begin{aligned} \forall \ell, \forall k \in \mathbb{J}(\ell) \cup \{\ell\}, \sin(\theta_\ell - \theta_k) &\approx (\theta_\ell - \theta_k), \\ \sin(\theta_\ell - \theta_k) &\approx (\theta_\ell - \theta_k). \end{aligned}$$

Substituting into the real power equality constraints (3.6), we obtain:

$$P_\ell = \sum_{k \in \mathbb{J}(\ell) \cup \{\ell\}} B_{\ell k}(\theta_\ell - \theta_k), \forall \ell.$$

Suppose there are no shunt elements in the system. Then, the real power flow on a line joining buses ℓ and k is given by:

$$\mathcal{P}_{\ell k} = B_{\ell k}(\theta_\ell - \theta_k).$$

For the rest of the question, we will continue to assume that there are no shunt elements in the system. Similarly to the analysis in Section 3.2.4, define the following matrices:

- let $\mathcal{B}$ be the diagonal matrix with diagonal elements equal to the susceptances of the series elements in the system, and
- the bus-to-line incidence matrix W, with columns ordered corresponding to the rows and columns of $\mathcal{B}$, and with each column having a single 1 and -1 entry at the locations of the buses joined by the corresponding series element in $\mathcal{B}$.

With these definitions, evaluate the following.

(i) Find an expression for the vector of real power flows, $\mathcal{P}$, on the lines in the system in terms of $\mathcal{B}, W$, and θ.

(ii) Find an expression for the vector of real power injections, P, at the buses in the system in terms of $\mathcal{B}, W$, and θ.

(iii) The matrix in the last part is singular; however, by deleting a row corresponding to the slack bus σ and a column corresponding to the reference bus ρ, a non-singular matrix can be obtained. Given a choice of slack bus σ and choice of reference bus ρ, express the vector of real power injections everywhere except the slack bus, $P_{-\sigma}$ in terms of $\mathcal{B}$, $W_{-\sigma}$, $W_{-\rho}$, and θ_ρ, where $W_{-\sigma}$ is the matrix W with row σ deleted, $W_{-\rho}$ is the matrix W with row ρ deleted, and θ_ρ is the vector θ with the entry ρ deleted.

(iv) Find an analogous expression for $\mathcal{P}$ in terms of $\mathcal{B}$, $W_{-\rho}$, and θ_ρ.

(v) Evaluate the matrix, C, of shift factors that evaluates flow $\mathcal{P}$ in terms of injections P_σ at all buses except the slack bus.

3.7 Consider again the modified one-line two-bus system in Section 3.11.4 as shown in in Figure 3.13, and repeated as Figure 3.20, with bus $\rho = 1$ is the angle reference bus, and admittance of the series element $\mathcal{Y}_{12} = 100 - 1000\sqrt{-1}$. Evaluate the exact loss expression (3.23) and the quadratic approximation (3.24) for each of the following angles θ_2 (in radians).

(i) $\theta_2 = 0.01$,
(ii) $\theta_2 = 0.05$,
(iii) $\theta_2 = 0.1$,
(iv) $\theta_2 = 0.5$.

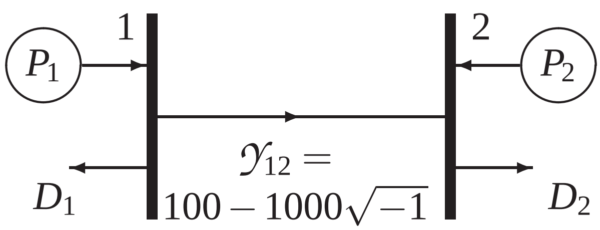

Fig. 3.20. One-line two-bus network.

3.8 Consider again the four-bus four-line example shown in Figures 3.9 and 3.12, and repeated in Figure 3.21. Section 3.10 used the DC power flow circuit interpretation to evaluate the shift factors to all the lines for injection at bus 1, withdrawal at bus 0. In this question, we will evaluate the shift factors for injections at the other buses. Evaluating by inspection is sufficient in all cases.

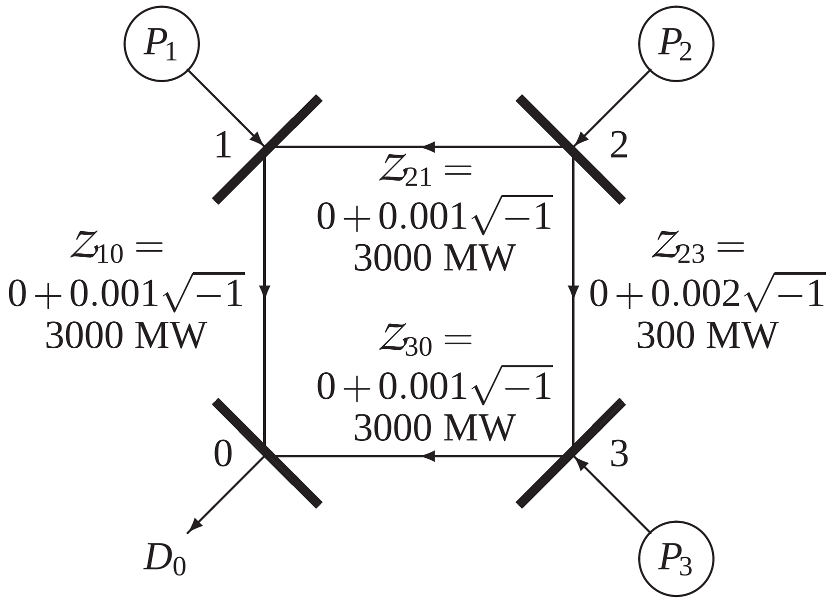

Fig. 3.21. Four-line four-bus network repeated from Figures 3.9 and 3.12.

(i) Evaluate the shift factors to all the lines for injection at bus 2 and withdrawal at bus 0.

(ii) Evaluate the shift factors to all the lines for injection at bus 3 and withdrawal at bus 0.

(iii) Consider a contingency on the line joining bus 1 to bus 0.

  (a) Evaluate the shift factors to all the remaining lines for injection at bus 1 and withdrawal at bus 0.

  (b) Evaluate the shift factors to all the remaining lines for injection at bus 2 and withdrawal at bus 0.

  (c) Evaluate the shift factors to all the remaining lines for injection at bus 3 and withdrawal at bus 0.

(iv) Consider a contingency on the line joining bus 1 to bus 2.

  (a) Evaluate the shift factors to all the remaining lines for injection at bus 1 and withdrawal at bus 0.

  (b) Evaluate the shift factors to all the remaining lines for injection at bus 2 and withdrawal at bus 0.

  (c) Evaluate the shift factors to all the remaining lines for injection at bus 3 and withdrawal at bus 0.