Course notes for EE394V Restructured Electricity Markets: Market Power

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5

Transmission constraints

- This material is based on:
  - Severin Borenstein, James Bushnell, and Steven Stoft, “The Competitive Effects of Transmission Capacity in a Deregulated Electricity Industry,” *RAND Journal of Economics*, 31(2):294–325, Summer 2000.
  - Carolyn A. Berry, Benjamin Hobbs, William A. Meroney, Richard P. O’Neill, and William R. Stewart, Jr., “Analyzing Strategic Bidding

Behavior in Transmission Networks." In H. Singh, Editor, *IEEE Tutorial on Game Theory Applications in Power Systems*, pages 7–32, IEEE Press, 1999.

– Lin Xu and Yixin Yu, "Transmission constrained linear supply function equilibrium in power markets: method and example," In *Proceedings of PowerCon 2002, International Conference on Power System Technology*, 3:1349–1354, October 2002.

– Electric Reliability Council of Texas, "ERCOT Nodal Protocols," 2007. Available from http://nodal.ercot.com/protocols/index.html.

– Lin Xu and Ross Baldick, "Transmission-constrained Residual Demand Derivative in Electricity Markets," *IEEE Transactions on Power Systems*, 22(4):1563–1573, November 2007.

– Ross Baldick, "Course notes for EE394V Restructured Electricity Markets: Locational marginal pricing," Fall 2006. Available from http://users.ece.utexas.edu/˜baldick/classes/394V/Locational.pdf and from http://users.ece.utexas.edu/˜baldick/classes/394V/Linearized.pdf

– Ross Baldick, "Applied Optimization: Formulation and Algorithms for Engineering Systems Slides," Fall 2006. Available from http://users.ece.utexas.edu/˜baldick/classes/380N/Inequality%20II.pdf

– Manho Joung and Ross Baldick, “The Competitive Effects of Ownership of Financial Transmission Rights in a Deregulated Electricity Industry.”

Outline

(i) Modeling market power, revisited,
(ii) Transmission constraints and geographical market power,
(iii) Shift factors and the DC power flow,
(iv) Offer-based transmission-constrained economic dispatch,
(v) Ad hoc analyses of market power with transmission constraints,
(vi) Consideration of incentives when transmission constraints bind,
(vii) Ownership of generation at multiple buses,
(viii) Pivotal offers,
(ix) Transmission and equilibrium analysis,
(x) Transmission, equilibrium, and transmission rights,
(xi) Summary.

5.1 Modeling market power, revisited

- Market power assessment approaches:

  (i) *Ad hoc* approaches based on indices such as HHI:
  - have also been extended to include transmission constraints, but
  - since foundation is *ad hoc*, results are unreliable,
  - will use example from ERCOT Nodal Protocols to illustrate.

  (ii) Empirical analyses to test if offers are above marginal costs or assess the change in prices due to deviation from competitive offers:
  - Joskow–Kahn paper,
  - IMM report,
  - transmission can be included,
  - effect of market power can be assessed in presence of transmission constraints, but
  - difficult to obtain insights into effect of transmission on competition.

Modeling market power, revisited, continued

(iii) Analysis of incentives to deviate from competitive prices:

- given hypothesis of profit maximizer,
- what would have been the best response or the mark-up,
- Hortaçsu and Puller paper,
- basic insight is that derivative of residual demand with respect to price (or derivative of inverse residual demand with respect to quantity) determines the incentives to mark-up above competitive:
  - if demand is very elastic (derivative of residual demand with respect to price is large) then profit maximizer will offer close to marginal, while
  - if demand is inelastic (derivative of residual demand with respect to price is small) then profit maximizer will offer above marginal.
- So far have not included transmission in this assessment.

Modeling market power, revisited, continued

- We will consider transmission constraints in assessment of incentives by generalizing the notion of residual demand to the transmission-constrained case:
  - we will first review the situation in the absence of transmission constraints,
  - then see how to generalize to case of transmission constraints.

5.1.1 Incentives in the absence of transmission constraints

5.1.2 Residual demand

- As previously, we consider the residual demand faced by a market participant:
  - the actual demand minus the supply of all the other participants.
- Suppose that the demand in a particular pricing interval is D:
  - we ignore price-responsiveness of demand, but it can be incorporated into the analysis.
- Consider a particular market participant k.
- Suppose that the total offered generation of all the *other* market participants besides k is specified by the function $q_{-k} : \mathbb{R} \to \mathbb{R}$:
  - At price P, the total offered generation of all the other market participants is $q_{-k}(P)$.
  - The *residual demand* faced by market participant k is $(D - q_{-k}(P))$.
- The inverse of the function $(D - q_{-k})$ is the inverse residual demand function faced by participant k, which we will denote $p^{\mathrm{d}}_{-k} : \mathbb{R} \to \mathbb{R}$.

5.1.3 Profits

- Consider operating profit, $\pi_k : \mathbb{R} \to \mathbb{R}$, for participant k, which is revenue minus costs:
  - Revenue equals the product of:
    - quantity, Q_k, multiplied by
    - the resulting price $p^{\text{d}}_{-k}(Q_k)$,
  - Total variable operating costs for participant k are $c_k : \mathbb{R} \to \mathbb{R}$.
- Operating profit for market participant k is:

$$\forall Q_k \in \mathbb{R}, \pi_k(Q_k) = Q_k p^{\text{d}}_{-k}(Q_k) - c_k(Q_k).$$

Profits, continued

- Assuming that:
  - sufficient conditions for maximization are satisfied,
  - functions p^{d}_{-k} and c_k are differentiable, and
  - generation capacity constraints are not binding at the profit maximizing condition,
- then we can find the maximum of profit by setting its derivative to zero:

$$
\begin{aligned}
0 &= \frac{\partial \pi_k}{\partial Q_k}(Q_k), \\
&= p^{\mathrm{d}}_{-k}(Q_k) + Q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k) - c'_k(Q_k),
\end{aligned}
$$

- where $c'_k = \dfrac{\partial c}{\partial Q_k}$ is the marginal costs.

5.1.4 Mark-up and market power index

- Re-arranging, we obtain the price-cost mark-up of price above marginal cost under the hypothesis that the generator was maximizing its profits:

$$p^{\mathrm{d}}_{-k}(Q_k) - c'_k(Q_k) = -Q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k). \tag{5.1}$$

- We have seen this basic result previously:
  - incentive for generator k to mark-up price above marginal cost depends on the derivative of the inverse residual demand faced by generator k.
- The right-hand side of (5.1) is a market power index:
  - if it is "large" according to some standard then a profit-maximizing generator has incentives to drive up prices "significantly" by withholding,
  - ignoring forward contracts.
- Any generator that is not at full production but such that the right-hand side of (5.1) is above a threshold would be flagged as of concern.
- In the context of "market power mitigation," such generators might then be subject to limits on offer prices.

5.1.5 Excess transfers above competitive

- If marginal costs roughly represent the level of competitive prices then the mark-up approximates the excess transfer of wealth, over and above competitive levels, from consumers to producers per MW of production.
- Multiplying by production Q_k, we obtain an *approximate* index of excess wealth transfer to participant k:

$$-(Q_k)^2 \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k). \tag{5.2}$$

- Since the marginal cost $c'_k(Q_k)$ of participant k at its production level may be below the competitive price, the excess wealth transfer may be less than implied by (5.2).
- It is profit maximizing for a firm to offer at (close to) marginal cost if the firm is "small:"
  - "small" means that its effect on price is small.

5.1.6 Calculation of index

- The right-hand side of (5.1) (or of (5.2)) can be evaluated using knowledge of the offers and the quantities and prices cleared in an offer-based electricity market.
- Example contexts:
  - (i) *ex ante* simulation of market operation over pricing intervals in a time horizon using a production cost simulator,
  - (ii) alongside the clearing of the actual market based on actual offers, or
  - (iii) based on historical information.

5.1.7 Forward contracts

- If the generator has a forward contract for quantity Q^{f}_k at price P^{f}_k then the profit function becomes:

$$\forall Q_k, \pi^{\mathrm{f}}_k(Q_k) = (Q_k - Q^{\mathrm{f}}_k)p^{\mathrm{d}}_{-k}(Q_k) + Q^{\mathrm{f}}_k P^{\mathrm{f}}_k - c_k(Q_k).$$

- Again setting the derivative of profit to zero:

$$\begin{aligned} 0 &= \frac{\partial \pi^{\mathrm{f}}_k}{\partial Q_k}(Q_k), \\ &= p^{\mathrm{d}}_{-k}(Q_k) + (Q_k - Q^{\mathrm{f}}_k)\frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k) - c'_k(Q_k). \end{aligned}$$

- Price-cost mark-up with a forward contract under the hypothesis that the generator was maximizing its profits:

$$p^{\mathrm{d}}_{-k}(Q_k) - c'_k(Q_k) = -(Q_k - Q^{\mathrm{f}}_k)\frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k). \tag{5.3}$$

- Incentives for mark-up are reduced with forward contracts:
  - as observed previously.

Forward contracts, continued

- The right-hand side of (5.3) provides an index for assessing the incentives to exercise market power:
  - relies on knowledge of forward market positions.
- In real-time market, the day-ahead positions are forward financial positions:

$$-(Q_k^{\text{real-time}} - Q_k^{\text{day-ahead}})\frac{\partial p^{\text{d}}_{-k}}{\partial Q_k}(Q_k^{\text{real-time}}).$$

- It is profit maximizing for a firm to offer at (close to) marginal cost if:
  - the firm is "small," or
  - the firm's net position is small.
- No explicit representation so far of transmission constraints.

5.2 Transmission constraints and geographical market power

5.2.1 *Non-thermal constraints and "reliability must run"*

- In some cases, a generator may be in a unique position when transmission constraints are limiting:
  - no other generator available to compete to supply.
- This is particularly the case with respect to non-thermal constraints, such as voltage constraints, since reactive power must primarily be supplied *locally*:
  - geographically limited competition of reactive power supply,
  - no explicit market prices for provision of reactive power (nor reactive power reserves) in any electricity markets,
  - so reactive power and voltage issues dealt with "out-of-market."

Non-thermal constraints and "reliability must run," continued

- There may be well-defined situations when a particular generator must run in order that demand be met:
  - when demand in an importing level is high,
  - a local generator providing reactive power may be "pivotal" in the sense that if it was not in-service, demand would have to be curtailed.
- "Reliability must run" contracts are a typical mechanism to deal with this type of market power:
  - essentially a forward contract at negotiated or regulated prices.

5.2.2 Thermal constraints

- In other cases, thermal constraints (or proxy thermal constraints) may be the limiting issue.
- These constraints may limit competition, but we may want to avoid regulated prices if (or whenever) possible:
  - must then analyze competitive conditions explicitly to see incentives for mark-up of price above marginal cost.
- Analysis will primarily focus on thermal constraints (and constraints that can be well approximated by proxy thermal constraints).
- When constraints are binding, it is common to say that "congestion" is occurring:
  - *not* like traffic congestion!
  - transmission congestion means that one or more transmission constraints are binding,
  - so limiting element *cannot* be operated at a higher level without risking cascading outages and blackout.

5.2.3 Radial system

- Consider a system with a single radial transmission constraint joining two zones.
- Whenever there is transmission congestion between the zones, the two zones are separated into two markets.

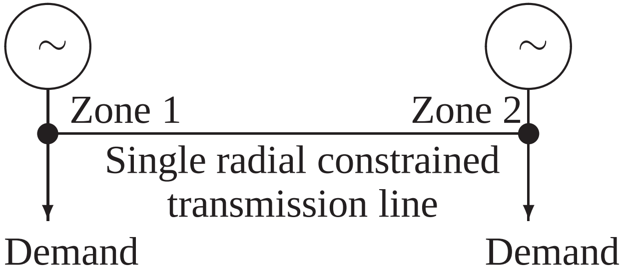


Fig. 5.1. Two zone network joined by radial transmission.

Transmission-constrained residual demand for a radial system, continued

- Suppose that the flow on the radial line is at its limit:
  - If there is only one generator in Zone 2 then the derivative of the inverse residual demand faced by that generator is given by the derivative of the inverse demand in Zone 2.
  - If demand in Zone 2 is inelastic then the the derivative of the inverse demand is large.
  - The incentive to mark-up price above marginal cost is large.
  - There is "geographical market power."
- Suppose that the flow on the radial line is not at its limit:
  - The derivative of the inverse residual demand faced by the generator in Zone 2 is due to:
    - the supply in Zone 1,
    - the demand in Zone 1, and
    - the demand in Zone 2.
  - Residual demand is more elastic in this case,
  - Smaller incentive to mark-up price above marginal cost.

5.2.4 Transmission-constrained residual demand for a radial system

- Whenever the transmission limit is binding, small changes in price in one zone cannot affect the flow to or from the other zone:
  - Analysis of residual demand involves considering each zone separately.
  - Participants in one zone can be considered separately from other zone.
  - Residual demand in each zone is due to offers and bids in that zone only.
  - Residual demand elasticity is lower than when transmission constraint does not bind.
- Analysis is valid in radial systems because of a particular feature of the "shift factors:"
  - the fraction of power flowing on a line due to injection at one zone and withdrawal at a zone.
- For a radial line, the shift factors are always either zero or one.
- Given that the constraint is binding, the two zone system can validly be analyzed as two separate markets.

Transmission-constrained residual demand for a radial system, continued

- Borenstein, Bushnell, and Stoft analyze a radial system.
- Central insights of Borenstein, Bushnell, and Stoft:
  - when transmission constraints are binding, residual demand will be less elastic,
  - increasing capacity of transmission links between markets can improve competitiveness in both markets by making residual demand of combined market more elastic than residual demand of individual markets.
- Also investigate more subtle issues regarding existence of pure strategy equilibrium when constraints bind.

5.2.5 More realistic systems

- Realistic systems are meshed.

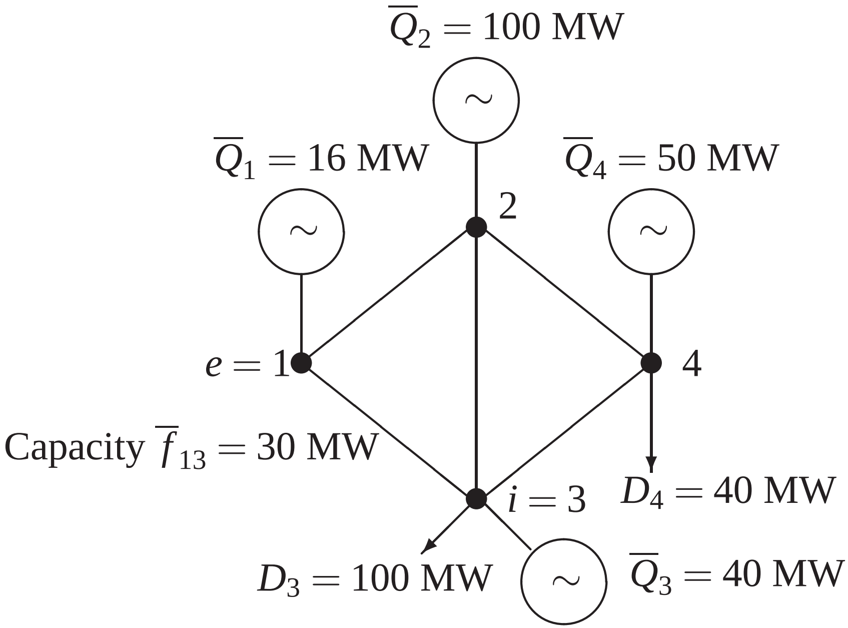


Fig. 5.2. Four bus, five line network based on an example from Berry, Hobbs, Meroney, O'Neill, and Stewart and in Lin Xu and Yixin Yu.

More realistic systems, continued

- Shift factors in a meshed system are almost always between zero and one.
- Market participants *cannot* be validly divided into being in one zone or the other.
- Residual demand at each bus can be affected by offers throughout system, even when transmission constraints bind.
- Nevertheless, central insight of Borenstein, Bushnell, and Stoft is relevant:
  – when transmission constraints bind, residual demand will be less elastic.
- Transmission constraints can exacerbate market power by reducing geographical extent of market:
  – as discussed qualitatively in IMM report.
- How to quantitatively analyze this issue in meshed systems?
- Need to analyze shift factors.

5.3 Shift factors and the DC power flow

5.3.1 Definition of shift factor

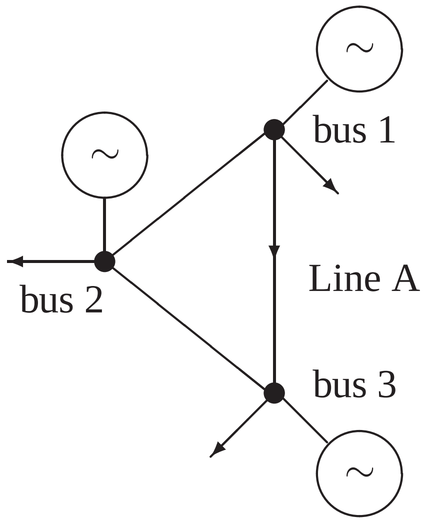
- For a given amount of power:
  - injected at a specified point of injection, bus k,
  - withdrawn at a specified point of withdrawal, bus ℓ,
- what is the fraction, $\sigma_{k\ell}$, of the amount power that flows on a particular line.
- Shift factors will vary with:
  - point of injection,
  - point of withdrawal, and
  - line.
- Values of shift factors calculated from Kirchhoff's laws and the transmission network parameters:
  - see derivation in EE394V: "Locational Marginal Pricing," Available from http://users.ece.utexas.edu/˜baldick/classes/394V/Linearized.pdf

5.3.2 DC power flow as commercial network model

- The commercial network model for both the ERCOT zonal market and the ERCOT nodal market uses the DC power flow approximation:
  - for a given network configuration, the shift factors are constant independent of the levels of flows.
- Enables flows on lines to be expressed as a linear function of "net injections" at buses.
- Net injection q_k at a bus is the difference between generation Q_k and demand D_k at each bus:
  - positive for a net generator,
  - negative for a net demand.

5.3.3 Example

- Consider the three bus three line network with all lines having equal admittance.
- Power injected at one bus and withdrawn at another is split between "long" and "short" paths in proportion to path admittance.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.3. Three bus, three line network and shift factors to line A.

Example, continued

- Let q_k be the net injection at bus k:
  - power balance requires that:

$$q_1 + q_2 + q_3 = 0.$$

  - we can pick out any one of the injections and express it in terms of the others:

$$q_3 = -q_1 - q_2,$$

  - we call q_3 the “reference” bus.
- If we:
  - inject q_1 at bus 1, and
  - inject q_2 at bus 2,
- then we must withdraw $(-q_3) = q_1 + q_2$ at bus 3.

Example, continued

- Therefore, using the definition of shift factors, the resulting flow on line A will be:

$$q_1\sigma_{13} + q_2\sigma_{23}.$$

- Define:

$$\begin{aligned} \hat{q} &= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \\ \hat{C} &= \begin{bmatrix} \sigma_{13} & \sigma_{23} \end{bmatrix}, \\ &= \begin{bmatrix} 2/3 & 1/3 \end{bmatrix}, \end{aligned}$$

  and let $\hat{d}$ equal the capacity of line A.
- Then we can write the transmission capacity constraint as:

$$\hat{C}\hat{q} \leq \hat{d}.$$

- We could pick any of the three buses to be the reference bus:
  - different choices will result in different representations of the transmission capacity constraints.

5.3.4 Multiple constraints

- In a typical network there may be many constraints that are potentially binding (including many contingency constraints):
  - that is, there are many limiting transmission elements.
- We can pick a reference bus and then calculate the shift factors to each limiting transmission element:
  - for injection at each bus, and
  - withdrawal at the reference bus.
- If we have r limiting transmission elements in a system with n buses then we can again express the constraints as:

$$\hat{C}\hat{q} \leq \hat{d}, \tag{5.4}$$

  with:
  - the matrix $\hat{C} \in \mathbb{R}^{r \times (n-1)}$ has rows that consist of shift factors to the limiting elements, with withdrawal at the reference bus,
  - the vector $\hat{q} \in \mathbb{R}^{n-1}$ are the net injections at all buses except the reference bus, and
  - the vector $\hat{d} \in \mathbb{R}^{r}$ consists of transmission element limits.

5.4 Offer-based transmission-constrained economic dispatch

- Also called “security-constrained economic dispatch” or “offer-based optimal power flow.”

5.4.1 Formulation

- Recall formulation of offer-based economic dispatch:
  - maximize the (revealed) surplus (or revealed benefits $\tilde{b}_k$ minus revealed costs $\tilde{c}_k$),
  - subject to the upper and lower bound constraints and to the power balance constraint.
- To simplify notation, consider all demands as “negative generation,” represent benefits as “negative costs,” and assume that each offer or bid is at a different bus:
  - derivative of a revealed cost $\tilde{c}_k$ at a bus k is the offer p_k at that bus,
  - we will consider the case of both offer and demand at a single bus in examples.
- Formulation then equivalent to minimizing revealed costs subject to constraints.

Formulation, continued

- Collect net generations q_k (including negative demands) together into vector $q \in \mathbb{R}^n$ of net injections, where we assume that there are n offers and bids.
- Let $\hat{q} \in \mathbb{R}^{n-1}$ be vector of net injections at buses other than reference bus.
- Using shift factors, flow on line can be expressed as a linear combination of entries of $\hat{q}$ as in (5.4).
- Upper and lower bound constraints on generation can also be expressed in this form:
  - upper and lower bound constraints on generation at the reference bus require constraints of form $Cq \leq d$,
  - for simplicity, we will assume that upper and lower bound constraints on generation at the reference bus are not binding and can be ignored.
- Offer-based transmission-constrained economic dispatch formulation:

$$\min_{q \in \mathbb{R}^n} \{ \sum_{k=1}^{n} \tilde{c}_k(q_k) | \mathbf{1}^{\dagger} q = 0, \hat{C}\hat{q} \leq \hat{d} \},$$

- where $\mathbf{1}$ is a vector of all ones and $\mathbf{1}^{\dagger} q = 0$ enforces power balance.

5.4.2 Solution

- Software, such at the MATLAB function `quadprog`, can be used to find the minimizer $q^\star \in \mathbb{R}^n$ of this problem.
- Recall that, under suitable conditions, a set of "first-order necessary conditions" characterize the minimizer:
  - software to solve the problem typically seeks a solution of the first-order necessary conditions.
- The first-order necessary conditions involve the "Lagrange multipliers" on the equality and inequality constraints:
  - see derivation in EE394V: "Locational Marginal Pricing," Available from http://users.ece.utexas.edu/˜baldick/classes/394V/Locational.pdf
  - the scalar $\hat{\lambda}^\star \in \mathbb{R}$ is the Lagrange multiplier on the equality constraint $\mathbf{1}^\dagger q = 0$ and represents the marginal value, in \$/MWh, of additional generation at the reference bus,
  - the vector $\hat{\mu}^\star \in \mathbb{R}^r_+$ is the vector of Lagrange multipliers on the inequality constraints and represents the sensitivity of the cost of dispatch to a reduction in a corresponding limit in $\hat{d}$.

Solution, continued

- First-order necessary conditions (ignoring upper and lower bound constraints on generation at reference bus):

$$
\begin{aligned}
\exists \hat{\lambda}^{\star} \in \mathbb{R}, \exists \hat{\mu}^{\star} \in \mathbb{R}^{r}, &\text{ such that:} \\
\forall k \text{ not the reference bus}, p_k(q_k^{\star}) - \hat{\lambda}^{\star} + [\hat{C}_k]^{\dagger} \hat{\mu}^{\star} &= 0; \\
\text{For the reference bus}, p_k(q_k^{\star}) - \hat{\lambda}^{\star} &= 0; \\
\forall \ell, \text{ either the } \ell\text{-th constraint is binding, or } \hat{\mu}_{\ell}^{\star} &= 0, \text{ or both;} \\
\mathbf{1}^{\dagger} q^{\star} &= 0; \\
\hat{C} \hat{q}^{\star} &\leq \hat{d}; \text{ and} \\
\hat{\mu}^{\star} &\geq \mathbf{0},
\end{aligned}
$$

- where $\hat{C}_k$ is the k-th column of $\hat{C}$, consisting of the shift factors associated with injection at bus k.
- "Complementary slackness constraints:" for each constraint ℓ, either:
  - (i) constraint ℓ is binding, or
  - (ii) the corresponding entry of $\hat{\mu}_{\ell}^{\star}$ is equal to zero, or
  - (iii) both.

5.4.3 Locational marginal prices

- The locational marginal prices (LMPs) at the buses are given by:

$$\text{LMP}_k = \begin{cases} \left(\hat{\lambda}^\star - [\hat{C}_k]^\dagger \hat{\mu}^\star\right), & \text{if } k \text{ is not the price reference bus,} \\ \hat{\lambda}^\star, & \text{if } k \text{ is the price reference bus.} \end{cases}$$

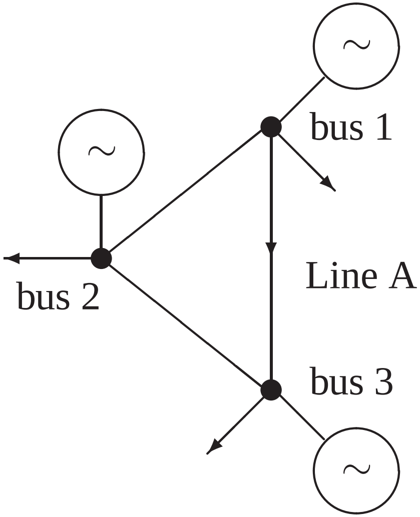
- where $\hat{C}_k$ is the k-th column of $\hat{C}$, consisting of the shift factors associated with injection at bus k.
- The LMPs or "nodal prices" are the market clearing prices at each bus:
  - all energy bought and sold at a bus is priced at the LMP in order to clear the market,
  - LMPs are equal to:
    - the price at the reference bus, minus
    - a shift factor-weighted combination of the entries of $\hat{\mu}^\star$.

Locational marginal prices, continued

- The values of $\hat{\lambda}^\star$ and $\hat{\mu}^\star$ will vary with the choice of reference bus:
  - different representations of constraints result in different values of Lagrange multipliers.
- However, the resulting LMPs are independent of the choice of reference bus:
  - market clearing prices are independent of arbitrary choice of reference bus.

5.4.4 Example

- If bus 3 is the reference bus and if the capacity of Line A is 10 MW then the inequality constraint is $\hat{C}\hat{q} \leq \hat{d}$,
- where: $\hat{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $\hat{C} = [\sigma_{13} \;\; \sigma_{23}] = [2/3 \;\; 1/3]$, and $\hat{d} = [10]$.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.4. Three bus, three line network and shift factors to line A.

Example, continued

- Suppose that there are inelastic demands D_k at each bus.
- Since we have both demand and generation at each bus k, we will explicitly use Q_k for the generation at bus k.
- Suppose that the offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- We consider two demand conditions:

  (i) $D_1 = D_2 = 0, D_3 = 11$ MW, and
  (ii) $D_1 = D_2 = 0, D_3 = 30$ MW.

Example, continued

- The first-order necessary conditions for $Q_1^\star$, $Q_2^\star$, and $Q_3^\star$ to be optimal generations are:

$$
\begin{aligned}
&\exists \hat{\lambda}^\star \in \mathbb{R}, \exists \hat{\mu}^\star \in \mathbb{R}, \text{ such that:} \\
p_1(Q_1^\star) - \hat{\lambda}^\star + (2/3)\hat{\mu}^\star &= 0; \\
p_2(Q_2^\star) - \hat{\lambda}^\star + (1/3)\hat{\mu}^\star &= 0; \\
p_3(Q_3^\star) - \hat{\lambda}^\star &= 0; \\
\text{either } [2/3 \quad 1/3] \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \end{bmatrix} - [10] = 0, \text{ or } \hat{\mu}^\star &= 0, \text{ or both;} \\
\mathbf{1}^\dagger \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \\ Q_3^\star - D_3 \end{bmatrix} &= 0; \\
[2/3 \quad 1/3] \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \end{bmatrix} &\leq [10]; \text{ and} \\
\hat{\mu}^\star &\geq 0.
\end{aligned}
$$

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 11$ **MW**

- Offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- We claim that:

$$\begin{aligned}
Q_1^\star &= 6 \text{ MW}, \\
Q_2^\star &= 3 \text{ MW}, \\
Q_3^\star &= 2 \text{ MW}, \\
\hat{\lambda}^\star &= 6 \text{ \$/MWh}, \\
\hat{\mu}^\star &= 0 \text{ \$/MWh},
\end{aligned}$$

- satisfy the first-order necessary conditions.
- The LMPs at all buses are equal to \$6/MWh.

Example, continued

- To see this, note that:

$$
\begin{aligned}
p_1(Q_1^\star) - \hat{\lambda}^\star + (2/3)\hat{\mu}^\star &= (6 \times 1) - 6 + ((2/3) \times 0), \\
&= 0; \\
p_2(Q_2^\star) - \hat{\lambda}^\star + (1/3)\hat{\mu}^\star &= (3 \times 2) - 6 + ((1/3) \times 0), \\
&= 0; \\
p_3(Q_3^\star) - \hat{\lambda}^\star &= (2 \times 3) - 6, \\
&= 0; \\
\hat{\mu}^\star &= 0;
\end{aligned}
$$

Example, continued

$$
\begin{aligned}
\mathbf{1}^\dagger \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \\ Q_3^\star - D_3 \end{bmatrix} &= (6-0)+(3-0)+(2-11), \\
&= 0; \\
[2/3 \quad 1/3] \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \end{bmatrix} &= [2/3 \quad 1/3] \begin{bmatrix} 6-0 \\ 3-0 \end{bmatrix}, \\
&= [5], \\
&\leq [10]; \text{ and} \\
\hat{\mu}^\star &= 0, \\
&\geq 0.
\end{aligned}
$$

- Note that the transmission constraint is not binding for this demand condition.
- Since $\hat{\mu}^\star = 0$, the LMPs are all equal to $\hat{\lambda}^\star = 6$ \$/MWh.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- Offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- We claim that:

$$
\begin{aligned}
Q_1^\star &= 10 \text{ MW}, \\
Q_2^\star &= 10 \text{ MW}, \\
Q_3^\star &= 10 \text{ MW}, \\
\hat{\lambda}^\star &= 30 \text{ \$/MWh}, \\
\hat{\mu}^\star &= 30 \text{ \$/MWh},
\end{aligned}
$$

- satisfy the first-order necessary conditions.
- The LMPs at the buses are:

Bus 1 $\hat{\lambda}^\star - [\hat{C}_1]^\dagger \hat{\mu}^\star = 30 - (2/3)30 =$ **\$10/MWh,**

Bus 2 $\hat{\lambda}^\star - [\hat{C}_2]^\dagger \hat{\mu}^\star = 30 - (1/3)30 =$ **\$20/MWh,**

Bus 3 $\hat{\lambda}^\star = \$30/\text{MWh}$.

Example, continued

- To see this, note that:

$$
\begin{aligned}
p_1(Q_1^\star) - \hat{\lambda}^\star + (2/3)\hat{\mu}^\star &= (10 \times 1) - 30 + ((2/3) \times 30), \\
&= 0; \\
p_2(Q_2^\star) - \hat{\lambda}^\star + (1/3)\hat{\mu}^\star &= (10 \times 2) - 30 + ((1/3) \times 30), \\
&= 0; \\
p_3(Q_3^\star) - \hat{\lambda}^\star &= (10 \times 3) - 30, \\
&= 0; \\
\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} Q_1^\star - D_1 \\ Q_2^\star - D_2 \end{bmatrix} - [10] &= \begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} - [10], \\
&= 0;
\end{aligned}
$$

Example, continued

$$
\begin{aligned}
\mathbf{1}^{\dagger}\begin{bmatrix} Q_1^{\star}-D_1 \\ Q_2^{\star}-D_2 \\ Q_3^{\star}-D_3 \end{bmatrix} &= (10-0)+(10-0)+(10-30), \\
&= 0; \\
[2/3 \quad 1/3]\begin{bmatrix} Q_1^{\star}-D_1 \\ Q_2^{\star}-D_2 \end{bmatrix} &= [2/3 \quad 1/3]\begin{bmatrix} 10-0 \\ 10-0 \end{bmatrix}, \\
&= 10, \\
&\leq [10]; \text{ and} \\
\hat{\mu}^{\star} &= 30, \\
&\geq 0.
\end{aligned}
$$

- Note that the transmission constraint is binding for this demand condition.

5.4.5 Dependence of LMPs on offers

- As demand varies, it is typical for the binding constraints to vary:
  - at low demand, perhaps no line constraints are binding, while
  - at high demand, several transmission and generator capacity constraints may be binding.
- As demand and offers vary, the binding constraints will vary.
- For any given offers and demand, the offer-based transmission-constrained economic dispatch will result in some particular subset of the constraints being binding:
  - suppose that r_B constraints (out of the total r constraints) are binding,
  - let $\hat{C}_B \in \mathbb{R}^{r_B \times (n-1)}$ be the rows of $\hat{C}$ corresponding to the binding constraints,
  - let $\hat{d}_B \in \mathbb{R}^{r_B}$ be the entries of $\hat{d}$ corresponding to the binding constraints, and
  - let $\hat{\mu}_B^\star$ be the entries of $\hat{\mu}^\star$ corresponding to the binding constraints.

Dependence of LMPs on offers, continued

- If, among other things, the solution is *not* at a "corner" then for small changes in the demand and/or the offers the set of binding constraints will stay the same:
  - constraints that are binding remain binding for small changes, while
  - constraints that are not binding remain not binding for small changes.
- The first-order necessary conditions for the changed demand and/or offers are then (again ignoring upper and lower bound constraints on generation at reference bus):

$$\exists \hat{\lambda}^\star \in \mathbb{R}, \exists \hat{\mu}_B^\star \in \mathbb{R}^{r_B}, \text{ such that:}$$

$$\begin{aligned} \forall k \text{ not the reference bus}, p_k(q_k^\star) - \hat{\lambda}^\star + [\hat{C}_{Bk}]^\dagger \hat{\mu}_B^\star &= 0; \\ \text{For the reference bus}, p_k(q_k^\star) - \hat{\lambda}^\star &= 0; \\ \mathbf{1}^\dagger q^\star &= 0; \\ \hat{C}_B \hat{q}^\star &= \hat{d}_B, \end{aligned}$$

- where $\hat{C}_{Bk}$ is the k-th column of $\hat{C}_B$, consisting of the shift factors associated with injection at bus k.

Dependence of LMPs on offers, continued

- These are a set of equations and can be solved for values of $q^\star$, $\hat{\lambda}^\star$, and $\hat{\mu}_B^\star$.
- Note that the entries of $\hat{\mu}^\star$ corresponding to non-binding constraints are zero, so we can obtain the values of all the entries in the vector $\hat{\mu}^\star$.
- Define C_B to be the matrix obtained from $\hat{C}_B$ by adding a column of zeros corresponding to the reference bus.
- Define $p : \mathbb{R}^n \to \mathbb{R}^n$ to be the vector consisting of the offers p_k at all buses.
- Then the first-order necessary conditions are:

$$\begin{aligned} p(q^\star) - \mathbf{1}\hat{\lambda}^\star + [C_B]^\dagger \hat{\mu}_B^\star &= \mathbf{0}; \\ \mathbf{1}^\dagger q^\star &= 0; \\ \hat{C}_B \hat{q}^\star &= \hat{d}_B, \end{aligned}$$

- where $\mathbf{0} \in \mathbb{R}^n$ is the vector of all zeros.
- Focusing on the first set of constraints, suppose that $r_B \leq n-1$ and that the rows of C_B are linearly independent:
  - otherwise, Lagrange multipliers are not unique.
- We consider relationship between $p(q^\star)$, $\hat{\lambda}^\star$, and $\hat{\mu}_B^\star$.

Dependence of LMPs on offers, continued

- On re-arranging the first set of equations, we have:

$$\begin{bmatrix} \mathbf{1} & -[C_B]^\dagger \end{bmatrix} \begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} = p(q^\star),$$

$$\begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} \begin{bmatrix} \mathbf{1} & -[C_B]^\dagger \end{bmatrix} \begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} = \begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} p(q^\star),$$

on multiplying on the left,

$$\begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} = \left[\begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} \begin{bmatrix} \mathbf{1} & -[C_B]^\dagger \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} p(q^\star), \tag{5.5}$$

on multiplying through by the inverse.

Dependence of LMPs on offers, continued

- Repeating (5.5):

$$\begin{bmatrix} \hat{\lambda}^{\star} \\ \hat{\mu}_B^{\star} \end{bmatrix} = \left[\begin{bmatrix} \mathbf{1}^{\dagger} \\ -C_B \end{bmatrix} \left[\mathbf{1} \quad -[C_B]^{\dagger} \right] \right]^{-1} \begin{bmatrix} \mathbf{1}^{\dagger} \\ -C_B \end{bmatrix} p(q^{\star}).$$

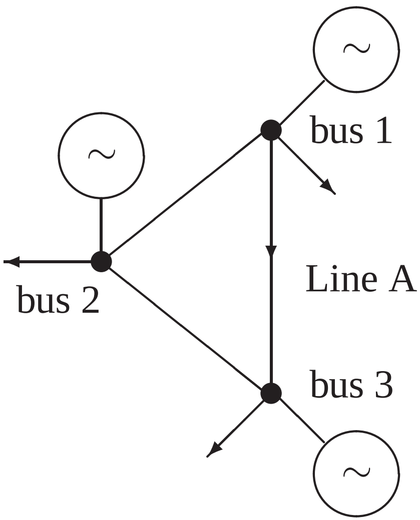
- That is, the Lagrange multipliers, and hence the LMPs, depend on the offer prices at the solution, $q^{\star}$, to the offer-based transmission-constrained economic dispatch with changed demand and/or offers:
  - effect of offer prices on LMPs is weighted by terms that depend on the shift factors,
  - note that the solution, $q^{\star}$, to offer-based transmission-constrained economic dispatch, and hence the offer prices $p(q^{\star})$, will generally change with changes in demand and/or offers.

Dependence of LMPs on offers, continued

- If a shift factor for injection at a generator is non-zero and the generator is not at maximum or minimum generation then its offer price will contribute to determining the LMPs at every bus:
  - when shift factors are between zero and one, situation is qualitatively *different* from the two zone model where shift factors were either zero or one,
  - in the two zone model, when the transmission constraint binds, generation offers in one zone do not contribute (directly) to determining the LMPs in the other zone,
  - offers indirectly contribute to determining the LMPs through determining whether or not constraint is binding,
  - intuition from two zone model must be used with caution!

5.4.6 Example

- Again consider the three bus example.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.5. Three bus, three line network and shift factors to line A.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 11$ **MW**

- Offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- Recall that:

$$\begin{aligned}
Q_1^\star &= 6 \text{ MW}, \\
Q_2^\star &= 3 \text{ MW}, \\
Q_3^\star &= 2 \text{ MW}, \\
\hat{\lambda}^\star &= 6 \text{ \$/MWh}, \\
\hat{\mu}^\star &= 0 \text{ \$/MWh},
\end{aligned}$$

- satisfy the first-order necessary conditions.
- The LMPs at all buses are equal to \$6/MWh.

Example, continued

- Consider relationship between offers and prices.
- Note that $\hat{\mu}_B^\star$ has no entries since no constraints are binding for demand of $D_3 = 11$ MW.
- Specializing (5.5) to this case, we obtain:

$$\begin{aligned}\hat{\lambda}^\star &= \left[\mathbf{1}^\dagger\mathbf{1}\right]^{-1}\mathbf{1}^\dagger p(Q^\star), \\ &= \frac{1}{3}\mathbf{1}^\dagger p(Q^\star),\end{aligned}$$

- so, $\hat{\lambda}^\star$, the price at each bus, is a linear combination of the (equal) offer prices at each bus.

Example, continued

- Suppose that the offer at bus 1 increases in price to:

$$\forall Q_1, p_1(Q_1) = Q_1 \times 2 \text{ \$/(MW)}^2\text{h}.$$

- We claim that:

$$\begin{aligned}
Q_1^\star &= 4.125 \text{ MW},\\
Q_2^\star &= 4.125 \text{ MW},\\
Q_3^\star &= 2.75 \text{ MW},\\
\hat{\lambda}^\star &= 8.25 \text{ \$/MWh, which is higher than before},\\
\hat{\mu}^\star &= 0 \text{ \$/MWh, which is the same as before.}
\end{aligned}$$

- satisfy the first-order necessary conditions.
- The LMPs at the buses are now all equal to \$8.25/MWh:
  - higher than before since offer at bus 1 has increased, but
  - dispatch is still such that offer prices at all buses are equal,
  - offer prices all equal to \$8.25/MWh.

Example, continued

- Increase in offer at bus 1 results in higher LMPs at all buses.
- The transmission constraint is still not binding.
- Moreover, we still have that:

$$\hat{\lambda}^{\star} = \left[(1/3) \quad (1/3) \quad (1/3) \right] p(Q^{\star}).$$

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- Offers at each bus are:

$$\begin{aligned} \forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}. \end{aligned}$$

- Recall that:

$$\begin{aligned} Q_1^\star &= 10 \text{ MW}, \\ Q_2^\star &= 10 \text{ MW}, \\ Q_3^\star &= 10 \text{ MW}, \\ \hat{\lambda}^\star &= 30 \text{ \$/MWh}, \\ \hat{\mu}^\star &= 30 \text{ \$/MWh}, \end{aligned}$$

- satisfy the first-order necessary conditions.

Example, continued

- Consider relationship between offers and prices.
- In this case, $\hat{\mu}_B^\star$ is the same as $\hat{\mu}_B^\star$ since the one constraint is binding.
- By (5.5), we have that:

$$
\begin{aligned}
\begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} &= \left[\begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} \begin{bmatrix} \mathbf{1} & -[C_B]^\dagger \end{bmatrix}\right]^{-1} \begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} p(Q^\star), \\
&= \begin{bmatrix} 3 & -1 \\ -1 & (5/9) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ -(2/3) & -(1/3) & 0 \end{bmatrix} p(Q^\star), \\
&= \begin{bmatrix} (5/6) & (3/2) \\ (3/2) & (9/2) \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -(2/3) & -(1/3) & 0 \end{bmatrix} p(Q^\star), \\
&= \begin{bmatrix} -(1/6) & (1/3) & (5/6) \\ -1 & 0 & 1 \end{bmatrix} p(Q^\star).
\end{aligned}
$$

- Note that the coefficients of the offer at bus 1 are *negative*, so that *increases* in offer prices at bus 1 *reduce* the values of $\hat{\lambda}^\star$ and $\hat{\mu}_B^\star$.

Example, continued

- Suppose that the offer at bus 1 increases in price to:

$$\forall Q_1, p_1(Q_1) = Q_1 \times 2 \text{ \$/(MW)}^2\text{h}.$$

- We claim that:

$$\begin{aligned}
Q_1^\star &= 9.23 \text{ MW}, \\
Q_2^\star &= 11.53 \text{ MW}, \\
Q_3^\star &= 9.23 \text{ MW}, \\
\hat{\lambda}^\star &= 27.69 \text{ \$/MWh, which is lower than before}, \\
\hat{\mu}^\star &= 13.85 \text{ \$/MWh, which is also lower than before.}
\end{aligned}$$

- satisfy the first-order necessary conditions, with LMPs:

Bus 1 $\hat{\lambda}^\star - [\hat{C}_1]^\dagger \hat{\mu}^\star = 27.69 - (2/3)13.85 =$ \$18.46/MWh, which is higher than before,

Bus 2 $\hat{\lambda}^\star - [\hat{C}_2]^\dagger \hat{\mu}^\star = 27.69 - (1/3)13.85 =$ \$23.08/MWh, which is higher than before,

Bus 3 $\hat{\lambda}^\star =$ \$27.69/MWh, which is lower than before.

Example, continued

- Increase in offer at bus 1 results in lower LMP at bus 3!
- The transmission constraint is still binding.
- Note that the offer at bus 1 on the "export" side of the constraint can affect the LMP at the demand at bus 3 on the "import" side of the constraint:
  - the system is not *divided* into independent "zones" by the transmission constraint!
  - offers in one ERCOT zone affect prices in other zones even when transmission constraints are binding,
  - we will return to this issue in the context of ad hoc analyses of market power.
- Moreover, at the new solution of offer-based transmission-constrained economic dispatch, we still have that:

$$\begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} = \begin{bmatrix} -(1/6) & (1/3) & (5/6) \\ -1 & 0 & 1 \end{bmatrix} p(Q^\star).$$

5.4.7 Sensitivity analysis

- In the last section, we considered the relationship between $p(q^\star)$, $\hat{\lambda}^\star$, and $\hat{\mu}_B^\star$ when demand and/or offers changed:
  - focused on changes in the values of $p(q^\star)$ rather than on changes in the values of $q^\star$, but
  - enabled qualitative analysis of the dependence of prices on offers.
- To apply the results quantitatively, we must calculate the changed value of $q^\star$ due to change in demand and/or offers:
  - as in previous example.
- In this section, we will apply sensitivity analysis to understand the relationship between $q^\star$, $\hat{\lambda}^\star$, and $\hat{\mu}_B^\star$:
  - will focus on injection at reference bus since that makes analysis simpler, but
  - can apply to any bus through change of reference bus.

Sensitivity analysis, continued

- We will calculate the derivative of the residual demand faced by a generator that is located at the reference bus:
  - the transmission-constrained residual demand derivative,
  - this will be the key to deriving an index of transmission-constrained market power that is analogous to (5.1).

Sensitivity analysis, continued

- Recall first-order necessary conditions, again focusing on binding constraints:

$$\exists \hat{\lambda}^\star \in \mathbb{R}, \exists \hat{\mu}_B^\star \in \mathbb{R}^{r_B}, \text{ such that:}$$

$$\begin{aligned}
\forall k \text{ not the reference bus}, p_k(q_k^\star) - \hat{\lambda}^\star + [\hat{C}_{Bk}]^\dagger \hat{\mu}_B^\star &= 0; \\
\text{For the reference bus}, p_k(q_k^\star) - \hat{\lambda}^\star &= 0; \\
\mathbf{1}^\dagger q^\star &= 0; \\
\hat{C}_B \hat{q}^\star &= \hat{d}_B.
\end{aligned}$$

- Now suppose that the generator at the reference bus were to slightly vary its offer, resulting in a different set of quantities and prices.
- As the offer from the generator at the reference bus changes, there would be changes in:
  - the injection at the reference bus,
  - the price $\hat{\lambda}^\star$ at the reference bus,
  - the injections elsewhere, and
  - the Lagrange multipliers $\hat{\mu}_B^\star$.

Sensitivity analysis, continued

- Equivalently, if the generator at the reference bus commits to meet the residual demand then, as the price $\hat{\lambda}^\star$ varies, the residual demand faced at the reference bus, $-\mathbf{1}^\dagger \hat{q}^\star$, will vary:
  - we ignore the *offer* at the reference bus and just consider the relationship between the injection at the reference bus and other injections and prices.
- We can think of $\hat{\lambda}^\star$ as an independent variable and think of $\hat{q}^\star$ and $\hat{\mu}^\star$ as dependent variables:
  - as the price at the reference bus varies, the residual demand, $-\mathbf{1}^\dagger \hat{q}^\star$, faced at the reference bus varies.
- Given, among other things, that we are not at a "corner" solution then "the implicit function theorem" enables us to evaluate the sensitivity of the dependence of $\hat{q}^\star$ and $\hat{\mu}^\star$ on $\hat{\lambda}^\star$:
  - see derivation in EE380N "Applied Optimization: Formulation and Algorithms for Engineering Systems Slides." Available from http://users.ece.utexas.edu/~baldick/classes/380N/Inequality%20II.pdf

Sensitivity analysis, continued

- We begin with the first-order necessary conditions, focusing on binding constraints, but:
  - ignoring the offer at the reference bus, since we are considering the dependence of the injection at the reference bus on the price at the reference bus, and
  - (temporarily) ignoring power balance, but will later use power balance to evaluate the derivative of residual demand.
- That is:

$$\begin{aligned} \hat{p}(\hat{q}^{\star}) - \mathbf{1}\hat{\lambda}^{\star} + [\hat{C}_B]^{\dagger}\hat{\mu}_B^{\star} &= \mathbf{0}; \\ \hat{C}_B\hat{q}^{\star} &= \hat{d}_B. \end{aligned}$$

- where $\hat{p} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is the vector consisting of the offers p_k at all buses except the reference bus, and
- where $\mathbf{0} \in \mathbb{R}^{n-1}$ is the vector of all zeros.

Sensitivity analysis, continued

- Viewing $\hat{q}^\star$ and $\hat{\mu}^\star$ as functions of $\hat{\lambda}^\star$, we can totally differentiate with respect to $\hat{\lambda}^\star$:

$$\begin{aligned}
\begin{bmatrix} p_1'(q_1^\star) & 0 & \cdots & 0 \\ 0 & p_2'(q_2^\star) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n'(q_n^\star) \end{bmatrix} \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} - \mathbf{1} + [\hat{C}_B]^\dagger \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star} &= \mathbf{0}; \\
\hat{C}_B \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} &= \mathbf{0}.
\end{aligned}$$

- where p_k' is the derivative of the offer p_k, and
- all matrices and vectors omit the reference bus.

Sensitivity analysis, continued

- Let:

$$\Lambda = \begin{bmatrix} p_1'(q_1^\star) & 0 & \cdots & 0 \\ 0 & p_2'(q_2^\star) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n'(q_n^\star) \end{bmatrix}^{-1}.$$

- Then, re-arranging the first equality, we obtain:

$$\frac{d\hat{q}^\star}{d\hat{\lambda}^\star} = \Lambda\left(\mathbf{1} - [\hat{C}_B]^\dagger \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star}\right).$$

Using the second equality,

$$\begin{aligned} \mathbf{0} &= \hat{C}_B \frac{d\hat{q}^\star}{d\hat{\lambda}^\star}, \\ &= \hat{C}_B \Lambda\left(\mathbf{1} - [\hat{C}_B]^\dagger \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star}\right), \\ &= \hat{C}_B \Lambda \mathbf{1} - \hat{C}_B \Lambda [\hat{C}_B]^\dagger \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star}. \end{aligned}$$

Sensitivity analysis, continued

$$\text{Re-arranging: } \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star} = \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger\right]^{-1} \hat{C}_B \Lambda \mathbf{1}.$$

$$\text{Finally: } \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} = \Lambda \left(\mathbf{1} - [\hat{C}_B]^\dagger \frac{d\hat{\mu}_B^\star}{d\hat{\lambda}^\star} \right),$$

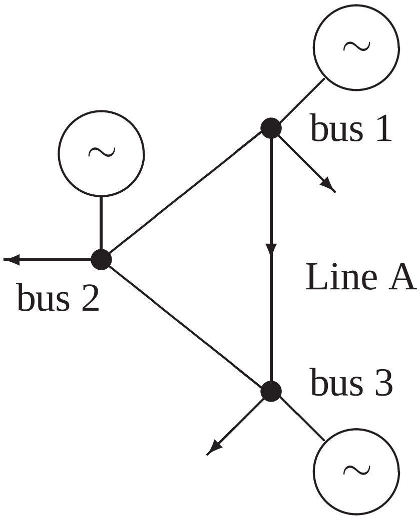
$$= \Lambda \left(\mathbf{1} - [\hat{C}_B]^\dagger \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger\right]^{-1} \hat{C}_B \Lambda \mathbf{1} \right).$$

- Now note that residual demand faced at the reference bus is, by definition, $-\mathbf{1}^\dagger \hat{q}^\star$.
- The derivative of the transmission-constrained residual demand faced at the reference bus is:

$$-\mathbf{1}^\dagger \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} = -\mathbf{1}^\dagger \Lambda \left(\mathbf{1} - [\hat{C}_B]^\dagger \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger\right]^{-1} \hat{C}_B \Lambda \mathbf{1} \right). \qquad (5.6)$$

5.4.8 Example

- Again consider the three bus example.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.6. Three bus, three line network and shift factors to line A.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 11$ **MW**

- Offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- Recall that:

$$\begin{aligned}
Q_1^\star &= 6 \text{ MW}, \\
Q_2^\star &= 3 \text{ MW}, \\
Q_3^\star &= 2 \text{ MW}, \\
\hat{\lambda}^\star &= 6 \text{ \$/MWh}, \\
\hat{\mu}^\star &= 0 \text{ \$/MWh},
\end{aligned}$$

- satisfy the first-order necessary conditions.
- The LMPs at all buses are equal to \$6/MWh.

Example, continued

- We calculate the residual demand derivative.
- We have that:

$$
\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_1, p_1'(Q_1) &= 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_2'(Q_2) &= 2 \text{ \$/(MW)}^2\text{h}, \\
\Lambda &= \begin{bmatrix} p_1'(q_1^\star) & 0 \\ 0 & p_2'(q_2^\star) \end{bmatrix}^{-1}, \\
&= \begin{bmatrix} 1 & 0 \\ 0 & (1/2) \end{bmatrix}.
\end{aligned}
$$

Example, continued

- In this case, there are no binding transmission constraints.
- Therefore, $\hat{C}_B$ has no rows and so (5.6) becomes:

$$\begin{aligned} -\mathbf{1}^\dagger \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} &= -\mathbf{1}^\dagger \Lambda \mathbf{1}, \\ &= -\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1/2) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ &= -(3/2). \end{aligned}$$

- Note that the residual demand derivative is negative since increasing price reduces the residual demand.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- Offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- Recall that:

$$\begin{aligned}
Q_1^\star &= 10 \text{ MW}, \\
Q_2^\star &= 10 \text{ MW}, \\
Q_3^\star &= 10 \text{ MW}, \\
\hat{\lambda}^\star &= 30 \text{ \$/MWh}, \\
\hat{\mu}^\star &= 30 \text{ \$/MWh},
\end{aligned}$$

- satisfy the first-order necessary conditions.

Example, continued

- We calculate the transmission-constrained residual demand derivative.
- We have that:

$$
\begin{aligned}
\hat{C}_B &= \left[(2/3) \quad (1/3) \right], \\
\hat{C}_B \Lambda [\hat{C}_B]^\dagger &= \left[(2/3) \quad (1/3) \right] \begin{bmatrix} 1 & 0 \\ 0 & (1/2) \end{bmatrix} \begin{bmatrix} (2/3) \\ (1/3) \end{bmatrix}, \\
&= [(1/2)], \\
\left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger \right]^{-1} &= [2], \\
[\hat{C}_B]^\dagger \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger \right]^{-1} \hat{C}_B \Lambda \mathbf{1} & \\
&= \begin{bmatrix} (2/3) \\ (1/3) \end{bmatrix} [2] \left[(2/3) \quad (1/3) \right] \begin{bmatrix} 1 & 0 \\ 0 & (1/2) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
&= \begin{bmatrix} (10/9) \\ (5/9) \end{bmatrix}.
\end{aligned}
$$

Example, continued

- By (5.6),

$$-\mathbf{1}^{\dagger}\frac{d\hat{q}^{\star}}{d\hat{\lambda}^{\star}} = -\mathbf{1}^{\dagger}\Lambda\left(\mathbf{1} - [\hat{C}_B]^{\dagger}\left[\hat{C}_B\Lambda[\hat{C}_B]^{\dagger}\right]^{-1}\hat{C}_B\Lambda\mathbf{1}\right),$$
$$= -[1 \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & (1/2) \end{bmatrix}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} (10/9) \\ (5/9) \end{bmatrix}\right),$$
$$= -(1/9).$$

- Note that this derivative is smaller in magnitude than in the case where transmission constraints were not binding:
  - illustrates general observation that residual demand becomes less elastic when transmission constraints bind, even if all derivatives of offers are constant.

5.4.9 Summary

- We have considered the optimality conditions for offer-based transmission-constrained economic dispatch.
- Showed that offers at buses on both "sides" of a transmission constraint can affect LMPs everywhere through (5.5):

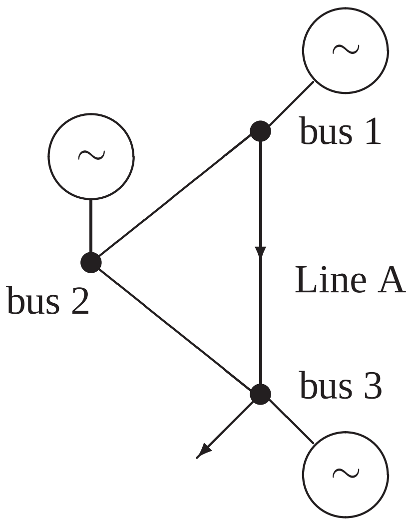
$$\begin{bmatrix} \hat{\lambda}^\star \\ \hat{\mu}_B^\star \end{bmatrix} = \left[\begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} \begin{bmatrix} \mathbf{1} & -[C_B]^\dagger \end{bmatrix} \right]^{-1} \begin{bmatrix} \mathbf{1}^\dagger \\ -C_B \end{bmatrix} p(q^\star).$$

- Derived the transmission-constrained residual demand (5.6):

$$-\mathbf{1}^\dagger \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} = -\mathbf{1}^\dagger \Lambda \left(\mathbf{1} - [\hat{C}_B]^\dagger \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger \right]^{-1} \hat{C}_B \Lambda \mathbf{1} \right).$$

Homework exercise, part a

- Suppose that all baseload generators (each 500 MW capacity) were located at bus 1; all intermediate generators (each 300 MW capacity) were located at bus 2; and all peaking generators (each 100 MW capacity) were located at bus 3.
- Line A has capacity 1800 MW.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.7. Three bus, three line network and shift factors to line A.

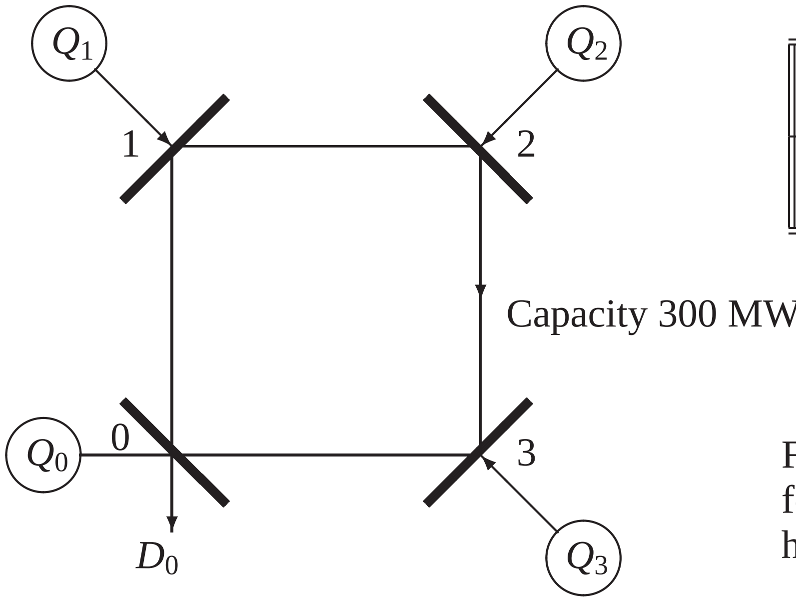
Homework exercise, part a, continued

- Suppose that the costs for the last homework exercise stayed exactly the same.
- Again assume that offers are required to be the same for each interval.
- Offers will be dispatched subject to transmission constraint that flow on line A is less than 1800 MW.
- All demand is at bus 3.
- Inverse demand at bus 3:

  Interval 1 $\forall Q, p^{\mathrm{d}}(Q) = \max\{50 - (Q - 2800)/2, 0\}$,
  Interval 2 $\forall Q, p^{\mathrm{d}}(Q) = \max\{75 - (Q - 3500)/2, 0\}$,
  Interval 3 $\forall Q, p^{\mathrm{d}}(Q) = \max\{500 - (Q - 4200)/2, 0\}$,
- where Q is in MW and $p^{\mathrm{d}}(Q)$ is in \$/MWh.
- Update your offers to maximize your profits.

Homework exercise, part b

- Consider the example four-line four-bus system shown.
- Bus 0 is the reference bus and location of demand:
  - injection at bus 3 and withdrawal at bus 0 causes “counterflow” on the 300 MW capacity line.



Shift factors to 300 MW line		
σ_{10}	σ_{20}	σ_{30}
0.2	0.4	−0.2

Fig. 5.8. Four-line four-bus network for homework exercise.

Homework exercise, part b

- Line limit constraint is:

$$\hat{C}\hat{q} \leq \hat{d}.$$

- where:

$$\begin{aligned} \hat{q} &= \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \\ \hat{C} &= \begin{bmatrix} \sigma_{10} & \sigma_{20} & \sigma_{30} \end{bmatrix}, \\ &= \begin{bmatrix} 0.2 & 0.4 & -0.2 \end{bmatrix}, \\ \hat{d} &= [300], \end{aligned}$$

- q_k is the net injection (equal to the generation Q_k) at buses $k = 1,2,3$.
- Net injection at bus $k = 0$ is:

$$q_0 = Q_0 - D_0.$$

Homework exercise, part b

- Suppose that the offers at the four buses are:

$$\begin{aligned}
\forall Q_0, p_0(Q_0) &= Q_0 \times 0.045 \text{ \$/(MW)}^2\text{h},\\
\forall Q_1, p_1(Q_1) &= Q_1 \times 0.04 \text{ \$/(MW)}^2\text{h} + 10 \text{ \$/MWh},\\
\forall Q_2, p_2(Q_2) &= Q_2 \times 0.035 \text{ \$/(MW)}^2\text{h},\\
\forall Q_3, p_3(Q_3) &= Q_3 \times 0.04 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- We consider two demand conditions:

  (i) $D_0 = 1000$ MW, and
  (ii) $D_0 = 4000$ MW.

- Use the excel solver or the MATLAB function `quadprog` to solve for offer-based transmission-constrained economic dispatch for each demand level.
- For each demand condition:
  - Show the dispatch, Lagrange multipliers, and LMPs.
  - Calculate the transmission-constrained residual demand derivative faced by the generator at the reference bus.

5.5 Ad hoc analyses of market power with transmission constraints

5.5.1 ERCOT market context

- In the upcoming ERCOT nodal market there are several *ad hoc* methods to assess market power.
- The "Element Competitiveness Index" (ECI) is styled as an *ex ante* test of competitiveness in the face of transmission constraints.
- Consider the system in figure 5.9, which is based on an example from Berry, Hobbs, Meroney, O'Neill, and Stewart and in Lin Xu and Yixin Yu:
  - four buses, $1, \ldots, 4$, each with generators,
  - buses 3 and 4 have demand,
  - all lines have equal impedance,
  - the line joining buses $e = 1$ and $i = 3$, has capacity $\overline{f}_{13} = 30$ MW, while
  - other lines have much larger capacity.

- Example system for ECI.

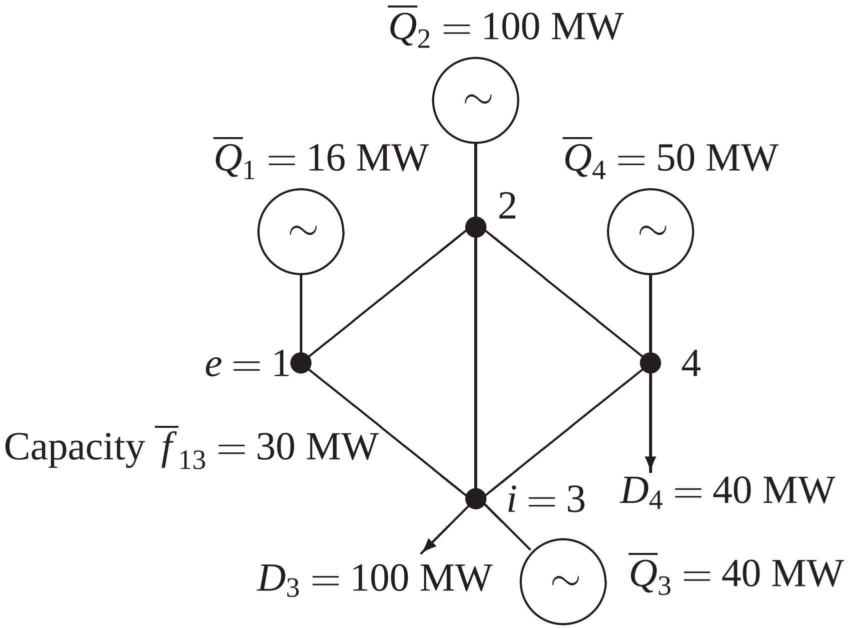


Fig. 5.9. Four bus, five line network based on an example from Berry, Hobbs, Meroney, O'Neill, and Stewart and in Lin Xu and Yixin Yu.

5.5.2 Shift factors for ECI test for example system

- The ECI test considers the DC shift factors for the various buses:
  - test consists of two parts,
  - we will perform the first part of the ECI test for the example system.
- For the $\overline{f}_{13} = 30$ MW capacity line, the required shift factors involve the "export" terminal of the line, bus $e = 1$, and the "import" terminal of the line, bus $i = 3$.
- Generation capacities, $\overline{Q}_k$ and forecast demands, D_k are also needed for the ECI calculation.

Bus	σ_{ki}	$\lvert\sigma_{ke}\rvert$	$\overline{Q}_k$	D_k
1	5/8	0	16	0
2	1/4	3/8	100	0
3	0	5/8	40	100
4	1/8	1/2	50	40

Table 5.1. *Data for ECI calculation for example system.*

5.5.3 First part of ECI test

(i) Determine the "effective load on the export side," D, by multiplying all load D_k at Electrical Buses k by the corresponding "import" shift factors σ_{ki}, so that:

$$\begin{aligned} D &= \sum_k D_k \sigma_{ki}, \\ &= 0 \times (5/8) + 0 \times (1/4) + 100 \times 0 + 40 \times (1/8), \\ &= 5\text{MW}. \end{aligned}$$

First part of ECI test, continued

(ii) Determine the "effective capacity needed to meet load and to supply power over the constraint on the export side" by:

(A) multiplying the generation capacity $\overline{Q}_k$ at each bus k by the corresponding "import" shift factors σ_{ki} to find the effective capacity $\overline{Q}_k^{\text{effective}\,e}$, so that:

$$
\begin{aligned}
\overline{Q}_1^{\text{effective}\,e} = 16 \times (5/8) &= 10 \text{ MW}, \\
\overline{Q}_2^{\text{effective}\,e} = 100 \times (1/4) &= 25 \text{ MW}, \\
\overline{Q}_3^{\text{effective}\,e} = 40 \times 0 &= 0 \text{ MW}, \\
\overline{Q}_4^{\text{effective}\,e} = 50 \times (1/8) &= 6.25 \text{ MW};
\end{aligned}
$$

(B) stacking the effective capacity in decreasing shift factor order (that is, bus 1, then bus 2, then bus 4, then bus 3); and then

(C) selecting sufficient effective capacity from the stack to meet the effective load on the export side plus the flow limit on the constraint, which is:

$$\begin{aligned} D + \overline{f}_{13} &= 5 + 30, \\ &= 35 \text{ MW}. \end{aligned}$$

Since the sum of the effective capacities at bus 1 and bus 2 is:

$$\begin{aligned} \overline{Q}_1^{\text{effective}\, e} + \overline{Q}_2^{\text{effective}\, e} &= 10 + 25, \\ &= 35 \text{ MW}, \end{aligned}$$

all of bus 1 and bus 2 effective capacity is necessary to meet the effective load plus the flow limit.
The generators at buses 1 and 2 are therefore not considered in determining the effective generation capacity available to resolve the constraint on the import side, leaving the generation at buses 3 and 4 available to resolve the constraint on the import side.

(iii) Determine the "effective generation capacities to resolve the constraint on the import side," $\overline{Q}_k^{\text{effective}\,i}$, by multiplying, for each Resource not excluded in the previous step and having shift factors greater than one-third of the highest Resource shift factor,

(A) the maximum capacity $\overline{Q}_k$, times

(B) the absolute value of shift factor of the bus to the export terminal $|\sigma_{ke}|$,

so that:

$$\overline{Q}_3^{\text{effective}\,i} = 40 \times (5/8) = 25 \text{ MW},$$
$$\overline{Q}_4^{\text{effective}\,i} = 50 \times (1/2) = 25 \text{ MW},$$

with total effective capacity:

$$\begin{aligned}\overline{Q}^{\text{effective}\,i} &= \overline{Q}_3^{\text{effective}\,i} + \overline{Q}_4^{\text{effective}\,i}, \\ &= 50 \text{ MW}.\end{aligned}$$

(iv) The ECI on the import side is equal to the sum of the square of the percentages of the effective capacities owned by each entity. Assuming that the generators at buses 3 and 4 are owned by different entities,

$$\begin{aligned} \text{ECI} &= \left(\frac{25}{50}\right)^2 + \left(\frac{25}{50}\right)^2, \\ &= 5000\%^2. \end{aligned}$$

(v) If the ECI is greater than 2,000 $\%^2$ on the import side then the constraint fails the competitive test for the month. Since the ECI is 5000 $\%^2$ in this case, the constraint fails the competitive test.

5.5.4 The ECI test

(i) The “effective load on the export side,” D, is the demand weighted by the shift factors:

- no first-principles justification of any relevance to market power assessment.

(ii) The “effective capacity needed to meet load and to supply power over the constraint on the export side” arbitrarily assigns the capacity of certain generators to meet the demand and to “fill up” the transmission capacity:

- ignores physical reality that generation collectively meets demand,
- no first-principles justification of any relevance to market power assessment,
- these generators are arbitrarily removed from further analysis of competitiveness with regard to offers.

ECI test, continued

(iii) The "effective generation capacities to resolve the constraint on the import side," $\overline{Q}_k^{\text{effective}\,i}$, is the generation capacity weighted by shift factors:

- no first-principles justification of any relevance to market power assessment.

(iv) The ECI test is an HHI test based on capacity aimed at deciding if there is "enough" competition in the transmission-constrained case:

- As discussed previously, there is no theoretical justification for capacity-based HHIs as a measure of market power:
  - when HHIs are based on market shares instead of capacities, there *is* a connection to the Cournot model,
  - but need to include residual demand elasticity and forward contract positions to assess market power.

(v) The HHI threshold is arbitrary.

- The second part of the ECI test is similar to the first part.

ECI test, continued

- The ECI test incorporates, through the shift factors, a proxy for the geographical extent of the market.
- However, the ECI test *omits* the fundamental drivers of market power:
  - the residual demand faced by market participants, and
  - the forward contract positions.
- For example, in step (ii)(C), the generators at buses 1 and 2 are arbitrarily removed from consideration in the final calculation of the ECI:
  - buses 3 and 4 are then essentially evaluated as being a duopoly market by the ECI test,
  - this ignores the fact that in this meshed system the offers at buses 1 and 2 also contribute to setting the LMP and also contribute to the residual demand at buses 3 and 4:
    - as discussed in the previous three bus example.

ECI test, continued

- The ECI test focuses on each line separately.
- Does not consider the effect of *interactions* between constraints on market power.
- In fact, the key economic issue is the incentives to market participants at particular *buses* due to potentially multiple interacting constraints:
  - The ECI test obscures the locus of the fundamental economic incentives.

5.6 Consideration of incentives when transmission constraints bind

- To model market power in the presence of transmission constraints, we focus on incentives.
- To do this we will use the derivative of the residual demand faced by market participants at an actual or estimated market clearing condition of offer-based transmission-constrained economic dispatch.
- We will need to consider offer information.
- We will also consider forward contract positions.

5.6.1 Transmission-constrained residual demand

- Section 5.4.7 showed how to calculate the derivative of the transmission-constrained residual demand.
- The analysis focused on the residual demand at the reference bus, but can be applied to any bus k.
- Write p^{d}_{-k} for the *inverse* transmission-constrained residual demand faced by a generator at bus k
- The derivative of p^{d}_{-k} is the reciprocal of the derivative of the transmission-constrained residual demand.

5.6.2 Profits

- Suppose that the generator at bus k has a forward contract for quantity Q_k^{f} at price p_k^{f}.
- In this case, the profit function becomes:

$$\forall Q_k \in \mathbb{R}, \pi_k^{\mathrm{f}}(Q_k) = (Q_k - Q_k^{\mathrm{f}}) p_{-k}^{\mathrm{d}}(Q_k) + Q_k^{\mathrm{f}} p_k^{\mathrm{f}} - c_k(Q_k).$$

- Assuming that:
  - sufficient conditions for maximization are satisfied,
  - functions p_{-k}^{d} and c_k are differentiable, and
  - generation capacity constraints are not binding at the profit maximizing condition,
- then we can find the maximum of profit by setting its derivative to zero.

5.6.3 Mark-up and market power index

- Re-arranging the maximum profit condition, we can obtain the transmission-constrained price-cost mark-up with a forward contract under the hypothesis that the generator was maximizing its profits:

$$p^{\mathrm{d}}_{-k}(Q_k) - c'_k(Q_k) = -(Q_k - Q^{\mathrm{f}}_k)\frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k). \tag{5.7}$$

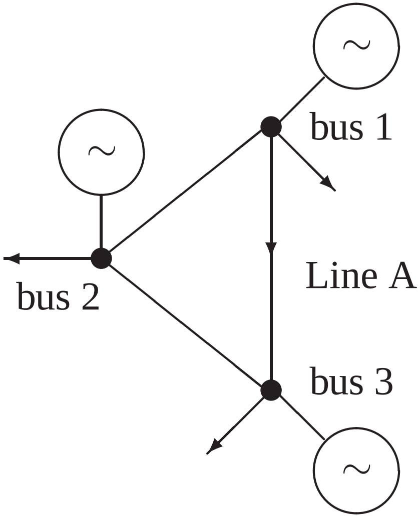
- The right-hand side of (5.7) is a transmission-constrained market power index:
  - if it is "large" according to some standard then a profit-maximizing generator has incentives to drive up prices "significantly" by withholding,
  - ignoring forward contracts.
- Again, any generator that is not at full production but such that the right-hand side of (5.7) is above a threshold would be flagged as of concern.

Mark-up and market power index, continued

- In the context of "market power mitigation," generators having a large value of the index (5.7) might be subject to limits on offer prices when transmission constraints are binding.
- In contrast to ECI, the index (5.7) has a concrete interpretation in terms of market power:
  - it represents the mark-up of price above marginal cost for a hypothetical profit maximizing generator.
- If forward contract information was not available then $-Q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k)$ could be used as an index instead:
  - however, any market power mitigation should be sensitive to the implications of forward contracting on market power.
- Excess wealth transfer can be estimated using (5.2).

5.6.4 Example

- Again consider the three bus example.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.10. Three bus, three line network and shift factors to line A.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 11$ **MW**

- Offers at each bus are:

$$\begin{aligned} \forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}. \end{aligned}$$

- Recall that:

$$\begin{aligned} Q_1^\star &= 6 \text{ MW}, \\ Q_2^\star &= 3 \text{ MW}, \\ Q_3^\star &= 2 \text{ MW}, \\ \hat{\lambda}^\star &= 6 \text{ \$/MWh}, \\ \hat{\mu}^\star &= 0 \text{ \$/MWh}, \end{aligned}$$

- satisfy the first-order necessary conditions.
- The LMPs at all buses are equal to \$6/MWh.

- From previous analysis, the derivative of the residual demand faced at the reference bus, bus 3, is $-(2/3)$.
- Therefore, at the market clearing conditions, the derivative of the inverse residual demand at bus 3 is:

$$\frac{\partial p^{\mathrm{d}}_{-3}}{\partial Q_k}(Q_k^\star) = -(3/2).$$

- Ignoring forward contracts, the index (5.7) is:

$$\begin{aligned} -Q_k^\star \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k^\star) &= -2 \times -(3/2), \\ &= 3 \text{ \$/MWh}. \end{aligned}$$

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- Offers at each bus are:

$$\begin{aligned} \forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\ \forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}. \end{aligned}$$

- Recall that:

$$\begin{aligned} Q_1^\star &= 10 \text{ MW}, \\ Q_2^\star &= 10 \text{ MW}, \\ Q_3^\star &= 10 \text{ MW}, \\ \hat{\lambda}^\star &= 30 \text{ \$/MWh}, \\ \hat{\mu}^\star &= 30 \text{ \$/MWh}, \end{aligned}$$

- satisfy the first-order necessary conditions.

Example, continued

- From previous analysis, the derivative of the residual demand faced at the reference bus, bus 3, is $-(1/9)$.
- Therefore, at the market clearing conditions the derivative of the inverse residual demand at bus 3 is:

$$\frac{\partial p^{\mathrm{d}}_{-3}}{\partial Q_k}(Q_k^\star) = -(9).$$

- Ignoring forward contracts, the index (5.7) is:

$$\begin{aligned} -Q_k^\star \frac{\partial p^{\mathrm{d}}_{-k}}{\partial Q_k}(Q_k^\star) &= -10 \times -(9), \\ &\approx 90 \text{ \$/MWh}. \end{aligned}$$

- Even with the slopes of the offers the same, when transmission constraints bind, market power increases significantly at bus 3 because:
  - the residual demand faced at bus 3 is less elastic, and
  - the generator at bus 3 is selling a greater quantity.

5.6.5 Second example

- As a second example of applying the index (5.7), consider the example that was used to illustrate the ECI.

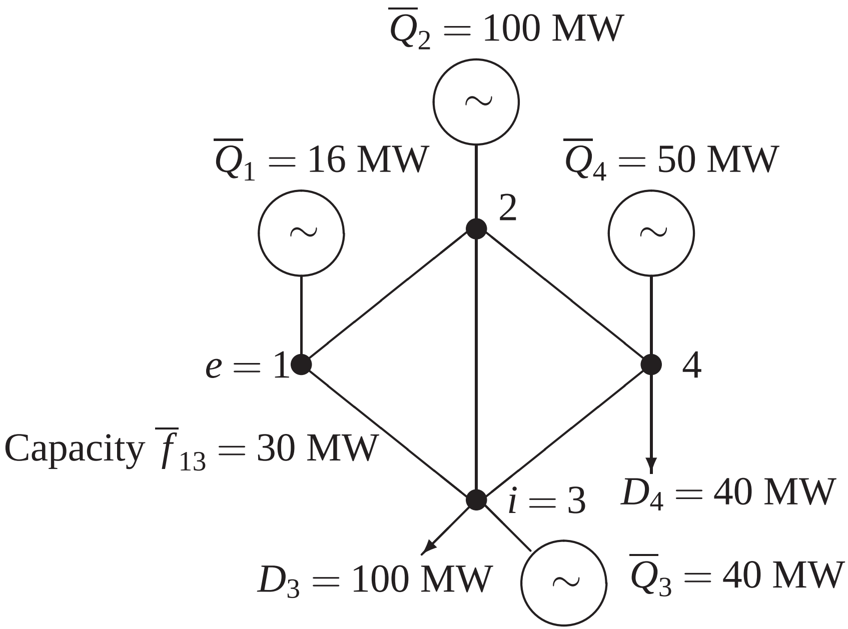


Fig. 5.11. Four bus, five line network based on an example from Berry, Hobbs, Meroney, O'Neill, and Stewart and in Lin Xu and Yixin Yu.

Second example, continued

- Xu and Baldick report residual demand derivatives at the transmission-constrained affine supply function equilibrium market clearing conditions as:

$$\begin{aligned}
\frac{\partial p^{\mathrm{d}}_{-1}}{\partial Q_1}(Q_1) &\approx -8 \text{ (\$/MWh)/MW}, \\
\frac{\partial p^{\mathrm{d}}_{-2}}{\partial Q_2}(Q_2) &\approx -0.9 \text{ (\$/MWh)/MW}, \\
\frac{\partial p^{\mathrm{d}}_{-3}}{\partial Q_3}(Q_3) &\approx -1.5 \text{ (\$/MWh)/MW}, \\
\frac{\partial p^{\mathrm{d}}_{-4}}{\partial Q_4}(Q_4) &\approx -3.7 \text{ (\$/MWh)/MW}.
\end{aligned}$$

- In this example, and in contrast to the arbitrary assumption in the ECI calculation, all four generators are marginal and all contribute to determining the residual demand at all buses.

Second example, continued

- Market clearing quantities at these buses are:

$$
\begin{aligned}
Q_1 &\approx 13 \text{ MW}, \\
Q_2 &\approx 93 \text{ MW}, \\
Q_3 &\approx 4 \text{ MW}, \\
Q_4 &\approx 30 \text{ MW}.
\end{aligned}
$$

Second example, continued

- The index (5.7) for these four buses, ignoring forward contracts, is:

$$
\begin{aligned}
-Q_1 \frac{\partial p^{\mathrm{d}}_{-1}}{\partial Q_1}(Q_1) &\approx 96 \text{ \$/MWh}, \\
-Q_2 \frac{\partial p^{\mathrm{d}}_{-2}}{\partial Q_2}(Q_2) &\approx 85 \text{ \$/MWh}, \\
-Q_3 \frac{\partial p^{\mathrm{d}}_{-3}}{\partial Q_3}(Q_3) &\approx 6 \text{ \$/MWh}, \\
-Q_4 \frac{\partial p^{\mathrm{d}}_{-4}}{\partial Q_2}(Q_4) &\approx 112 \text{ \$/MWh}.
\end{aligned}
$$

- With a \$10/MWh mark-up threshold, generators 1, 2, and 4 would be flagged for market power mitigation, whereas generator 3 would not.
- Although there is considerable market power in this small market, consistent with the ECI test, the diagnosis of market power specifically at buses 1, 2, and 4 contrasts with the ECI result that flagged buses 3 and 4 as a duopoly.

5.7 Ownership of generation at multiple buses

- So far assumed that any particular firm owns a generator or generators at only one bus, bus k.
- In reality, in LMP markets, assets will be owned at multiple buses and the LMPs at these different buses are related.
- For convenience, suppose that a firm owns generators at buses $k = 1, \ldots, s$, with $s < n$.
- Collect the production quantities $q_k, k = 1, \ldots, s$ into a vector $\check{q} \in \mathbb{R}^s$.
- The inverse residual demand faced by a generator at bus k now depends on the whole vector $\check{q}$, so that $p^{\mathrm{d}}_{-k} : \mathbb{R}^s \to \mathbb{R}$.
- The prices at each of the buses $k = 1, \ldots, s$, depend on the vector $\check{q}$.

5.7.1 Jacobian

- Collecting the inverse residual demands $p^{\mathrm{d}}_{-k}, k = 1, \ldots, s$, together into a vector function $p^{\mathrm{d}} : \mathbb{R}^s \to \mathbb{R}^s$, we can consider the dependence of the vector of inverse residual demands at buses $k = 1, \ldots, s$ on the generation $\check{q}$.
- Paralleling the earlier development, the incentives faced by the firm at the market clearing condition will involve the *Jacobian* of p^{d}; that is, $\frac{\partial p^{\mathrm{d}}}{\partial \check{q}}$.
- Calculation of the Jacobian is an extension of the sensitivity analysis in section 5.4.7 to the vector case, involving s calculations, one for each bus.

5.7.2 Index for ownership at multiple buses

- We consider how to extend the market power index (5.3) to the case of a firm owning generators at buses $k = 1, \ldots, s$ using the Jacobian.
- The production cost function of the firm is specified by the cost functions $c_k : \mathbb{R} \to \mathbb{R}, k = 1, \ldots, s$.
- Profit for the market participant is now specified by:

$$\forall \check{q} \in \mathbb{R}^s, \pi(\check{q}) = \sum_{k=1}^{s} q_k p^{\mathrm{d}}_{-k}(\check{q}) - c_k(q_k).$$

Index for ownership at multiple buses, continued

- As previously, assuming that:
  - sufficient conditions for maximization are satisfied,
  - p^{d} and $c_k, k = 1, \ldots, s$, are differentiable, and
  - generation capacity constraints are not binding at the profit maximizing condition,
- we can find the maximum of profit by setting the vector of its partial derivatives equal to zero:

$$\mathbf{0} = \frac{\partial \pi}{\partial \check{q}}(\check{q}).$$

Index for ownership at multiple buses, continued

- Focusing on the partial derivative with respect to q_ℓ, we obtain:

$$0 = \frac{\partial \pi}{\partial q_\ell}(\check{q}) = p^{\mathrm{d}}_{-\ell}(\check{q}) + \sum_{k=1}^{s} q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_\ell}(\check{q}) - c'_\ell(q_\ell),$$

- where $c'_\ell = \dfrac{\partial c_\ell}{\partial q_\ell}$ is the marginal cost of the generator owned by the firm at bus ℓ.
- Re-arranging, we obtain the price-cost mark-up at bus ℓ under the hypothesis that the generator was maximizing its profits:

$$p^{\mathrm{d}}_{-\ell}(\check{q}) - c'_\ell(q_\ell) = -\sum_{k=1}^{s} q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_\ell}(\check{q}). \tag{5.8}$$

- There is a value of the index (5.8) for each of the generators owned by the firm.

Index for ownership at multiple buses, continued

- Since cross-derivatives $\frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_\ell}$ for $k \neq \ell$ will typically be positive, it can be the case that, at some buses, profit maximization corresponds to a *mark-down* rather than a mark-up:
  - That is, for some buses ℓ, the index (5.8) may be negative.
  - For example, Hogan and Cardell *et al.* describe a case where a firm owning a generator at a bus e on the export side of a constraint offers at below marginal cost in order to congest the line and be able to offer well above marginal cost at a bus i on the import side.
  - That is, $\left(-\sum_{k=1}^{s} q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_e}(\check{q}) \right)$ is negative, while $\left(-\sum_{k=1}^{s} q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_i}(\check{q}) \right)$ is significantly positive.
  - The mark-up at each bus does not, in this case, give a full picture of the excess transfer of wealth from consumers to the firm.

Index for ownership at multiple buses, continued

- As a measure of market power for this case, consider (5.2), which estimated the excess transfer from consumers to a single generator.
- Generalizing (5.2), the net excess transfer from consumers to the firm, over and above competitive levels, is approximated by:

$$-\sum_{\ell=1}^{s} q_\ell \sum_{k=1}^{s} q_k \frac{\partial p^{\mathrm{d}}_{-k}}{\partial q_\ell}(\check{q}) = -[\check{q}]^\dagger \frac{\partial p^{\mathrm{d}}}{\partial \check{q}}(\check{q})\check{q}.$$

- As with (5.2), this must be viewed as only an approximate estimate of excess wealth transfer above competitive levels since competitive prices at each bus ℓ may deviate from the marginal costs c'_ℓ.

Index for ownership at multiple buses, continued

- Exercising market power by strategically congesting and forcing price on the importing side very high is not evident in the two zone model that assumes *separate* consideration of each market when constraints bind.
- That is, this mode of exercising market power represents a case that cannot easily be qualitatively analyzed in a two zone model.
- Moreover, in the context of an *ad hoc* index such as described in Section 5.5, the arbitrary removal of particular generators based on shift factors means that the significance of generator offers on the export side *interacting* with offers on the import side might be overlooked.

5.8 Pivotal offers

- The analysis so far considers the “small signal” issue of whether the slope of the residual demand is such that the first-order necessary conditions for profit maximization imply “economic” withholding that would significantly increase price over marginal costs.
- As discussed in the IMM report, there is also a concern that the “large signal” action of “physically” withholding capacity would lead to infeasibility.
- In the four bus system in Figure 5.11, for example, the 100 MW generator at bus 2 is pivotal since removing it from the system would leave only 106 MW of generation capacity but 140 MW of demand.
- Transmission constraints can make firms pivotal that would otherwise not be pivotal.
- Analysis of this case requires some sort of explicit consideration of withholding of capacity from market:
  - for example, check whether offer-based transmission-constrained dispatch is feasible if offer of a firm is removed from market.

5.9 Transmission and equilibrium analysis

- In the discussion of Nash equilibrium, we only briefly discussed the effect of transmission constraints.
- Borenstein, Bushnell, and Stoft (BBS) consider a Nash equilibrium analysis of a radial system:
  - we will follow their analysis, but recognize that non-radial systems can behave somewhat differently from radial systems,
  - as in example of strategically congesting to force prices very high on import side of constraint.
- Key observation in BBS is that transmission capacity between two zones can have "value" in mitigating market power even if no power flows between the zones:
  - the presence of the line causes prices to decline compared to the case of separated markets;
  - therefore, the societal benefit may not be linked to the level of usage of the line.

5.9.1 Symmetric two-firm model

- We will follow the notation of BBS, although it differs somewhat from the notation we have previously used.
- Assume two identical markets, N and S.
- Inverse-demand in each market is given by $P(Q^S)$ and $P(Q^N)$, respectively, where Q^S and Q^N are the quantities consumed in S and N, respectively.
- Firms n and s, located in N and S respectively, have identical costs $C(q^n)$ and $C(q^s)$, where q^n and q^s are the generations by the firms n and s.
- Assume generation quantities q^n and q^s are the strategic variables, so that the model is Cournot.

Symmetric two-firm model, continued

- If there were no transmission capacity between the two zones then each firm would act as a monopolist in its own market:
  - n would maximize $P(q^n)q^n - C(q^n)$,
  - s would maximize $P(q^s)q^s - C(q^s)$,
  - because of symmetry, each firm would produce the same,
  - the price would be the same in both markets and demand would be the same in both markets.

Symmetric two-firm model, continued

- If there were infinite (or large) transmission capacity between the two zones the market would be *merged* and there would be a Cournot duopoly:
  - because of symmetry, each firm would produce the same,
  - the price would be the same in both markets and demand would be the same in both markets,
  - so the flow on the line would be zero!
  - the "merged-market Cournot duopoly."
- Prices in the merged-market Cournot duopoly are lower than monopoly prices in each separate market:
  - the "separated-market monopoly prices,"
  - presence of the line decreases prices (and increases quantities) compared to separated-market monopoly prices despite no net flow on the line.

Symmetric two-firm model, continued

- We consider capacities K that are neither zero nor very large.
- First suppose that generation is equal in each zone so that $q^n = q^s$:
  - because of demand symmetry, there would be equal demand in each zone and zero flow on the line.
- More generally, if $q^n \approx q^s$ then the the flow on the line would be less than K:
  - demand and prices would still be equal in each zone.

Symmetric two-firm model, continued

- However, if $q^n < q^s - 2K$ we claim that the demand cannot be equal in each zone:
  - suppose demand in each zone was $Q^N = Q^S = (q^n + q^s)/2$,
  - so flow from S to N would be:

$$
\begin{aligned}
Q^N - q^n &= (q^n + q^s)/2 - q^n, \\
&= (q^s - q^n)/2, \\
&> K,
\end{aligned}
$$

  - which exceeds transmission capacity.
  - So demand cannot be equal, flow from S to N must equal K.
  - Inverse demand faced by n has slope of inverse demand in zone N alone.

Symmetric two-firm model, continued

- Similarly, if $q^n > q^s + 2K$ then:
  - flow from N to S equals K.
  - Inverse demand faced by n has slope of inverse demand in zone N alone.
- Figure 1 of BBS shows the variation in the slope of the inverse demand faced by n as q^n varies:
  - for $q^s - 2K \leq q^n \leq q^s + 2K$, flow is not at limit, inverse demand faced by n is due to demand response of merged markets,
  - for $q^n < q^s - 2K$, flow is at limit from S to N, inverse demand faced by n is due to demand response in N alone,
  - for $q^n > q^s + 2K$, flow is at limit from N to S, inverse demand faced by n is due to demand response in N alone.
- Symmetric observations apply to inverse demand faced by s as q^s varies.

5.9.2 *Best response*

- Recall the best response calculation in Hortaçsu and Puller:
  - calculated the supply function that was the best response to a *particular* choice of supply functions of other market participants.
- Could also calculate the best quantity response of firm n to the generation of firm s:
  - the "best response curve" for n is the graph of the best quantity response versus various values of q^s,
- Similarly, can calculate the best quantity response of firm s to the generation of firm n:
  - the "best response curve" for s.

5.9.3 Equilibrium

- The intersection of the best response curves, if there is one, is the Nash equilibrium.
- In the case of a small value of K:
  - for low values of q^s, best response of n is to generate enough to cause flow on line to equal capacity and to be monopoly supplier to local demand in N,
  - for intermediate values of q^s, the best response of n is to increase production as q^s increases,
  - above a threshold value of q^s, the best response of n involves a drop in production.
- Situation is illustrated in figure 2 of BBS:
  - best response of n to q^s is shown as $BR^n(q^s)$,
  - best response of s to q^n is shown as $BR^s(q^n)$.
- There is no intersection and therefore no pure strategy equilibrium.

Equilibrium, continued

- In the case of a intermediate values of K:
  - the sloping section of best response may intersect the merged-market Cournot duopoly best response before the threshold is reached for a discontinuous drop in production.
- Shown in figure 3 of BBS.
- Still may be no intersection of best responses and therefore no pure strategy equilibrium.
- For larger values of K, above a threshold $K^\star$, there is a pure strategy Nash equilibrium:
  - the Nash equilibrium is equal to the merged-market Cournot duopoly equilibrium as shown in figure 4 of BBS.
- BBS characterizes threshold $K^\star$.

5.9.4 Example

- Suppose that demand in each market is $P(Q) = 10 - Q$.
- Suppose marginal costs are are zero.
- Separated-market monopoly:
  - with no line joining the markets, profit would be:

$$\begin{aligned} P(q^n)q^n &= 10q^n - (q^n)^2, \text{ for firm } n\text{, and} \\ P(q^s)q^s &= 10q^s - (q^s)^2, \text{ for firm } s. \end{aligned}$$

  - Profit maximizing conditions yield $q^n = q^s = 5$ and prices 5 in both markets.

Example, continued

- Merged-market Cournot duopoly:
  - with a large capacity line joining the markets, each generator supplies half of demand in each market, profit would be:

$$P((q^n+q^s)/2)q^n, \text{ for firm } n\text{, and}$$
$$P((q^n+q^s)/2)q^s, \text{ for firm } s.$$

  - Profit maximizing conditions yield $q^n = q^s = 6.67$ and prices 3.33 in both markets.
  - Threshold capacity is $K^\star = 0.57$.
  - Note that the increase in production of the merged-market Cournot duopoly over the separated-market monopoly is 1.67, which exceeds $K^\star$.

5.9.5 Summary

- For large enough K, above a threshold $K^\star$, there is a pure strategy equilibrium corresponding to the merged-market Cournot duopoly:
  - increase in production over the separated-market monopoly production is greater than $K^\star$.
- For values of K less than $K^\star$ there is no pure strategy equilibrium:
  - simulation of mixed strategy equilibrium indicates that increase in production over the separated-market monopoly production increases with K and exceeds K.

5.9.6 Extensions

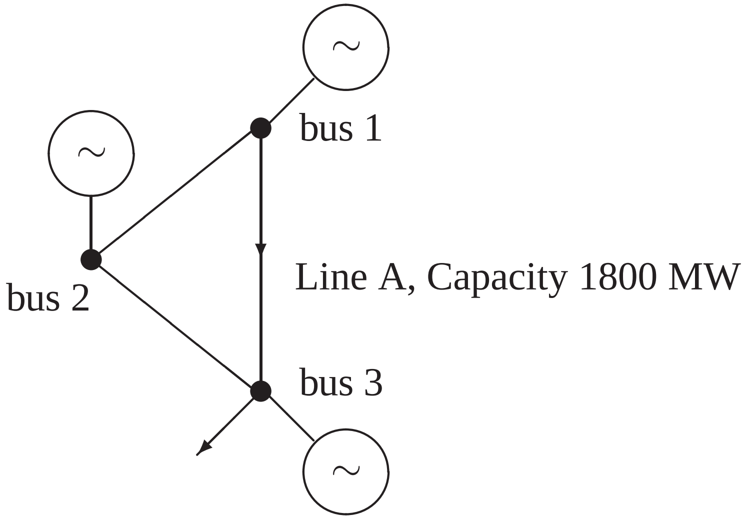
- More realistic systems have asymmetric markets:
  - Separated-market monopoly prices differ in N and S.
  - In this case, a pure strategy equilibrium can exist for small values of capacity K.
  - At small values of capacity, the pure strategy will involve power flowing from lower-price to higher-price market.
  - At high values of capacity, the merged-market Cournot duopoly equilibrium will occur.
  - At intermediate values of capacity then, depending on particulars of market and value of K:
    - there may be an asymmetric pure-strategy equilibrium with quantities lower than the merged-market Cournot duopoly, the "passive/aggressive equilibrium, as shown in figure 5 of BBS,
    - there may be no pure-strategy equilibrium, as shown in figure 6 of BBS, or
    - there may be both the merged-market Cournot duopoly and the passive/aggressive equilibrium, as shown in figure 7 of BBS.

5.9.7 Application to California market

- Represent California as two zones, North and South, (represents most significant transmission constraints in California).
- Consider demand in September and December.
- Analysis suggests that at lowest demand levels in December there is no pure strategy Cournot equilibrium.
- At other demand levels in September and December there is a pure strategy Cournot equilibrium with flow at limit in South to North direction.

Homework exercise, part a: Due Tuesday, December 4, by 5pm

- Continue to suppose that all baseload generators are located at bus 1; all intermediate generators are located at bus 2; and all peaking generators are located at bus 3.
- Update your offers to maximize your profits.

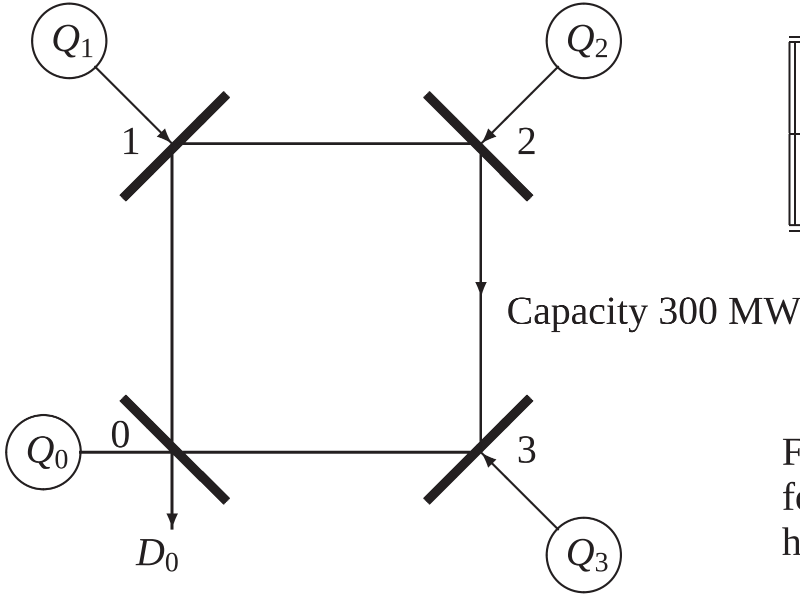


Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.12. Three bus, three line network and shift factors to line A.

Homework exercise, part b: Solution

- Consider the example four-line four-bus system shown.
- Bus 0 is the reference bus and location of demand:
  - injection at bus 3 and withdrawal at bus 0 causes "counterflow" on the 300 MW capacity line.



Shift factors to 300 MW line		
σ_{10}	σ_{20}	σ_{30}
0.2	0.4	−0.2

Fig. 5.13. Four-line four-bus network for homework exercise.

Homework exercise, part b: Solution, continued

- Line limit constraint is:

$$\hat{C}\hat{q} \leq \hat{d}.$$

- where:

$$\begin{aligned}
\hat{q} &= \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \\
\hat{C} &= \begin{bmatrix} \sigma_{10} & \sigma_{20} & \sigma_{30} \end{bmatrix}, \\
&= \begin{bmatrix} 0.2 & 0.4 & -0.2 \end{bmatrix}, \\
\hat{d} &= [300],
\end{aligned}$$

- q_k is the net injection (equal to the generation Q_k) at buses $k = 1, 2, 3$.
- Net injection at bus $k = 0$ is:

$$q_0 = Q_0 - D_0.$$

Homework exercise, part b: Solution, continued

- Suppose that the offers at the four buses are:

$$\begin{aligned}
\forall Q_0, p_0(Q_0) &= Q_0 \times 0.045 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_1, p_1(Q_1) &= Q_1 \times 0.04 \text{ \$/(MW)}^2\text{h} + 10 \text{ \$/MWh}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 0.035 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 0.04 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- We consider two demand conditions:

  (i) $D_0 = 1000$ MW, and

  (ii) $D_0 = 4000$ MW.

- For each demand condition, we calculate:
  - the dispatch, Lagrange multipliers, and LMPs.
  - the transmission-constrained residual demand derivative faced by the generator at the reference bus.

Homework exercise, part b: Solution, continued

(i) $D_0 = 1000$ MW.

- We can use MATLAB or, alternatively, directly seek a solution of the first-order necessary conditions.
- For this demand, let us guess that the transmission constraint is not binding.
- Under this assumption, the LMPs are the same at all buses.
- Solving the first-order necessary conditions results in:

$$\begin{aligned}\hat{\lambda}^\star &= 12.4016,\\ Q_0^\star &= 275.59,\\ Q_1^\star &= 60.04,\\ Q_2^\star &= 354.33,\\ Q_3^\star &= 310.04.\end{aligned}$$

- Given these generations, the flow on the constrained line is 91.7 MW, which is consistent with the guess.
- Therefore, we have found the solution and the LMPs are $12.40/MWh at each bus.

Homework exercise, part b: Solution, continued

- In this case, there are no binding transmission constraints.
- Therefore, $\hat{C}_B$ has no rows and so (5.6) becomes:

$$
\begin{aligned}
-\mathbf{1}^\dagger \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} &= -\mathbf{1}^\dagger \Lambda \mathbf{1}, \\
&= -78.57 \text{ MW/(\$/MWh)}.
\end{aligned}
$$

- Note that the residual demand derivative is negative since increasing price reduces the residual demand.

Homework exercise, part b: Solution, continued

(ii) $D_0 = 4000$ MW.

- For this demand, let us guess that the transmission constraint is binding.
- Solving the first-order necessary conditions results in:

$$\begin{aligned}
\hat{\lambda}^\star &= 45, \\
\hat{\mu}^\star &= 25, \\
Q_0^\star &= 1000, \\
Q_1^\star &= 750, \\
Q_2^\star &= 1000, \\
Q_3^\star &= 1250.
\end{aligned}$$

- Given these generations, the flow on the constrained line is 300 MW, which is consistent with the guess.

Homework exercise, part b: Solution, continued

- Therefore, we have found the solution and the LMPs are:

$$\begin{aligned}
\mathrm{LMP}_0^\star &= \hat{\lambda}^\star = 45, \\
\mathrm{LMP}_1^\star &= \hat{\lambda}^\star - (0.2) \times \hat{\mu}^\star = 40, \\
\mathrm{LMP}_2^\star &= \hat{\lambda}^\star - (0.4) \times \hat{\mu}^\star = 35, \\
\mathrm{LMP}_3^\star &= \hat{\lambda}^\star - (-0.2) \times \hat{\mu}^\star = 50.
\end{aligned}$$

- In this case $\hat{C}_B = \hat{C}$ and by (5.6),

$$\begin{aligned}
-\mathbf{1}^\dagger \frac{d\hat{q}^\star}{d\hat{\lambda}^\star} &= -\mathbf{1}^\dagger \Lambda \left(\mathbf{1} - [\hat{C}_B]^\dagger \left[\hat{C}_B \Lambda [\hat{C}_B]^\dagger \right]^{-1} \hat{C}_B \Lambda \mathbf{1} \right), \\
&= -58.69 \text{ MW/(\$/MWh)}.
\end{aligned}$$

- Note that this derivative is smaller in magnitude than in the case where transmission constraints were not binding:
  - again illustrates general observation that residual demand becomes less elastic when transmission constraints bind, even if all derivatives of offers are constant.

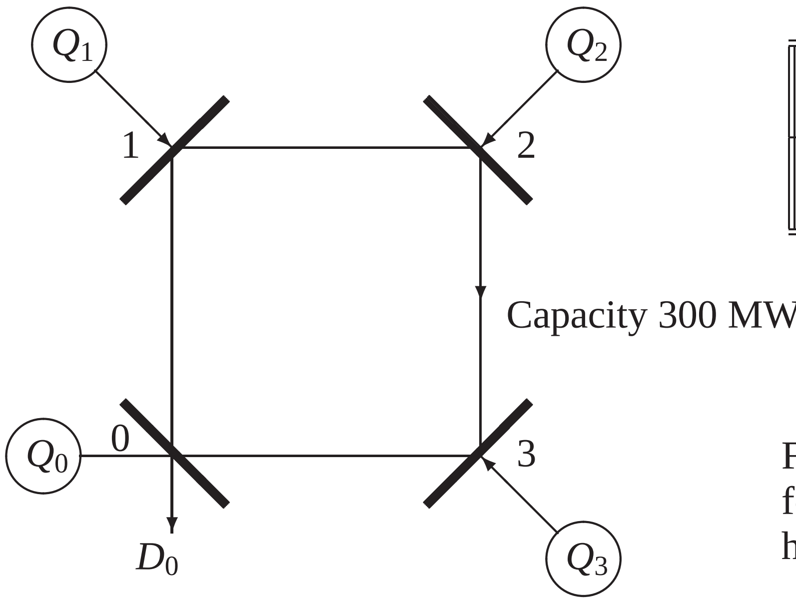
Homework exercise, part b: Due Wednesday, December 5

- Again consider the example four-line four-bus system:

  (i) $D_0 = 1000$ MW, and

  (ii) $D_0 = 4000$ MW.

- For each demand condition calculate the index (5.7) for the generator at the reference bus.



Shift factors to 300 MW line		
σ_{10}	σ_{20}	σ_{30}
0.2	0.4	-0.2

Fig. 5.14. Four-line four-bus network for homework exercise.

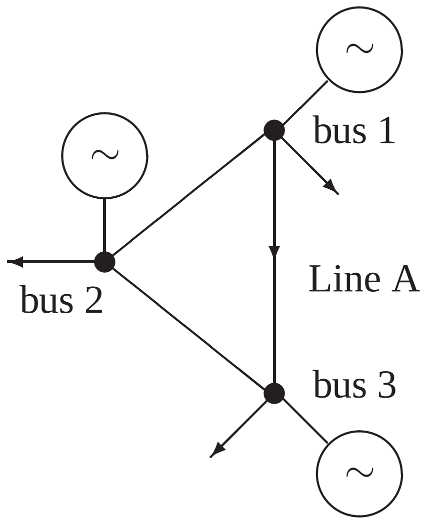
5.10 Transmission, equilibrium, and transmission rights

5.10.1 Congestion rent

- When transmission constraints bind, the total payment by demand based on LMPs is greater than the total payment to generators based on LMPs.
- This difference is called the "congestion rent:"
  - as discussed in the transmission-constrained homework exercise, part a,
  - the difference is sometimes called the "merchandising surplus" and sometimes (erroneously) called the "congestion cost."
- Since demand pays the ISO and the ISO pays the generators, the congestion rent accrues to the ISO:
  - we will see that the ISO pays out the congestion rent to transmission rights holders.

5.10.2 Example

- Again consider the three bus example.
- We have: $\hat{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \hat{C} = [\sigma_{13} \;\; \sigma_{23}] = [2/3 \;\; 1/3]$, and $\hat{d} = [10]$.



Shift factors to line A		
σ_{12}	σ_{13}	σ_{23}
1/3	2/3	1/3

Fig. 5.15. Three bus, three line network and shift factors to line A.

Example, continued

- There are inelastic demands D_k at each bus.
- The offers at each bus are:

$$\begin{aligned}
\forall Q_1, p_1(Q_1) &= Q_1 \times 1 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_2, p_2(Q_2) &= Q_2 \times 2 \text{ \$/(MW)}^2\text{h}, \\
\forall Q_3, p_3(Q_3) &= Q_3 \times 3 \text{ \$/(MW)}^2\text{h}.
\end{aligned}$$

- Consider the two demand conditions:

  (i) $D_1 = D_2 = 0, D_3 = 11$ MW, and
  (ii) $D_1 = D_2 = 0, D_3 = 30$ MW.

Demand $D_1 = D_2 = 0, D_3 = 11$ **MW**

- The solution is:

$$\begin{aligned} Q_1^\star &= 6 \text{ MW}, \\ Q_2^\star &= 3 \text{ MW}, \\ Q_3^\star &= 2 \text{ MW}, \\ \hat{\lambda}^\star &= 6 \text{ \$/MWh}, \\ \hat{\mu}^\star &= 0 \text{ \$/MWh}. \end{aligned}$$

- The LMPs at all buses are equal to \$6/MWh.
- Since total demand equals total generation, the demand payments of 11 MW $\times$ \$6/MWh $=$ 66 \$/h are equal to the payments to generation of:

$$(6+3+2) \text{ MW} \times \$6/\text{MWh} = 66 \text{ \$/h},$$

- Congestion rent is zero when there is no congestion.

Example, continued

Demand $D_1 = D_2 = 0, D_3 = 30$ **MW**

- The solution is:

$$
\begin{aligned}
Q_1^\star &= 10 \text{ MW},\\
Q_2^\star &= 10 \text{ MW},\\
Q_3^\star &= 10 \text{ MW},\\
\hat{\lambda}^\star &= 30 \text{ \$/MWh},\\
\hat{\mu}^\star &= 30 \text{ \$/MWh}.
\end{aligned}
$$

- The LMPs at the buses are:
  Bus 1 $\hat{\lambda}^\star - [\hat{C}_1]^\dagger \hat{\mu}^\star = 30 - (2/3)30 = \$10/\text{MWh},$
  Bus 2 $\hat{\lambda}^\star - [\hat{C}_2]^\dagger \hat{\mu}^\star = 30 - (1/3)30 = \$20/\text{MWh},$
  Bus 3 $\hat{\lambda}^\star = \$30/\text{MWh}.$
- The congestion rent is:

$$
\begin{aligned}
30 \times 30 \text{ \$/h} - [(10 \times 10) + (10 \times 20) + (10 \times 30) \text{ \$/h}] &= 900 - 600 \text{ \$/h},\\
&= 300 \text{ \$/h}.
\end{aligned}
$$

5.10.3 Financial transmission rights

- Consider a generator with a forward contract for sale of energy to demand.
- In the discussion of forward contracts, we ignored transmission and showed that the forward contract hedged the position of the generator and the demand for the contract quantity:
  - implicitly predicated on both the generator and demand being exposed to the same LMP.
- Unless the generator and demand are at the same bus, they will be exposed to different LMPs whenever transmission constraints are binding:
  - in ERCOT nodal, even generator and demand at the same bus are not exposed to the same prices, since generators are paid the LMP, but demand is paid a demand-weighted average of demand LMPs in its zone.
- The forward *energy* contract holders are exposed to the difference in LMPs between their buses:
  - forward energy contract does not hedge against transmission price risk.

Financial transmission rights, continued

- For example, suppose that a generator and demand agree to a forward contract for 10 MW at \$50/MWh:
  - that is, the generator commits to providing 10 MW to demand at a net price paid by demand of \$50/MWh.
- If the LMP at the demand is LMP_d then the demand will pay to the ISO:

$$10 \text{ MW } \times \mathrm{LMP}_\mathrm{d}.$$

  - To make the *net* payment by the demand (sum of demand payments to the ISO and to the generator) equal to 10 MW $\times$ \$50/MWh, the demand should pay to the generator:

$$10 \text{ MW} \times (50 \text{ \$/MWh} - \mathrm{LMP}_\mathrm{d}).$$

  - the forward contract is implemented as an agreement by the demand to pay this amount, called a "contract for differences."

Financial transmission rights, continued

- If the LMP at the generator is LMP_g then the net payment to the generator will be due to the energy payment by the ISO and the contract for differences payment by the demand:

$$
\begin{aligned}
& 10\text{ MW} \times \text{LMP}_\text{g} + 10\text{ MW} \times (50\text{ \$/MWh} - \text{LMP}_\text{d}) \\
= \ & 10\text{ MW} \times [50\text{ \$/MWh} - (\text{LMP}_\text{d} - \text{LMP}_\text{g})].
\end{aligned}
$$

- If there is no congestion then $\text{LMP}_\text{d} = \text{LMP}_\text{g}$ and the net payment to the generator is 10 MW $\times$50 \$/MWh.
- If transmission limits are binding then $\text{LMP}_\text{d} \neq \text{LMP}_\text{g}$ and the net payment to the generator is different to 10 MW $\times$ \$50/MWh:
  - for most generators most of the time, when transmission limits are binding, $\text{LMP}_\text{d} > \text{LMP}_\text{g}$ and the net payment to the generator will be *less* than 10 MW $\times$ \$50/MWh,
  - the generator is exposed to the contract quantity multiplied by the difference in LMPs,
  - the generator is exposed to "transmission price risk," where the transmission price is defined by the difference in LMPs.

Financial transmission rights, continued

- To hedge this transmission price risk, “financial transmission rights” (FTRs) were invented:
  - forward financial contract for transmission prices,
  - FTRs pay out based on differences in LMPs to hedge the transmission prices,
  - key insight is to use the congestion rent to fund the FTRs.
- Called “congestion revenue rights” (CRRs) in ERCOT:
  - see details in “Course notes for EE394V Restructured Electricity Markets: Transmission pricing and hedging,” Fall 2006. Available from http://users.ece.utexas.edu/~baldick/classes/394V/Transmission.pdf

5.10.4 *Effect on incentives*

- Recall that forward financial energy contracts affect market power.
- Similarly, FTRs affect market power.
- Discussed in Manho Joung and Ross Baldick, "The Competitive Effects of Ownership of Financial Transmission Rights in a Deregulated Electricity Industry."

5.11 Summary

(i) Modeling market power, revisited,
(ii) Transmission constraints and geographical market power,
(iii) Shift factors and the DC power flow,
(iv) Offer-based transmission-constrained economic dispatch,
(v) Ad hoc analyses of market power with transmission constraints,
(vi) Consideration of incentives when transmission constraints bind,
(vii) Ownership of generation at multiple buses,
(viii) Pivotal offers,
(ix) Transmission and equilibrium analysis,
(x) Transmission, equilibrium, and transmission rights.