Transmission-Constrained Inverse Residual Demand Jacobian Matrix in Electricity Markets

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***Abstract*—A generation firm in an electricity market may own multiple generators located at multiple locations. This paper generalizes the concept of transmission-constrained residual demand from a single generator's perspective to that of a generation firm. We calculate the derivative of a generation firm's inverse residual demand function, i.e., the Jacobian matrix, based on a multi-parameter sensitivity analysis of the optimal power flow solution, and characterize some of its properties. This Jacobian matrix provides valuable information, such as in characterizing a generation firm's profit maximizing strategy. We apply the bundle-Newton method utilizing the Jacobian matrix to find a generation firm's maximum profit. The effectiveness and performance of the algorithm is demonstrated with the IEEE 118-bus system example. The Jacobian matrix and the profit maximizing algorithm are helpful for market participants to bid into electricity markets, and for market monitors to analyze firm-based strategic behaviors.**

***Index Terms*—Electricity market, optimal offer, residual demand.**

NOMENCLATURE

G	Set of generators, with each segment of an offer represented by a distinct element.
L	Set of loads.
$\mathsf{A} \subset \mathsf{G}$	Set of generators owned by firm A.
$\bar{\mathsf{A}} \subset \mathsf{G}$	Set of generators not owned by firm A, $\bar{\mathsf{A}} = \mathsf{G} \backslash \mathsf{A}$.
$\mathsf{f} \subset \bar{\mathsf{A}}$	Set of generators with binding/fixed quantity offers.
$\mathsf{v} \subset \bar{\mathsf{A}}$	Set of generators with offers having nonzero slopes.
$\mathsf{z} \subset \bar{\mathsf{A}}$	Set of generators with binding/fixed price offers.
$\mathbf{q}$	Power injection vector.
$O_g(q_g)$	Generator g's total offer cost function, whose derivative, $O'_g(q_g)$, is generator g's offer function.
$C_g(q_g)$	Generator g's production cost function.
$\mathbf{H}$	Shift factor matrix with rows corresponding to the transmission constraints and columns corresponding to power injections.
$\mathbf{q}^{\min}$	Vector of generator operating lower limits.
$\mathbf{q}^{\max}$	Vector of generator operating upper limits.
$\mathbf{Z}$	Vector of the transmission capacity limits.
λ	Lagrange multiplier for the energy balance constraint.
$\boldsymbol{\mu}$	Lagrange multiplier vector for the transmission constraints.
b	Set of binding transmission constraints.
$\mathbf{0}$	Vector or matrix of all zeros.
$\mathbf{1}$	Vector or matrix of all ones.

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I. INTRODUCTION

A SUPPLIER'S residual demand function is defined as the total system demand minus the total supply from all competitors at each given price. The residual demand has been widely used by economists and researchers to analyze strategic behaviors in oligopoly markets, such as electricity markets [1]–[6]. However, these analyses did not rigorously account for the impact of the transmission network in electricity markets on the market prices and the residual demand.

Transmission constraints play a central role in electricity markets [7], [8]. Locational marginal prices (LMPs) are designed to properly value the transmission network through the electricity prices. LMPs have gained great popularity in electricity market design in the last decade. Currently all electricity markets in the U.S. are LMP based.

The concept of residual demand needs to be redefined to represent LMPs. Reference [8] characterized the concept of transmission-constrained residual demand for LMP-based electricity markets. The transmission-constrained residual demand is implicitly defined through the market clearing engine, which is an optimal power flow (OPF) program, and thus it can rigorously capture the impact of transmission networks on a generator's residual demand function.

Conceptually, if a generator g offers a fixed supply quantity offer q_g, and the OPF solves at LMP p_g for g, then (p_g, q_g) is a point on g's residual demand function. Tracing out the whole transmission-constrained residual demand function involves solving the OPF multiple times with different q_g levels. Repeatedly solving the OPF can reveal the implicit residual demand function, but computationally it is not a practical approach. Production level OPFs can have thousands of variables and hundreds of constraints [9], and are not easy to solve. Solving the OPF hundreds of times may be computationally overwhelming.

Without tracing out the whole residual demand function, [8] characterizes locally the derivative of a generator's residual demand function, i.e., the transmission-constrained residual demand derivative (TCRDD). The TCRDD provides very valuable information. Reference [10] proposes a framework to utilize the TCRDD to find the profit maximizing strategies for a generator. The TCRDD method decouples the overall profit maximization problem into two subproblems: the OPF and TCRDD calculation, and the profit maximization based on TCRDD. The two subproblems are solved iteratively, which captures the network characteristics accurately.

Compared with other approaches such as the mathematical program with equilibrium constraints (MPEC) method [11], which solves the profit maximization integrated with the OPF, the TCRDD approach solves the profit maximization problem separately using the TCRDD instead of jointly with the complicated full network model. In addition, the existing advanced OPF solvers could be reused in the TCRDD calculation. Due to these advantages, the TCRDD method is a promising alternative to the MPEC method. However, one limitation in [8] and [10] is that the transmission-constrained residual demand and TCRDD only applies to a single generator (physical unit or generation portfolio) located at one pricing location (node or zone).

In an electricity market, a generation firm may own multiple generators at different pricing locations. The residual demand needs to be generalized from a single generator's perspective to that of a generation firm to be more useful in an LMP-based electricity market. We make this generalization in this paper. Specifically, we derive a generation firm's transmission-constrained inverse residual demand Jacobian (TCIRDJ) matrix.[1] Unlike [8], we choose to work on a generation firm's inverse residual demand function instead of the residual demand function, because it might be more natural to consider the prices as functions of quantities in electricity markets. This is consistent with the convention in [12]. The TCIRDJ can be used in the framework in [10] to replace the TCRDD, and makes the framework also applicable to generation firms. We implement the framework with the bundle-Newton method to find the profit maximizing strategy for a generation firm. The performance of this method will be tested with an IEEE 118-bus system.

The organization of the rest of the paper is as follows. Section II characterizes a generation firm's inverse residual demand and derives the TCIRDJ based on sensitivity analysis of the OPF. Section III characterizes some of the TCIRDJ properties. Section IV proposes applying the bundle-Newton method to find a generation firm's profit maximizing strategy. Section V tests the proposed bundle-Newton method in the IEEE 118-bus system, and compares its performance with the MPEC method, and Section VI concludes.

II. Generation Firm's Transmission-Constrained Inverse Residual Demand and the TCIRDJ

The transmission-constrained inverse residual demand and the TCIRDJ characterization in this paper can be viewed as a generalization of [12]. Reference [12] makes the following assumptions: the OPF only models DC power flow constraints, there are no generator output capacity bounds, and the generator offers are strictly monotonic and smooth functions. As discussed in [10], production level OPF models more constraints than just DC power flow, including generator output capacity bounds, security constraints, and nomograms. In addition, generators can submit constant price offer segments, which are not strictly monotonic. We deal with all these constraints and issues in this paper.

We consider an offer-based electricity market cleared by an OPF program minimizing the total generation offer cost. To be consistent with most electricity markets in the U.S., we assume piecewise quadratic offer cost functions, so the offer functions are piecewise linear. The OPF model follows (5)–(8) in [10]:

$$\min_{\mathbf{q}_{\mathsf{A}},\mathbf{q}_{\bar{\mathsf{A}}}} \quad \sum_{a\in\mathsf{A}} O_a(q_a) + \sum_{\bar{a}\in\bar{\mathsf{A}}} O_{\bar{a}}(q_{\bar{a}}) \tag{1}$$

$$\text{s.t.} \quad \mathbf{H}_{\bullet\mathsf{A}}\mathbf{q}_{\mathsf{A}} + \mathbf{H}_{\bullet\bar{\mathsf{A}}}\mathbf{q}_{\bar{\mathsf{A}}} + \mathbf{H}_{\bullet\mathsf{L}}\mathbf{q}_{\mathsf{L}} \le \mathbf{Z} \tag{2}$$

$$\mathbf{1}_{\mathsf{A}}^{\mathrm{T}}\mathbf{q}_{\mathsf{A}} + \mathbf{1}_{\bar{\mathsf{A}}}^{\mathrm{T}}\mathbf{q}_{\bar{\mathsf{A}}} + \mathbf{1}_{\mathsf{L}}^{\mathrm{T}}\mathbf{q}_{\mathsf{L}} = 0 \tag{3}$$

$$\mathbf{q}_{\mathsf{G}}^{\min} \le \mathbf{q}_{\mathsf{G}} \le \mathbf{q}_{\mathsf{G}}^{\max} \tag{4}$$

where

- (2) consists of transmission constraints;
- (3) is the power balance constraint.

The meaning of the variables and parameters are explained in the Nomenclature section. As a general convention in this paper, we use a subscript on a vector or matrix to represent its specific meaning and dimension. For example, $\mathbf{H}_{\bullet\mathsf{A}}$ represents the shift factor matrix from generators in set A to all transmission constraints; $\mathbf{1}_{\mathsf{A}}^{\mathrm{T}}$ represents the transpose of an all-one vector of dimension $|\mathsf{A}|$, the cardinality of set A.

As discussed in [10], (2) can model not only pre-contingency transmission constraints, but also post-contingency security constraints, nomogram constraints, and all kinds of other linear constraints. Denote the OPF solution by $(\hat{\mathbf{q}}_{\mathsf{A}}, \hat{\mathbf{q}}_{\bar{\mathsf{A}}}, \hat{\boldsymbol{\mu}}, \hat{\lambda})$.

After the OPF is solved, we know which offers are binding at constant quantities, not binding, or binding at constant prices, so we partition set $\bar{\mathsf{A}}$ into three subsets: $\bar{\mathsf{A}} = \mathsf{f} \cup \mathsf{v} \cup \mathsf{z}$ so that

$$\mathbf{q}_{\bar{\mathsf{A}}} = \begin{bmatrix} \mathbf{q}_{\mathsf{f}} \\ \mathbf{q}_{\mathsf{v}} \\ \mathbf{q}_{\mathsf{z}} \end{bmatrix}$$

where $\mathbf{q}_{\mathsf{f}}, \mathbf{q}_{\mathsf{v}}$, and $\mathbf{q}_{\mathsf{z}}$ are the sub-vectors of $\mathbf{q}_{\bar{\mathsf{A}}}$ corresponding to offers at binding quantities, not binding, and binding at constant prices, respectively. After the OPF is solved, we also know which transmission constraints are binding, and denote them by the set b.

Now we re-construct a special OPF based on (1)–(4) in the following way:

- view $\mathbf{q}_{\mathsf{A}}$ as a parameter vector;
- view $\mathbf{q}_{\mathsf{f}} = \hat{\mathbf{q}}_{\mathsf{f}}$ as a constant vector, and omit the generation capacity constraints;
- view $\mathbf{q}_{\mathsf{v}}$ and $\mathbf{q}_{\mathsf{z}}$ as the decision variable vectors;
- include power balance constraint and binding transmission constraints.

The so-constructed OPF is called the residual OPF, and its Karush-Kuhn-Tucker (KKT) conditions are

$$\mathbf{O}_{\mathsf{v}}'(\mathbf{q}_{\mathsf{v}}) + \mathbf{1}_{\mathsf{v}}\lambda + \mathbf{H}_{\mathsf{bv}}^{\mathrm{T}}\boldsymbol{\mu}_{\mathsf{b}} = \mathbf{0}_{\mathsf{v}}$$
$$\mathbf{O}_{\mathsf{z}}'(\mathbf{q}_{\mathsf{z}}) + \mathbf{1}_{\mathsf{z}}\lambda + \mathbf{H}_{\mathsf{bz}}^{\mathrm{T}}\boldsymbol{\mu}_{\mathsf{b}} = \mathbf{0}_{\mathsf{z}}$$

[1]Reference [12] refers to the TCIRDJ as the price response matrix.

$$\begin{aligned}&\mathbf{H}_{\mathsf{bv}}\mathbf{q}_{\mathsf{v}}+\mathbf{H}_{\mathsf{bz}}\mathbf{q}_{\mathsf{z}}+\mathbf{H}_{\mathsf{bL}}\mathbf{q}_{\mathsf{L}}+\mathbf{H}_{\mathsf{bA}}\mathbf{q}_{\mathsf{A}}=\mathbf{Z}_{\mathsf{b}}\\&\mathbf{1}_{\mathsf{v}}^{T}\mathbf{q}_{\mathsf{v}}+\mathbf{1}_{\mathsf{z}}^{T}\mathbf{q}_{\mathsf{z}}+\mathbf{1}_{\bar{\mathsf{A}}}^{T}\mathbf{q}_{\mathsf{A}}+\mathbf{1}_{\mathsf{L}}^{T}\mathbf{q}_{\mathsf{L}}+\mathbf{1}_{\mathsf{f}}^{T}\mathbf{q}_{\mathsf{f}}=0\end{aligned}\tag{5}$$

where

- $\mathbf{O}'_{\mathsf{v}}(\mathbf{q}_{\mathsf{v}})=\nabla_{\mathbf{q}_{\mathsf{v}}}(\sum_{\bar{a}\in\bar{\mathsf{A}}}O_{\bar{a}}(q_{\bar{a}}))$;
- $\mathbf{O}'_{\mathsf{z}}(\mathbf{q}_{\mathsf{z}})=\nabla_{\mathbf{q}_{\mathsf{z}}}(\sum_{\bar{a}\in\bar{\mathsf{A}}}O_{\bar{a}}(q_{\bar{a}}))$;
- $\mathbf{H}_{\mathsf{bA}}$, $\mathbf{H}_{\mathsf{bv}}$ and $\mathbf{H}_{\mathsf{bz}}$ are the shift factor matrices from generators in set A, v, and z, respectively, to the binding constraints.

One can verify that (5) is a subset of the original OPF's KKT conditions. Therefore, $(\hat{\mathbf{q}}_{\mathsf{A}},\hat{\mathbf{q}}_{\mathsf{f}},\hat{\mathbf{q}}_{\mathsf{v}},\hat{\mathbf{q}}_{\mathsf{z}},\hat{\boldsymbol{\mu}},\hat{\lambda})$ satisfies (5).

The inverse residual demand function is a price function of supply quantities that specifies the LMPs for all generators in the set A. It is implicitly defined by the residual OPF in the vicinity of $\hat{\mathbf{q}}_{\mathsf{A}}$ as characterized in the following proposition.

Proposition 1: If the columns of $\mathbf{H}_{\mathsf{bz}}$ are linearly independent, i.e.,

$$\operatorname{rank}(\mathbf{H}_{\mathsf{bz}})=|\mathsf{z}|\tag{6}$$

and the binding constraints of the residual OPF are linearly independent, i.e.,

$$\operatorname{rank}\left(\begin{bmatrix}\mathbf{H}_{\mathsf{bv}}&\mathbf{H}_{\mathsf{bz}}\\\mathbf{1}_{\mathsf{v}}^{T}&\mathbf{1}_{\mathsf{z}}^{T}\end{bmatrix}\right)=|\mathsf{b}|+1\tag{7}$$

then there exists a vector price function $\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})$ in the vicinity of $\hat{\mathbf{q}}_{\mathsf{A}}$ that specifies the LMPs for the generators in set A.

Proof: The KKT conditions (5) can be viewed as

$$\mathbf{F}(\mathbf{q}_{\mathsf{A}},\mathbf{q}_{\mathsf{v}},\mathbf{q}_{\mathsf{z}},\boldsymbol{\mu},\lambda)=\mathbf{0}$$

where $\mathbf{F}(\bullet)$ is a continuously differentiable function. The Jacobian of $\mathbf{F}(\bullet)$ evaluated at $(\hat{\mathbf{q}}_{\mathsf{v}},\hat{\mathbf{q}}_{\mathsf{z}},\hat{\boldsymbol{\mu}},\hat{\lambda})$ is

$$\begin{bmatrix}\frac{\partial\mathbf{F}}{\partial\mathbf{q}_{\mathsf{v}}}&\frac{\partial\mathbf{F}}{\partial\mathbf{q}_{\mathsf{z}}}&\frac{\partial\mathbf{F}}{\partial\boldsymbol{\mu}}&\frac{\partial\mathbf{F}}{\partial\lambda}\end{bmatrix}=\begin{bmatrix}\mathbf{O}''_{\mathsf{vv}}&\mathbf{0}_{\mathsf{vz}}&\mathbf{H}_{\mathsf{bv}}^{T}&\mathbf{1}_{\mathsf{v}}\\\mathbf{0}_{\mathsf{zv}}&\mathbf{0}_{\mathsf{zb}}&\mathbf{H}_{\mathsf{bz}}^{T}&\mathbf{1}_{\mathsf{z}}\\\mathbf{H}_{\mathsf{bv}}&\mathbf{H}_{\mathsf{bz}}&\mathbf{0}_{\mathsf{bb}}&\mathbf{0}_{\mathsf{b}}\\\mathbf{1}_{\mathsf{v}}^{T}&\mathbf{1}_{\mathsf{z}}^{T}&\mathbf{0}_{\mathsf{b}}^{T}&0\end{bmatrix}$$

where

$$\mathbf{O}''_{\mathsf{vv}}=\nabla^{2}_{\mathbf{q}_{\mathsf{v}}\mathbf{q}_{\mathsf{v}}}\left(\sum_{\bar{a}\in\bar{\mathsf{A}}}O_{\bar{a}}(q_{\bar{a}})\right).$$

We first prove the Jacobian of $\mathbf{F}(\bullet)$ evaluated at $(\hat{\mathbf{q}}_{\mathsf{v}},\hat{\mathbf{q}}_{\mathsf{z}},\hat{\boldsymbol{\mu}},\hat{\lambda})$ is invertible.

Let us assume

$$\begin{bmatrix}\mathbf{O}''_{\mathsf{vv}}&\mathbf{0}_{\mathsf{vz}}&\mathbf{H}_{\mathsf{bv}}^{T}&\mathbf{1}_{\mathsf{v}}\\\mathbf{0}_{\mathsf{zv}}&\mathbf{0}_{\mathsf{zz}}&\mathbf{H}_{\mathsf{bz}}^{T}&\mathbf{1}_{\mathsf{z}}\\\mathbf{H}_{\mathsf{bv}}&\mathbf{H}_{\mathsf{bz}}&\mathbf{0}_{\mathsf{bb}}&\mathbf{0}_{\mathsf{b}}\\\mathbf{1}_{\mathsf{v}}^{T}&\mathbf{1}_{\mathsf{z}}^{T}&\mathbf{0}_{\mathsf{b}}^{T}&0\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{v}}\\\boldsymbol{\gamma}_{\mathsf{z}}\\\boldsymbol{\gamma}_{\mathsf{b}}\\\gamma_{1}\end{bmatrix}=\mathbf{0}\tag{8}$$

and prove

$$\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{v}}\\\boldsymbol{\gamma}_{\mathsf{z}}\\\boldsymbol{\gamma}_{\mathsf{b}}\\\gamma_{1}\end{bmatrix}=\mathbf{0}$$

which proves that the column vectors are linearly independent, and thus the matrix is invertible.

Multiplying both sides of the first two row equations in (8) by $[\boldsymbol{\gamma}_{\mathsf{v}}^{T}\quad\boldsymbol{\gamma}_{\mathsf{z}}^{T}]$, we have

$$\boldsymbol{\gamma}_{\mathsf{v}}^{T}\mathbf{O}''_{\mathsf{vv}}\boldsymbol{\gamma}_{\mathsf{v}}+[\boldsymbol{\gamma}_{\mathsf{v}}^{T}\quad\boldsymbol{\gamma}_{\mathsf{z}}^{T}]\begin{bmatrix}\mathbf{H}_{\mathsf{bv}}^{T}&\mathbf{1}_{\mathsf{v}}\\\mathbf{H}_{\mathsf{bz}}^{T}&\mathbf{1}_{\mathsf{z}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{b}}\\\gamma_{1}\end{bmatrix}=\mathbf{0}.\tag{9}$$

The last two row equations in (8) are

$$\begin{bmatrix}\mathbf{H}_{\mathsf{bv}}&\mathbf{H}_{\mathsf{bz}}\\\mathbf{1}_{\mathsf{v}}^{T}&\mathbf{1}_{\mathsf{z}}^{T}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{v}}\\\boldsymbol{\gamma}_{\mathsf{z}}\end{bmatrix}=\mathbf{0}.\tag{10}$$

Transposing both sides of (10), we have

$$[\boldsymbol{\gamma}_{\mathsf{v}}^{T}\quad\boldsymbol{\gamma}_{\mathsf{z}}^{T}]\begin{bmatrix}\mathbf{H}_{\mathsf{bv}}^{T}&\mathbf{1}_{\mathsf{v}}\\\mathbf{H}_{\mathsf{bz}}^{T}&\mathbf{1}_{\mathsf{z}}\end{bmatrix}=\mathbf{0}$$

which zeros out the second term on the left-hand side of (9). Therefore, (9) becomes

$$\boldsymbol{\gamma}_{\mathsf{v}}^{T}\mathbf{O}''_{\mathsf{vv}}\boldsymbol{\gamma}_{\mathsf{v}}=\mathbf{0}.\tag{11}$$

Because $\mathbf{O}''_{\mathsf{vv}}$ is positive definite by definition, (11) implies $\boldsymbol{\gamma}_{\mathsf{v}}=\mathbf{0}$.

Setting $\boldsymbol{\gamma}_{\mathsf{v}}=\mathbf{0}$ in (10) results in

$$\mathbf{H}_{\mathsf{bz}}\boldsymbol{\gamma}_{\mathsf{z}}=\mathbf{0}$$

which implies $\boldsymbol{\gamma}_{\mathsf{z}}=\mathbf{0}$ by (6).

Setting $\boldsymbol{\gamma}_{\mathsf{v}}=\mathbf{0}$ in the first two row equations of (8) results in

$$\begin{bmatrix}\mathbf{H}_{\mathsf{bv}}^{T}&\mathbf{1}_{\mathsf{v}}\\\mathbf{H}_{\mathsf{bz}}^{T}&\mathbf{1}_{\mathsf{z}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{b}}\\\gamma_{1}\end{bmatrix}=\mathbf{0}\tag{12}$$

which implies

$$\begin{bmatrix}\boldsymbol{\gamma}_{\mathsf{b}}\\\gamma_{1}\end{bmatrix}=\mathbf{0}$$

by (7).

We have proved the Jacobian of $\mathbf{F}(\bullet)$ evaluated at $(\hat{\mathbf{q}}_{\mathsf{v}},\hat{\mathbf{q}}_{\mathsf{z}},\hat{\boldsymbol{\mu}},\hat{\lambda})$ is invertible. By the implicit function theorem, there exist unique functions $\mathbf{q}_{\mathsf{v}}(\mathbf{q}_{\mathsf{A}})$, $\mathbf{q}_{\mathsf{z}}(\mathbf{q}_{\mathsf{A}})$, $\boldsymbol{\mu}(\mathbf{q}_{\mathsf{A}})$, and $\lambda(\mathbf{q}_{\mathsf{A}})$ in the vicinity of $\hat{\mathbf{q}}_{\mathsf{A}}$. Similar to [10], the LMPs for the generators in the set A are specified by an implicit function $\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})$ as follows:

$$\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})=-\mathbf{1}_{\mathsf{A}}\lambda(\mathbf{q}_{\mathsf{A}})-\mathbf{H}_{\mathsf{bA}}^{T}\boldsymbol{\mu}_{\mathsf{b}}(\mathbf{q}_{\mathsf{A}}).\tag{13}$$

■

Generally, (6) is satisfied if we only include generators in set z such that the rows $\mathbf{H}_{\mathsf{bz}}$ are linearly independent; (7) is satisfied if we only include linearly independent constraints in the residual OPF. By doing so, we can guarantee the existence of the function $\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})$.

The price function $\mathbf{P}_{\mathsf{A}}(\bullet)$ in (13) is the inverse residual demand function, and its Jacobian $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ is the TCIRDJ. The TCIRDJ is a constant for each specific combination of $(\mathsf{v},\mathsf{f},\mathsf{z},\mathsf{b})$, so $\mathbf{P}_{\mathsf{A}}(\bullet)$ is piecewise linear with each piece corresponding to a specific combination of $(\mathsf{v},\mathsf{f},\mathsf{z},\mathsf{b})$.

The (g,a) element of the matrix is defined by differentiating both sides of the gth row of (13) with respect to q_{a} :

$$\frac{\partial P_g}{\partial q_a} = -\frac{\partial \lambda}{\partial q_a} - \mathbf{H}_{\mathsf{b}g}^{\mathrm{T}} \frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a}, \quad \forall g, a \in \mathsf{A}. \tag{14}$$

We calculate $(\partial\lambda)/(\partial q_a)$ and $(\partial\boldsymbol{\mu}_{\mathsf{b}})/(\partial q_a)$ by a sensitivity analysis on (5).

Differentiating both sides of (5) with respect to $q_a, \forall a \in \mathsf{A}$, we get

$$\begin{aligned} \mathbf{O}''_{\mathsf{v}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} + \mathbf{1}_{\mathsf{v}} \frac{\partial \lambda}{\partial q_a} + \mathbf{H}_{\mathsf{bv}}^{\mathrm{T}} \frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a} &= \mathbf{0}_{\mathsf{v}} \\ \mathbf{1}_{\mathsf{z}} \frac{\partial \lambda}{\partial q_a} + \mathbf{H}_{\mathsf{bz}}^{\mathrm{T}} \frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a} &= \mathbf{0}_{\mathsf{z}} \\ \mathbf{H}_{\mathsf{bv}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} + \mathbf{H}_{\mathsf{bz}} \frac{\partial \mathbf{q}_{\mathsf{z}}}{\partial q_a} &= -\mathbf{H}_{\mathsf{b}a} \\ \mathbf{1}_{\mathsf{v}}^{\mathrm{T}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} + \mathbf{1}_{\mathsf{z}}^{\mathrm{T}} \frac{\partial \mathbf{q}_{\mathsf{z}}}{\partial q_a} &= -1. \end{aligned} \tag{15}$$

We can calculate each element of the TCIRDJ by solving (15) for $(\partial\lambda)/(\partial q_a)$ and $(\partial\boldsymbol{\mu}_{\mathsf{b}})/(\partial q_a)$, and substituting them into (14).

As a special case, if $\mathsf{z} = \emptyset$, then $(\partial\lambda)/(\partial q_a)$ and $(\partial\boldsymbol{\mu}_{\mathsf{b}})/(\partial q_a)$ have clean closed form solutions. In this case, (15) becomes

$$\mathbf{O}''_{\mathsf{v}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} + \mathbf{1}_{\mathsf{v}} \frac{\partial \lambda}{\partial q_a} + \mathbf{H}_{\mathsf{bv}}^{\mathrm{T}} \frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a} = \mathbf{0}_{\mathsf{v}} \tag{16}$$

$$\mathbf{H}_{\mathsf{bv}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} = -\mathbf{H}_{\mathsf{b}a} \tag{17}$$

$$\mathbf{1}_{\mathsf{v}}^{\mathrm{T}} \frac{\partial \mathbf{q}_{\mathsf{v}}}{\partial q_a} = -1. \tag{18}$$

Now define

$$\begin{aligned} \mathbf{M} &= \mathbf{H}_{\mathsf{bv}} (\mathbf{O}''_{\mathsf{v}})^{-1} \mathbf{H}_{\mathsf{bv}}^{\mathrm{T}} \\ N &= \mathbf{1}_{\mathsf{v}}^{\mathrm{T}} (\mathbf{O}''_{\mathsf{v}})^{-1} \mathbf{1}_{\mathsf{v}}. \end{aligned}$$

By assumption (7), $\mathrm{rank}(\mathbf{H}_{\mathsf{bv}}) = |\mathsf{b}|$ with $\mathsf{z} = \emptyset$, so $\mathbf{M}$ is invertible.

Multiplying both sides of (16) on the left by $\mathbf{H}_{\mathsf{bv}}(\mathbf{O}''_{\mathsf{v}})^{-1}$, substituting in (17), and then multiplying both sides on the left by $\mathbf{M}^{-1}$, we get

$$\frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a} = -\mathbf{M}^{-1}\mathbf{H}_{\mathsf{bv}} - \mathbf{M}^{-1}\mathbf{H}_{\mathsf{bv}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{1}_{\mathsf{v}} \frac{\partial \lambda}{\partial q_a}. \tag{19}$$

Multiplying both sides of (16) on the left by $\mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}$, substituting in (18), we get

$$-1 + N \frac{\partial \lambda}{\partial q_a} + \mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{H}_{\mathsf{bv}}^{\mathrm{T}} \frac{\partial \boldsymbol{\mu}_{\mathsf{b}}}{\partial q_a} = 0. \tag{20}$$

Solving (19) and (20), we get

$$\frac{\partial \lambda}{\partial q_a} = -V^{-1}\left(1 - \mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{H}_{\mathsf{bv}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{H}_{\mathsf{b}a}\right) \tag{21}$$

where

$$V = -N + \mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{H}_{\mathsf{bv}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{H}_{\mathsf{bv}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{1}_{\mathsf{v}}. \tag{22}$$

The derivative $(\partial\boldsymbol{\mu}_{\mathsf{b}})/(\partial q_a)$ can be calculated by substituting (21) into (19).

After simplification

$$\frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}} = -\mathbf{H}_{\mathsf{bA}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{H}_{\mathsf{bA}} + \mathbf{U}^{\mathrm{T}}\mathbf{U}V^{-1} \tag{23}$$

where

$$\mathbf{U} = \mathbf{1}_{\mathsf{A}} - \mathbf{H}_{\mathsf{bA}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{H}_{\mathsf{bv}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{1}_{\mathsf{v}}.$$

If the generation firm has only one generator, and the generator is located at the slack bus s, then $\mathbf{H}_{\mathsf{bA}} = \mathbf{0}$, and $\mathbf{U} = 1$. In this case

$$\frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}} = -\frac{\partial \lambda}{\partial q_a} = V^{-1}$$

where V is the TCRDD formula [8, (29)]. This verifies that, for this special single generator case, the TCIRDJ is equal to the inverse of the TCRDD, which is true by the inverse function theorem.

III. Properties of the TCIRDJ

In this section, we are going to prove that the TCIRDJ is symmetric and negative semi-definite, generalizing the special case proved in [12].

Proposition 2: The TCIRDJ $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ is symmetric and negative semi-definite.

Proof: The proof consists of two parts: first we prove $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ is symmetric and negative definite under the condition $\mathsf{z} = \emptyset$, and then prove it is symmetric and negative semi-definite for $\mathsf{z} \neq \emptyset$.

1) Under the condition $\mathsf{z} = \emptyset$. Directly from the TCIRDJ $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ formula (23)

$$\frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}} = \left(\frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}}\right)^{\mathrm{T}}$$

so it is symmetric.

Because $(\mathbf{O}''_{\mathsf{v}})^{-1}$ is positive definite, $\mathbf{M}$ is also positive definite. Therefore, we only need to prove $V \leq 0$ to prove $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ is negative definite.

Similar to [8] where the TCRDD was proved to be less than or equal to zero, we prove $V \leq 0$ by a weighted least squares (WLS) formulation. Consider the WLS problem[2] specified by

$$\begin{aligned} \mathbf{X} &= \mathbf{H}_{\mathsf{bv}}^{\mathrm{T}} \\ \mathbf{Y} &= \mathbf{1}_{\mathsf{v}} \\ \mathbf{W} &= (\mathbf{O}''_{\mathsf{v}})^{-1}. \end{aligned}$$

The least sum of squares error (SSE) is

$$\begin{aligned} \mathrm{SSE}^{\mathrm{WLS}} &= \mathbf{Y}^{\mathrm{T}}\mathbf{W}\mathbf{Y} - \mathbf{Y}^{\mathrm{T}}\mathbf{W}\mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{W}\mathbf{Y} \\ &= \mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{1}_{\mathsf{v}} \\ &\quad - \mathbf{1}_{\mathsf{v}}^{\mathrm{T}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{H}_{\mathsf{bv}}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{H}_{\mathsf{bv}}(\mathbf{O}''_{\mathsf{v}})^{-1}\mathbf{1}_{\mathsf{v}} \\ &= -V \\ &\geq 0. \end{aligned}$$

Therefore, $V \leq 0$, and the formula of $(\partial\mathbf{P}_{\mathsf{A}})/(\partial\mathbf{q}_{\mathsf{A}})$ in (23) is negative definite.

2) Under the condition $\mathsf{z} \neq \emptyset$. Denote the TCIRDJ solved from (15) by TCIRDJ^*. We construct a sequence as follows. Start with all positive bid slopes for generators in set z, and cut the slope values in set z by half each time. This process results in a sequence that the slope values in set z are all pos-

[2]See [8] Appendix A for basics about the WLS interpretation of TCRDD. More detailed discussion about WLS can be found in [13].

itive, and monotonically approaches $\mathbf{0}_{\mathsf{z}}$. For the kth point in the sequence, a corresponding TCIRDJ^k can be calculated by solving (23), and it is symmetric and negative definite as proved in 1). This TCIRDJ sequence $\mathrm{TCIRDJ}^k, k = 0, 1, 2, \ldots$, converges to TCIRDJ^*, because the sequence of the coefficients of (16)–(18) converges to the coefficients of (15). Therefore, the limit matrix TCIRDJ^* is symmetric and negative semi-definite by continuity. ■

IV. Maximizing a Generation Firm's Profit Using the TCIRDJ

A generation firm's profit depends on the competitors' bidding strategies as well as the transmission network constraints. Given the competitors' bidding strategies and the network model, a generation firm's profit is a function of its own bidding strategies. Under this assumption, the bidding strategy that maximizes the profit function is called the "best response" in a Game Theory context. The section calculates the "best response" for a generation firm.[3]

Reference [10] proposed a framework to use the derivative of the inverse residual demand to find a generator's profit maximizing strategy. This framework avoids solving the profit optimization coupled with the full network model as in the MPEC method [11], and thus has a computational advantage. In addition, existing advanced OPF solvers could be reused in the framework. In [10], the implementation of the framework is limited to the special case of a single generator located at one pricing location. With the TCRDD for a single generator being generalized to the TCIRDJ for a generation firm, the framework can also be implemented to find a generation firm's profit maximizing strategy.

As demonstrated in [10], the inverse residual demand function for a single generator may have kinks. For a generator firm owning multiple generators at multiple pricing locations, there will be kinks on the inverse residual demand function when the combination of $(\mathsf{v}, \mathsf{f}, \mathsf{z}, \mathsf{b})$ changes. Similar to optimizing a single generator's profit based on TCRDD, one major challenge is to deal with these kinks on the inverse residual demand function. The special bisection search algorithm proposed in [10] does not work for a higher dimensional inverse residual demand function for a generation firm. We are going to apply a bundle-Newton method proposed in [14] to optimize a generation firm's bidding strategy based on the TCIRDJ.

A. Bundle Newton Method

The bundle idea and bundle Newton method are reviewed in the Appendix. Without loss of generality, we assume that the decision variables are the output quantities of firm A's generators, i.e., $\mathbf{x} = \mathbf{q}_{\mathsf{A}}$.[4]

The objective function $f(\bullet)$ is the negative profit function

$$f(\mathbf{q_A}) = -\sum_{a \in \mathsf{A}} (P_a(q_a)q_a - C_a(q_a)).$$

[3]Typically, the network model is available from independent system operators. For example, CAISO and ERCOT make their network model available to their market participants. The competitors' strategies can be estimated based on production costs and historic bid data. How to make these estimates accurately is out of the scope of this paper.

[4]Reference [10] discussed different choices of strategic variables, and the implications in profit maximizing versus Nash Equilibrium.

Because $\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})$ is piece-wise linear, if we assume $C_a(q_a)$ is piecewise quadratic for any $a \in \mathsf{A}$, then $f(\mathbf{q_A})$ is piecewise quadratic.

One subgradient of $f(\bullet)$ is

$$\mathbf{g}(\mathbf{q}_{\mathsf{A}}) = -\left(\mathbf{P}_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}}) + \frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}}(\mathbf{q}_{\mathsf{A}})\mathbf{q}_{\mathsf{A}} - \mathbf{C}'_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})\right) \quad (24)$$

where

$$\mathbf{C}'_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}}) = \nabla_{\mathbf{q}_{\mathsf{A}}}\left(\sum_{a \in \mathsf{A}} C_a(q_a)\right)$$

and one approximation to the Hessian is

$$\mathbf{G}(\mathbf{q}_{\mathsf{A}}) = -\left(2\frac{\partial \mathbf{P}_{\mathsf{A}}}{\partial \mathbf{q}_{\mathsf{A}}}(\mathbf{q}_{\mathsf{A}}) - \mathbf{C}''_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}})\right) \quad (25)$$

where

$$\mathbf{C}''_{\mathsf{A}}(\mathbf{q}_{\mathsf{A}}) = \nabla^2_{\mathbf{q}_{\mathsf{A}}\mathbf{q}_{\mathsf{A}}}\left(\sum_{a \in \mathsf{A}} C_a(q_a)\right).$$

B. Algorithm

The bundle-Newton algorithm to find a generation firm's maximum profit given other firms' bids using the TCIRDJ is as follows.

Parameters:

- $\gamma > 0$ is the parameter used in (30) to define $\beta^{k,j}$;
- $\epsilon > 0$ is the convergence tolerance;
- $m_L \in (0, 0.5)$ is the relative objective function value improvement threshold; and
- $\mathbf{q}_{\mathsf{A}}^0$ is the starting point.

Procedure:

Step 1) Set initial value $\mathbf{x}^0 = \mathbf{q}_{\mathsf{A}}^0$, and iteration number $k = 0$.
Step 2) Solve OPF with $\mathbf{q}_{\mathsf{A}} = \mathbf{x}^k$, and calculate f^k and TCIRDJ.
Step 3) Calculate $\mathbf{g}^k$ from (24) and $\mathbf{G}^k$ from (25), and add them to the bundle.
Step 4) Solve (31), and denote the solution by $(\mathbf{d}^k, v^k)$.
Step 5) If $v^k < \epsilon$, optimal solution found with $\mathbf{x}^* = \mathbf{x}^k$, $f^* = f^k$, and stop.
Step 6) Solve OPF with $\mathbf{q}_{\mathsf{A}} = \mathbf{x}^k + \mathbf{d}^k$, and calculate $f(\mathbf{x}^k + \mathbf{d}^k)$ and TCIRDJ.
Step 7) If $f(\mathbf{x}^k + \mathbf{d}^k) - f^k \leq m_L v^k$, make a serious step, $\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}^k$; otherwise, make a null step $\mathbf{x}^{k+1} = \mathbf{x}^k$.
Step 8) Add $\mathbf{x}^{k+1}$ to the bundle.
Step 9) Set $k = k + 1$, and repeat steps 3) to 7).

Step 7) makes the decision between a serious step and null step. If there is relatively significant (compared with threshold m_L) objective function value decrease, update $\mathbf{x}^k$ to $\mathbf{x}^k + \mathbf{d}^k$, then proceed to $\mathbf{x}^k + \mathbf{d}^k$; otherwise, stay at current solution $\mathbf{x}^k$, and add $\mathbf{x}^k + \mathbf{d}^k$ into the bundle in step 8). The algorithm above is a special implementation of the standard algorithm in [14]. The parameter $\bar{t}$ in [14] is set at 1, which avoids the line search part in the standard algorithm. The line search can find a better step size in the direction $\mathbf{d}^k$ at the cost of multiple objective

TABLE I
GENERATOR DATA (EXCERPT)

generator	bus	$q^{\min}$	$q^{\max}$	marginal cost
1	1	0	100	$0.020q + 40$
2	4	0	100	$0.020q + 40$
3	6	0	100	$0.020q + 40$
4	8	0	100	$0.020q + 40$
5	10	0	550	$0.044q + 20$
30	69	0	805.2	$0.039q + 20$

function evaluations. In our case, the objective function evaluation is computationally intensive, because it involves solving the OPF. The bundle information is more important, because the objective function is piecewise quadratic, so if the bundle is large enough to include the relevant pieces at the optimizer, then locally $f^k_{\mathrm{PQ}}(\mathbf{x})$ exactly models $f(\mathbf{x})$. Therefore, in terms of performance, it is more efficient to enrich the bundle information than it is to seek a better step size. Similar to other nonlinear programming algorithms such as algorithms for MPECs, this algorithm aims at finding a local optimum. Similar to [10], one can run the algorithm with different starting points to explore a broader region to get closer to the global optimum.

V. COMPUTATIONAL EXAMPLE

We use the MATPOWER package [15] as our OPF solver, and apply the algorithm to the IEEE 118-bus test system included in the package. Similar to [10], we enforce 200-MW capacity limits on branches 30–17, 26–30, and 38–37 on the IEEE 118-bus base case from [15] so they are likely to be binding. All other branches have sufficiently large capacity limits. The total system load is 4242 MW.

A. Results for a Two-Generator Firm

We optimize the profit for a fictitious generation firm A, which owns two generators: generator 5 located at bus 10 with 550-MW capacity, and generator 30 located at bus 69 with 805.2-MW capacity. Part of the generator data is listed in Table I. Assume all generators other than generators 5 and 30 offer at their true marginal cost.

We test our algorithm with four different starting points: (200, 200), (300, 500), (450, 250), and (450, 550). The optimization trajectories are listed in Table II. The "step" column indicates a serious step by "S", a null step by "N", and optimal solution by "O". We also plot the contour of the profit function and the optimization trajectory with starting point (200, 200) in Fig. 1 to visually check the results.

The algorithm reliably finds the optimizer for each of the starting points within 3 to 6 total steps including both serious steps and null steps. The number of steps also implies the number of OPFs solved, because there is exactly one OPF solved in each step in order to evaluate the profit function, and calculate the TCIRDJ. For large scale problems, iteratively solving the OPF is the most computationally intense part for this algorithm, so the effort to find the optimizer is roughly proportional to the number of steps taken.

Another observation is that because the bundle method depends on history, even if different optimization paths intersect at a certain iteration, their subsequent trajectories may not be the same. For example, the solution at iteration 1 starting from (300, 500) coincides with the solution at iteration 1 starting from (450, 250) as observed in Table II, but after that, the two trajectories diverge. Nevertheless, finally they approach the same optimizer.

TABLE II
OPTIMIZATION TRAJECTORIES WITH DIFFERENT STARTING POINTS

iter.	q_5	q_{30}	profit	step
starting from $(200, 200)$				
0	200.00	200.00	6509.6	S
1	435.27	499.92	8260.2	S
2	356.00	495.31	9191.5	S
3	356.59	434.17	9192.3	O
starting from $(300, 500)$				
0	300.00	500.00	9015.0	S
1	374.59	445.67	9126.4	N
2	374.59	445.67	9126.4	S
3	367.12	439.99	9182.1	S
4	363.67	437.36	9190.1	S
5	360.80	437.36	9191.5	S
6	356.59	434.17	9192.3	O
starting from $(450, 250)$				
0	450.00	250.00	8293.7	S
1	374.59	445.67	9126.4	S
2	364.94	439.29	9185.5	S
3	361.17	436.86	9191.4	S
4	356.60	434.18	9192.3	O
starting from $(450, 550)$				
0	450.00	550.00	7694.0	S
1	326.63	399.15	9084.1	S
2	347.83	419.48	9169.1	S
3	359.71	428.04	9191.4	S
4	356.60	434.18	9192.3	O

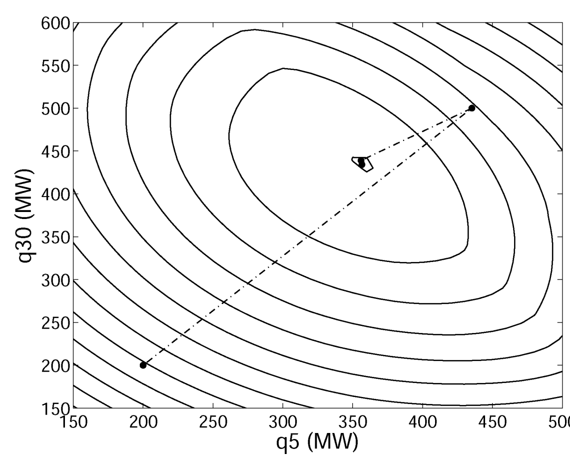

Fig. 1. Optimization trajectory starting from (200, 200).

Similar to [10], for benchmarking purpose, we also solve the same problem using the MPEC method with the NLPEC solver [16], which solves an MPEC problem as an equivalent sequence of nonlinear programming problems parameterized by a scalar μ. The NLPEC execution time is dependent on the initial value of μ. Based on our experience, an initial value of 1 for μ achieves good performance and robustness, and we use it for all the MPEC tests in this section. The comparison is listed in Table III. Although both approaches could find the optimizer with different starting points, the average run time of the TCIRDJ approach is about 1 s versus 10 s for the MPEC approach. We are going to further explore the performance difference between the TCIRDJ approach and the MPEC approach in the next section.

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TABLE III
COMPARISON WITH MPEC

	TCIRDJ			MPEC		
(g_5^0, g_{30}^0)	steps	run time	profit	iters	run time	profit
(200,200)	3S	0.87 s	9192.3	90	9.84 s	9192.3
(300.500)	5S+1N	1.22 s	9192.3	108	11.0 s	9192.3
(450,250)	4S	0.98 s	9192.3	95	10.1 s	9192.3
(450,550)	4S	0.90 s	9192.3	120	12.2 s	9192.3

TABLE IV
PERFORMANCE TEST

	TCIRDJ			MPEC		
gens	steps	run time	profit	iters	run time	profit
1–5	3S	0.63 s	4144.8	427	52.5 s	4144.8
1–10	2S+3N	1.52 s	5023.1	669	80.3 s	5021.6
1–15	4S+2N	2.16 s	10111	815	223.5 s	10111
1–20	2S	0.88 s	10454	842	111.5 s	10454

B. Performance Test

We test the performance of the algorithm by increasing the number of generators owned by generation firm A from five generators to 20 generators. All tests start from competitive output levels. The results are summarized in Table IV. The TCIRDJ approach finishes each of these scenarios within 2.5 s with less than six total steps. Although the number of generators has increased by five to ten times compared to the previous section, the algorithm only takes about twice the run time of the two-generator case. The performance of the algorithm depends less on the number of generators owned by the generation firm, and more on how kinky the profit function is. Generally speaking, a smooth profit function with more decision variables can be much easier to optimize than a profit function with many kinks but less decision variables. Because of this, the 20-generator case requires even less steps than the two-generator case.

The MPEC approach in comparison takes about 50 times more run time than the TCIRDJ approach, and the run time seems more sensitive to the number of decision variables. If the number of generators increases five to ten times, the MPEC approach will need ten to 20 times the amount of the original run time.

Through these tests, we can see the TCIRDJ method performs better than the MPEC method, and the advantage is expected to be more prominent as the size of the problem increases. However, we stress that more tests, especially on large scale systems, are needed to rigorously benchmark the TCIRDJ approach and MPEC approach.

VI. CONCLUSION

In electricity markets, especially nodal electricity markets, a generation firm may own multiple generators located at multiple buses and exposed to different prices. In this context, we generalize the residual demand concept from a single generator's perspective to a generation firm's perspective. Particularly, the TCRDD for a single generator is generalized to the TCIRDJ for a generation firm. We derived the TCIRDJ based on a multi-parameter sensitivity analysis of the OPF, and characterized some of its properties. The TCIRDJ provides useful insights about a generation firm's strategic behavior. Based on the TCIRDJ and the framework proposed in [10], we apply the bundle-Newton algorithm to optimize a generation firm's profit. The algorithm is tested in the IEEE 118-bus system, and it performs very well in terms of robustness and efficiency. Compared with the MPEC approach, the TCIRDJ approach has a significant computational advantage, as well as the advantage of being able to reuse existing OPF solvers. The framework of finding profit maximizing strategies based on the TCIRDJ and the bundle-Newton algorithm provides an effective and promising approach for generation firms' to bid into electricity markets, and for market monitors to understand the bidding behaviors.

APPENDIX
BUNDLE NEWTON METHOD

The bundle concept is widely applied to non-differentiable function optimization. The bundle idea is closely related to the cutting plane method, where the objective function is approximated by a piecewise linear function based on cutting planes [17], [18]. Reference [14] proposes the bundle-Newton method by generalizing the bundle idea to a second-order approximation, where the objective function is approximated by a piecewise quadratic function. The bundle-Newton method is suitable for finding a generation firm's maximum profit, because the profit function of a generation firm is piecewise quadratic as discussed earlier.

Following the convention of [18], we consider a minimization problem

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}) \\ &\text{s.t. } \underline{\mathbf{x}} \le \mathbf{x} \le \bar{\mathbf{x}} \end{aligned} \tag{26}$$

where $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ is a Lipschitz continuous non-smooth convex function. In the following discussion, to simplify notation, we are going to exclude the bound constraints, and bear in mind that the bound constraints can be easily incorporated into the algorithm. Assume a subgradient of $f(\bullet)$, $\mathbf{g}(\bullet)$, and an approximation of the Hessian $\nabla^2 f(\mathbf{x})$, $\mathbf{G}(\mathbf{x})$, are available. The bundle concept has two features [18]:

1) Make use of, at iteration k, the bundle information

$$(f(\mathbf{x}^k), \mathbf{g}(\mathbf{x}^k), \mathbf{G}(\mathbf{x}^k)), \ldots, (f(\mathbf{x}^1), \mathbf{g}(\mathbf{x}^1), \mathbf{G}(\mathbf{x}^1))$$

collected so far using iterates $\mathbf{x}^1, \ldots, \mathbf{x}^k$ to build a model of the objective function $f(\mathbf{x})$.

2) If, due to the kinked structure of f, this model does not characterize f accurately enough, then mobilize more subgradient and second-order information.

To limit the bundle size, some of the points in the bundle can be discarded. In other words, the points in the bundle $\{\mathbf{y}^j\}_{j\in J_k}$ can include a subset of $\{\mathbf{x}^0, \mathbf{x}^1, \ldots, \mathbf{x}^{k-1}, \mathbf{x}^k\}$ and their aggregations [14].

In the bundle-Newton method [14], the bundle information at iteration k results in the piecewise quadratic approximation of f :

$$f_{\mathrm{PQ}}^k(\mathbf{x}) = \max_{j\in J_k} \left\{ f_{\mathrm{Q}}^j(\mathbf{x}) \right\} \tag{27}$$

where

$$f_{\mathrm{Q}}^j(\mathbf{x}) = f(\mathbf{y}^j) + (\mathbf{g}^j)^{\mathrm{T}}(\mathbf{x}-\mathbf{y}^j) + 0.5(\mathbf{x}-\mathbf{y}^j)^{\mathrm{T}}\mathbf{G}^j(\mathbf{x}-\mathbf{y}^j).$$

Define $\mathbf{d} = \mathbf{x} - \mathbf{x}^k$, and

$$\alpha^{k,j} = f(\mathbf{x}^k) - f_{\mathrm{Q}}^j(\mathbf{x}^k)$$

then

$$f_{\mathrm{Q}}^j(\mathbf{x}) = f(\mathbf{x}^k) + (\mathbf{g}^j)^{\mathrm{T}}\mathbf{d} + 0.5\mathbf{d}^{\mathrm{T}}\mathbf{G}^j\mathbf{d} - \alpha^{k,j}. \quad (28)$$

Similar to common bundle method, we can solve iteratively

$$\min_{\mathbf{x}} f_{\mathrm{PQ}}^k(\mathbf{x}) = \min_{\mathbf{d}} f_{\mathrm{PQ}}^k(\mathbf{x}^k + \mathbf{d}) \quad (29)$$

in order to find the minimizer of f, if the piecewise quadratic function $f_{\mathrm{PQ}}^k(\mathbf{x})$ locally approximates $f(\mathbf{x})$ from below. This can be achieved by replacing $\alpha^{k,j}$ with $\beta^{k,j}$ defined as follows:

$$\beta^{k,j} = \max\{|\alpha^{k,j}|, \gamma\|\mathbf{x}^k - \mathbf{y}^j\|\} \quad (30)$$

where γ is a small positive number [14]. With this modification, we have

$$\min_{\mathbf{x}} f_{\mathrm{PQ}}^k(\mathbf{x}) \le f_{\mathrm{PQ}}^k(\mathbf{x}^k) = f(\mathbf{x}^k) - \min_{j\in J^k} \beta^{k,j} \le f(\mathbf{x}^k)$$

which guarantees $f_{\mathrm{PQ}}^k(\mathbf{x})$ is an approximation to $f(\mathbf{x})$ from below in the vicinity of $\mathbf{x}^k$. Now we can solve iteratively

$$\begin{aligned} &\min_{\mathbf{d}} v \\ &\text{s.t. } f(\mathbf{x}^k) + (\mathbf{g}^j)^{\mathrm{T}}\mathbf{d} + 0.5\mathbf{d}^{\mathrm{T}}\mathbf{G}^j\mathbf{d} - \beta^{k,j} \le v, \\ &\qquad \forall j \in J^k \quad (31) \end{aligned}$$

which is equivalent to (29), in order to find the minimizer of f. As discussed in [14], (31) can be solved efficiently using sequential quadratic programming (SQP).

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