Variation of distribution factors with loading

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***Abstract*—Power transfer distribution factors depend on the operating point and topology of an electric power system. However, it is known empirically that, for a fixed topology, the power transfer distribution factors are relatively insensitive to the operating point. We demonstrate this result theoretically for systems of arbitrary topology with losses, but only for the special case of having reactive compensation sufficient to keep voltages constant at all busses. We also analyze a power to current distribution factor that more closely relates to thermal constraints. We provide empirical corroboration for the theoretical result.**

Keywords: Power transfer distribution factors, shift factors.

I. Introduction

An (incremental) power transfer distribution factor (PTDF) is the relative change in power flow on a particular line due to a change in injection and corresponding withdrawal at a pair of busses. PTDFs depend on the topology of the electric power system, the behavior of controllable transmission system elements as their limits are approached, and on the operating point [1]. That is, PTDFs change when an outage of a line occurs, if a controllable element reaches its control limits, and also as the pattern of injections and withdrawals change the loadings on the lines in the system.

For the case of identical radial parallel lines, however, the PTDFs are completely independent of line loading. Moreover, it is known empirically that, given a fixed topology and ignoring controllable device limits, the PTDFs are relatively insensitive to the levels of injections and withdrawals. See, for example, [2][3, §3.9] for empirical studies of the variation of PTDFs for certain systems.

In this paper we develop theoretical insight into this empirical observation by showing, in corollary 3, that the PTDFs are approximately constant in a system with losses and arbitrary topology but having reactive compensation sufficient to keep voltage magnitudes constant at all busses. When the hypotheses of corollary 3 do not hold, the PTDFs can be expected to vary significantly as loadings change.

The relative insensitivity of the PTDFs is due fundamentally to the fact that the $\sin$ function has a Taylor expansion with zero quadratic term, so that linearization of the $\sin$ function about zero angle results in an error that is cubic and higher order in the angle, not a quadratic error. Consequently, the variation of the PTDFs with net power injection is quadratic and higher order rather than linear. Moreover, in some circumstances the power flow equations yield PTDFs that are exactly constant independent of injections [4].

The significance of the relative insensitivity of PTDFs to loading is that, in the context of flowgate right schemes for transmission rights [5], [6], capacity to flow power on a line or a group of lines is sold or leased to users of the transmission system. A transmission system user wishing to inject power at one point and withdraw it at another may want to purchase enough capacity on the line so as to hedge its congestion costs. If the PTDFs for a particular line vary significantly with the flows on the other lines then it is more difficult to predict the amount of capacity needed on each binding "flowgate." Either the risk due to variation of the PTDFs must be borne amongst the sellers and buyers of the transmission capability or conservative capacity limits must be used to compensate for variation of the PTDFs. If the PTDFs are relatively constant, however, then presumably the appropriate power flow capacity on each flowgate could be reserved to hedge the transmission congestion costs.

However, a further issue is that relative constancy of the PTDFs may not be the best measure of the lack of risk of unhedged transmission requirements. This is because, for example, in a thermally limited line the fundamental limiting factor is not literally the power flow down the line but rather the resistive losses in the line, which are proportional to the square of the magnitude of the current. Similarly, in a steady-state stability limited line, the angle across the line (or between a generation center and a demand center) is the limiting factor.

To consider current flow in a thermally limited line, we investigate power to current magnitude distribution factors (PIDFs) that measure the (incremental) effect of a change in power injection on the magnitude of the current in a line. (A similar development is possible for stability limited lines by considering the effect of a change in power injection on the angle across a line.)

PIDFs relate more closely to thermal constraints than do PTDFs, and we will see that they have similar properties to PTDFs. Under the DC power flow approximation, PTDFs and PIDFs are proportional to each other. However, in a nonlinear setting, the conditions for the PIDFs to be relatively constant as line loadings vary are more stringent than those for PTDFs, pointing to technical requirements that must be satisfied for flowgate rights on thermally limited transmission lines.

The results for systems of general topology are dependent on the assumption of there being voltage support at all busses sufficient to maintain constant voltage. The results will not hold where voltage constraints are binding, since by definition there is inadequate reactive support to maintain constant voltage. (However, it should be pointed out that in this case, the thermal limits themselves are not constant, since they are ultimately derived from the current rating and the operating voltage.) The results are unlikely to hold on lower voltage parts of the transmission system but may be applicable to higher voltage lines having sufficient reactive support and moderate ratios of

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resistance to reactance.

The structure of the paper is as follows. Section II presents a brief literature survey. In section III, we discuss the assumptions, the power flow formulation, and formal definition of PTDF. We then analyze PTDFs in section IV. In section V, we define and characterize PIDFs. A refinement of the PTDF calculation is presented in section VI. In section VII, we present some numerical results for the Electric Reliability Council of Texas (ERCOT) system that confirm the theoretical results. We conclude in section VIII.

II. Literature survey

Several authors discuss PTDFs in the context of approximating power flows. For example, Baughman and Schweppe use distribution factors to approximate flows as a function of injections and after a change in the topology of the network [7]. Sauer formulates PTDFs for linear load flows in [8]. Ng describes PTDFs for calculating the change in flows on lines given changes in generation and conforming changes in load at the busses [9]. Wood and Wollenberg describe the calculation of PTDFs using the DC power flow approximation in [10, Appendix 11A] and also discuss the calculation of PTDFs for outage conditions [10, §11.3.2]. The evaluation of PTDFs at an operating point from the Jacobian of the power flow equations is described in [10, §13.3].

Grijalva analyzes in detail the variation of PTDFs in a three bus, three line example system with voltages maintained constant and also discusses how the PTDFs vary with loading. Grijalva shows that if voltages are maintained constant at all busses then, as loading increases from zero injection conditions, the PTDFs that were largest at zero injection tend to decrease, while the PTDFs that were smallest at zero injection tend to increase [3, §3.9].

Generalizing the three bus, three line system, Grijalva discusses PTDFs from a given point of injection and a given point of withdrawal to each of the lines in a cutset of the power system, observing that in a lossless system the sum of the PTDFs across all lines in a cutset must be equal to one. Therefore, increases in PTDFs to some lines must be accompanied by decreases in the PTDFs to other lines. Grijalva observes that PTDFs begin to change significantly as steady-state stability limits are approached since the PTDF is zero for the condition of maximum transfer across a line when the angle across it is $\pi/2$ [3, §3.3 and Figure 3.3].

Grijalva also considers higher order terms in a Taylor expansion of the PTDFs using a rectangular representation for the voltage phasor and evaluates a quadratic approximation to the solution of the power flow [3, §3.9]. We take an analogous approach in section IV; however, we use a polar representation of the voltage phasor and consider a Taylor expansion about the zero injection operating point and also in terms of the entries in the real part of the admittance matrix. This allows for convenient explicit evaluation of the linear terms in the Taylor expansion for arbitrary topology systems.

Liu and Gross conduct an empirical study of the variation of PTDFs with injections and with other changes [2]. They show that for the system considered the PTDFs typically change by a relatively small amount as the levels of injections and withdrawals change.

Sauer *et al.* introduce and analyze various distribution factors in [11], including two that are closely related to the PIDFs that we consider. In particular, they define distribution factors of current injections to current flows (current transfer distribution factors or CTDFs), noting that the CTDFs are customarily converted to PTDFs. The PIDFs that we define are similar in flavor to the CTDFs except that our interest is in the effect of power injections on current flows.

Sauer *et al.* also consider angle distribution factors under outage conditions, generalizing the distribution factors that we consider to the line outage case [11, §5]. The analysis that we present concerning the relative constancy of distribution factors could be applied to the outage distribution factors considered in [11] and also to the various other distribution factors defined there.

Finally, Fradi *et al.* consider non-linear allocation of quantities to transactions [12]. They emphasize the variation of PTDFs. In contrast, we consider the conditions under which the PTDFs are relatively constant.

III. Assumptions and formulation

The material in this section is based on [10], [13] and is mostly standard. We develop it in detail so that we can precisely state the results to follow. We consider the single phase equivalent of a power system having $n+1$ busses. Bus number 0 is the reference bus and will be assumed to have reference angle of zero, while the other busses are labeled 1 through n. We use the symbols h, k, ℓ, m, s, t, u to index the busses.

For the analysis in sections IV–V we will have to assume that voltage magnitudes are constant (so that each bus, besides the reference bus, is a PV bus [13, §10.2].) We will consider the net power injections at each bus and the voltage angles at each bus explicitly. Consequently, we will explicitly represent net power injections and angles as arguments in the functions that we define to formulate the power flow equations. The voltage magnitudes will not be represented explicitly as arguments, but will be considered parameters.

We consider the effect of losses. We will explicitly represent the entries in the real part of the bus admittance matrix as arguments in the functions that we define.

Let the (ℓ, m) entry of the bus admittance matrix [13] be $G_{\ell m} + jB_{\ell m}$. Collect the entries in the real part of the bus admittance matrix together into a vector $G \in \mathbb{R}^{L+n}$, where L is the number of lines in the network. Let the net power injected by generation and demand at node ℓ be P_ℓ, so that for generator busses, $P_\ell > 0$. Let the voltage magnitude at bus ℓ be $|v_\ell|$ and its angle be Θ_ℓ. Collect the power injections at all the busses, except the reference bus, together into a vector $P \in \mathbb{R}^n$ and collect the angles at all busses, except the reference bus, together into a vector $\Theta \in [-\pi, \pi]^n$.

For every bus ℓ (including the reference bus) define functions

$p_\ell : \mathbb{R}^n \times \mathbb{R}^{L+n} \times \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall P \in \mathbb{R}^n, \forall G \in \mathbb{R}^{L+n}, \forall \Theta \in \mathbb{R}^n,$$
$$p_\ell(P, G, \Theta) = \sum_{m \in \mathbb{K}(\ell) \cup \{\ell\}} |v_\ell||v_m| \times$$
$$[G_{\ell m} \cos(\Theta_\ell - \Theta_m) + B_{\ell m} \sin(\Theta_\ell - \Theta_m)] - P_\ell,$$

where $\mathbb{K}(\ell)$ is the set of busses directly connected to bus ℓ by a line.

Collect the functions p_ℓ for each bus ℓ, except the reference bus, into a vector function $p : \mathbb{R}^n \times \mathbb{R}^{L+n} \times \mathbb{R}^n \to \mathbb{R}^n$. Then, given a vector of net injections P and a vector of entries in the real part of the bus admittance matrix G, solving the power flow is equivalent to solving for Θ in:

$$p(P, G, \Theta) = \mathbf{0}, \tag{1}$$

where:

- $\mathbf{0}$ is the vector of all zeros and
- the injection at the reference bus can be calculated once the vector of angles Θ is known.

Consider a vector of net injections $P^\star$, values of the real parts of the bus admittance matrix $G^\star$, and a corresponding solution $\Theta^\star$ of the power flow equations (1). We consider the properties of the solution as the vector of net injections is varied about $P^\star$ and as we consider changes to $G^\star$.

We first note that p is infinitely partially differentiable with respect to Θ. Suppose that the Jacobian $\frac{\partial p}{\partial \Theta}(P^\star, G^\star, \Theta^\star)$ is non-singular. Then by the implicit function theorem [14, §4.4] there exists an infinitely partially differentiable function $\theta : \mathbb{R}^n \times \mathbb{R}^{L+n} \to \mathbb{R}^n$ such that in some neighborhood $\mathcal{N}$ of $(P, G) = (P^\star, G^\star)$, the power flow equations (1) has a solution satisfying:

$$\forall (P, G) \in \mathcal{N}, p(P, G, \theta(P, G)) = \mathbf{0}.$$

That is, as is well-known, the power flow equations have a well-behaved solution in this neighborhood.

Consider the flow along a line joining busses ℓ and m. Neglecting shunt conductance in a line, we can evaluate the power flowing from bus ℓ into the line joining bus ℓ and m by the function $p_{\ell m} : \mathbb{R}^{L+n} \times \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\forall G \in \mathbb{R}^{L+n}, \forall \Theta \in \mathbb{R}^n, p_{\ell m}(G, \Theta) = |v_\ell||v_m| \times$$
$$[G_{\ell m} \cos(\Theta_\ell - \Theta_m) + B_{\ell m} \sin(\Theta_\ell - \Theta_m)] - |v_\ell|^2 G_{\ell m}.$$

If there are losses in the system, so that $G_{\ell m} < 0$, then the flow will be different at different points along the line. As a representative flow for the line joining ℓ to m, we take the average of the flows at the two ends of the line. That is, define $\tilde{p}_{\ell m} : \mathbb{R}^{L+n} \times \mathbb{R}^n \to \mathbb{R}$ by:

$$\begin{aligned}
&\forall G \in \mathbb{R}^{L+n}, \forall \Theta \in \mathbb{R}^n, \tilde{p}_{\ell m}(G, \Theta) \\
&= \tfrac{1}{2}(p_{\ell m}(\Theta) - p_{m\ell}(\Theta)), \\
&= |v_\ell||v_m| B_{\ell m} \sin(\Theta_\ell - \Theta_m) - \tfrac{1}{2}(|v_\ell|^2 - |v_m|^2) G_{\ell m}.
\end{aligned}$$

(We would obtain essentially the same results in the theorems below if we considered the sending end flow or the receiving end flow.) To relate the representative flow to the net injections, we define the function $\hat{p}_{\ell m} : \mathbb{R}^n \times \mathbb{R}^{L+n} \to \mathbb{R}$ by:

$$\forall (P, G) \in \mathcal{N}, \hat{p}_{\ell m}(P, G) = \tilde{p}_{\ell m}(G, \theta(P, G)).$$

Consider a bus k and a line joining busses ℓ and m. We consider the effect on the representative flow along the line joining ℓ and m of a change in the net injection at bus k from the level $P_k^\star$ (and assuming a corresponding change in the net withdrawal at the reference bus to maintain a solution of the power flow equations.) Following Wood and Wollenberg [10], the (incremental) power transfer distribution factor (PTDF) from injection at bus k to flow on the line joining ℓ to m is the sensitivity:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star, G^\star) = \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G^\star, \Theta^\star) \frac{\partial \theta}{\partial P_k}(P^\star, G^\star).$$

For brevity, we call this sensitivity "the PTDF from k to line ℓm."

In general, transactions may involve a change in injection at a bus k and a corresponding change at another bus h (that may not be the reference bus.) In this case, and if the system is lossless, then the PTDF from injection at bus k and withdrawal at bus h to flow on the line joining ℓ to m is the difference of sensitivities:

$$\begin{aligned}
&\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star, G^\star) - \frac{\partial \hat{p}_{\ell m}}{\partial P_h}(P^\star, G^\star) \\
&= \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G^\star, \Theta^\star) \left(\frac{\partial \theta}{\partial P_k}(P^\star, G^\star) - \frac{\partial \theta}{\partial P_h}(P^\star, G^\star) \right).
\end{aligned}$$

For brevity, we call this sensitivity "the PTDF from kh to line ℓm."

In the following section we calculate the PTDFs from k to line ℓm.

IV. Power transfer distribution factors

We calculate an estimate of the PTDF, under the assumption that all the voltage magnitudes are constant. That is, we assume that all busses are PV busses [13, §10.2] with adequate reactive support to maintain constant voltage. This assumption is not realistic; however, it can be approximately true. Moreover, after corollary 3 and in section VI we will comment on weakening this assumption slightly.

Consider a system with entries in the real part of the admittance matrix $G^\star$ and consider the PTDF from k to line ℓm at some operating point $P^\star, \Theta^\star$ such that the Jacobian $\frac{\partial p}{\partial \Theta}(P^\star, G^\star, \Theta^\star)$ is non-singular. Again, using the implicit function theorem, we can solve the power flow equations in a neighborhood $\mathcal{N}$ of $(P^\star, G^\star)$ for a solution θ as a function of P and G.

Suppose that this neighborhood $\mathcal{N}$ of $(P^\star, G^\star)$ includes a line segment joining $(\mathbf{0}, \mathbf{0})$ and $(P^\star, G^\star)$. Then, by Taylor's theorem applied to the derivative of $\hat{p}_{\ell m}$, and assuming that the voltage magnitudes are constant, the PTDF satisfies:

$$\begin{aligned}
\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star, G^\star) &= \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}) + \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0}, \mathbf{0}) P^\star \\
&\quad + \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial G}(\mathbf{0}, \mathbf{0}) G^\star + o(P^\star, G^\star),
\end{aligned} \tag{2}$$

where $o(P^\star, G^\star)$ is a function such that $\frac{\|o(P^\star,G^\star)\|}{\|(P^\star,G^\star)\|} \rightarrow 0$ as $\|(P^\star, G^\star)\| \rightarrow 0$. The error in neglecting the term $o(P^\star, G^\star)$ in (2) is *quadratic* and higher order in $(P^\star, G^\star)$.

In (2), $\dfrac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0})$ is the PTDF from k to line ℓm for zero injections. For a lossless system, $\dfrac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0})$ is the PTDF from k to line ℓm calculated according to the DC power flow approximation. For a system with losses, $\dfrac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0})$ can be evaluated with a similar calculation that uses the complex part of line admittances instead of the inverse of the inductive reactances. For convenience, by "DC PTDF" we will mean $\dfrac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0})$.

In the development that follows, we will show that the coefficients $\dfrac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0}, \mathbf{0})$ and $\dfrac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial G}(\mathbf{0}, \mathbf{0})$ of the linear terms in the expression (2) for the PTDF are all zero. That is, the PTDF evaluated at $P^\star$ is equal to a constant plus terms that are quadratic and higher order in $(P^\star, G^\star)$. This accounts for the relative constancy of the PTDFs at low to medium load if voltage constraints are not binding.

We evaluate the terms in the PTDF in the following:

Lemma 1: Consider a line ℓm and a bus k. We have the following expressions for the derivatives:

$$
\begin{aligned}
\forall G, \Theta, \frac{\partial(\Theta_\ell - \Theta_m)}{\partial \Theta}(\Theta) &= [\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\forall G, \Theta, \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G, \Theta) &= |v_\ell||v_m| \times \\
&\quad B_{\ell m} \cos(\Theta_\ell - \Theta_m)[\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\forall G, \Theta, \frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(G, \Theta) &= -|v_\ell||v_m| \times \\
&\quad B_{\ell m} \sin(\Theta_\ell - \Theta_m)[\mathbf{I}_\ell - \mathbf{I}_m][\mathbf{I}_\ell - \mathbf{I}_m]^\dagger, \\
\frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0}, \mathbf{0}) &= \mathbf{0}, \\
\forall G, \Theta, \frac{\partial^2 \tilde{p}_{\ell m}}{\partial G \partial \Theta}(G, \Theta) &= \mathbf{0}, \\
\forall P, G, \Theta, \frac{\partial p}{\partial P_k}(P, G, \Theta) &= -\mathbf{I}_k, \\
\forall P, G, \Theta, \forall t, \frac{\partial^2 p}{\partial P_k \partial P_t}(P, G, \Theta) &= \mathbf{0}, \\
\forall P, G, \Theta, \forall u, v, \frac{\partial^2 p}{\partial P_k \partial G_{uv}}(P, G, \Theta) &= \mathbf{0}, \\
\forall P, G, \Theta, \forall t, \frac{\partial^2 p}{\partial \Theta \partial P_t}(P, G, \Theta) &= \mathbf{0}, \\
\forall P, G, \Theta, \forall s, t, \frac{\partial p_s}{\partial \Theta_t}(P, \Theta) &= \\
&\begin{cases} \sum_{u \in \mathbb{K}(s)} |v_s||v_u| \times \\ [-G_{su} \sin(\Theta_s - \Theta_u) + B_{su} \cos(\Theta_s - \Theta_u)], \\ \text{if } t = s, \\ |v_s||v_t| \times \\ [G_{st} \sin(\Theta_s - \Theta_t) - B_{st} \cos(\Theta_s - \Theta_t)], \\ \text{if } t \in \mathbb{K}(s), \end{cases} \\
\forall P, \forall s, t, u, \frac{\partial^2 p_s}{\partial \Theta_t \partial \Theta_u}(P, \mathbf{0}, \mathbf{0}) &= 0, \\
\forall P, \forall s, t, u, v, \frac{\partial^2 p_s}{\partial \Theta_t \partial G_{uv}}(P, \mathbf{0}, \mathbf{0}) &= 0,
\end{aligned}
$$

where we note that:

- superscript † denotes transpose,
- $\mathbf{0}$ denotes a vector or matrix of all zeros, and
- $\mathbf{I}_\ell$ is the vector with all zeros except for a one in the ℓ-th place.

Proof: All of the terms follow from definition of the functions, direct calculation, and substitution. □

Corollary 2: Consider a line ℓm and a bus k. If $\dfrac{\partial p}{\partial \Theta}(\mathbf{0}, \mathbf{0}, \mathbf{0})$ is non-singular then $\dfrac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $\dfrac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial G}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

Proof: We note that for a lossless system, $P = \mathbf{0}$ and $\Theta = \mathbf{0}$ is a solution of the power flow equations (1). That is, $p(\mathbf{0}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$. Again using the implicit function theorem, there is an infinitely partially differentiable function $\theta : \mathbb{R}^n \times \mathbb{R}^{L+n} \rightarrow \mathbb{R}^n$ such that in some neighborhood $\mathcal{N}_0$ of $(P, G) = (\mathbf{0}, \mathbf{0})$, the power flow equations (1) have a solution satisfying:

$$\forall (P, G) \in \mathcal{N}, p(P, G, \theta(P, G)) = \mathbf{0}.$$

Totally differentiating $p(P, G, \theta(P, G)) = \mathbf{0}$ with respect to

P_k and evaluating at $(P, G) = (\mathbf{0}, \mathbf{0})$, we obtain:

$$\begin{aligned} \mathbf{0} &= \frac{\partial p}{\partial P_k}(\mathbf{0},\mathbf{0},\mathbf{0}) + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}), \\ &= -\mathbf{I}_k + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}), \end{aligned}$$

so that, since $\frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})$ is non-singular by hypothesis, $\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0})$ is well-defined.

Let t be a bus. Totally differentiating $p(P, G, \theta(P, G)) = \mathbf{0}$ with respect to P_k and then with respect to P_t and evaluating at $(P, G) = (\mathbf{0}, \mathbf{0})$, we obtain:

$$\begin{aligned} \mathbf{0} &= \frac{\partial^2 p}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0},\mathbf{0}) + \frac{\partial^2 p}{\partial P_k \partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_t}(\mathbf{0},\mathbf{0}) \\ &\quad + \left[\left[\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0})\right]^\dagger \frac{\partial^2 p_s}{\partial \Theta^2}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_t}(\mathbf{0},\mathbf{0})\right]_{s=1,\ldots,n} \\ &\quad + \frac{\partial^2 p}{\partial P_t \partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}) \\ &\quad + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0}), \\ &= \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0}), \end{aligned}$$

by lemma 1, where the terms of the form $[\cdots]_{s=1,\ldots,n}$ mean a vector having s-th entry given by the term inside the square brackets. Again, since $\frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})$ is non-singular, we have $\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0}) = \mathbf{0}$.

Let uv be a line. Totally differentiating $p(P, G, \theta(P, G)) = \mathbf{0}$ with respect to P_k and then with respect to G_{uv} and evaluating at $(P, G) = (\mathbf{0}, \mathbf{0})$, we obtain:

$$\begin{aligned} \mathbf{0} &= \frac{\partial^2 p}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0},\mathbf{0}) + \frac{\partial^2 p}{\partial P_k \partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial G_{uv}}(\mathbf{0},\mathbf{0}) \\ &\quad + \left[\left[\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0})\right]^\dagger \frac{\partial^2 p_s}{\partial \Theta^2}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial G_{uv}}(\mathbf{0},\mathbf{0})\right]_{s=1,\ldots,n} \\ &\quad + \frac{\partial^2 p}{\partial G_{uv} \partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}) \\ &\quad + \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0}), \\ &= \frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0}), \end{aligned}$$

by lemma 1. Again, since $\frac{\partial p}{\partial \Theta}(\mathbf{0},\mathbf{0},\mathbf{0})$ is non-singular, we have $\frac{\partial^2 \theta}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0}) = \mathbf{0}$.

Also, by lemma 1, $\frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0},\mathbf{0}) = \mathbf{0}$. By direct calculation:

$$\begin{aligned} &\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0},\mathbf{0}) \\ &\quad = \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}), \\ &\forall t, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0}) \\ &\quad = \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial P_t}(\mathbf{0},\mathbf{0}) \\ &\qquad + \left[\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0})\right]^\dagger \frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_t}(\mathbf{0},\mathbf{0}), \\ &\forall u, v, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0}) \\ &\quad = \frac{\partial^2 \tilde{p}_{\ell m}}{\partial G_{uv} \partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0}) \\ &\qquad + \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(\mathbf{0},\mathbf{0})\frac{\partial^2 \theta}{\partial P_k \partial G_{uv}}(\mathbf{0},\mathbf{0}) \\ &\qquad + \left[\frac{\partial \theta}{\partial P_k}(\mathbf{0},\mathbf{0})\right]^\dagger \frac{\partial^2 \tilde{p}_{\ell m}}{\partial \Theta^2}(\mathbf{0},\mathbf{0})\frac{\partial \theta}{\partial G_{uv}}(\mathbf{0},\mathbf{0}). \end{aligned}$$

Substituting in from the terms previously calculated, we obtain: $\forall t, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial P_t}(\mathbf{0}) = \mathbf{0}$ and $\forall u, v, \frac{\partial^2 \hat{p}_{\ell m}}{\partial P_k \partial G_{uv}}(\mathbf{0}) = \mathbf{0}$. □

Corollary 3: Consider a system with the entries in the real part of the bus admittance matrix specified by $G^\star$ and reactive compensation such that all bus voltage magnitudes are constant. Also consider an operating point $P^\star$ sufficiently close to the condition of zero net injection such that for all operating points on the line segment joining $(\mathbf{0}, \mathbf{0})$ and $(P^\star, G^\star)$ we have that:

- the solution of the power flow equations are well-defined and unique and
- the Jacobian $\frac{\partial p}{\partial \Theta}$ is non-singular.

Then the incremental PTDFs at the operating point $P^\star$ for the system with entries in the real part of the bus admittance matrix $G^\star$ differ from the PTDFs calculated from the DC load flow by an error that is quadratic and higher order in $(P^\star, G^\star)$. That is,

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star, G^\star) = \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0},\mathbf{0}) + o(P^\star, G^\star). \tag{3}$$

□

Using a network equivalencing argument, we can slightly weaken the requirement of constancy of voltage in corollary 3 to only requiring that all generator and load buses and busses ℓ and m are held at constant voltage.

To interpret corollary 3, we consider the conditions when the error between the incremental PTDF $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P, G)$ and the DC PTDF $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0},\mathbf{0})$ becomes large. We have the following, based on Grijalva [3, §3.3]:

Lemma 4: Consider a line ℓm and a bus k. Also consider any vector of net injections P and values of real parts of the admittance matrix G such that the angle across the line ℓm is $\Theta_\ell - \Theta_m = \pi/2$. Then:

- $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P, G) = 0$ and
- the representative flow $\hat{p}_{\ell m}$ on line ℓm is at its maximum value given by:

$$\overline{p}_{\ell m} = |v_\ell||v_m|B_{\ell m} - \tfrac{1}{2}(|v_\ell|^2 - |v_m|^2)G_{\ell m}.$$

Proof: The first result follows from the definition of the PTDF on substituting the value $\Theta_\ell - \Theta_m = \pi/2$ into the expression for $\frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G, \Theta)$ from lemma 1. The second result follows from the definition of $\hat{p}_{\ell m}$. □

The value $\overline{p}_{\ell m}$ defined in lemma 4 is the steady-state stability limit of line ℓm for the given value of G. We will now use corollary 3 and lemma 4 to roughly estimate when we can expect the difference between the incremental and DC PTDFs to be smaller than, say, 5%. The condition will involve limiting the flow on line ℓm to being a fraction of $\overline{p}_{\ell m}$.

Neglecting cubic and higher order terms in (3), we have that for any P and G satisfying the conditions on $P^\star$ and $G^\star$ in corollary 3:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P, G) - \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}) \approx \begin{bmatrix} P \\ G \end{bmatrix}^\dagger E \begin{bmatrix} P \\ G \end{bmatrix}, \quad (4)$$

where $E \in \mathbb{R}^{(L+2n)\times(L+2n)}$ is a coefficient matrix characterizing the quadratic terms in (3). Now, in lemma 4, set $G = 5G^\star$ and $P = \overline{P}$, where $\overline{P}$ is any vector of net injections such that the angle across the line ℓm is $\Theta_\ell - \Theta_m = \pi/2$ so that the flow on line ℓm is at its steady state stability limit. That is:

$$\begin{aligned} \overline{p}_{\ell m} &= |v_\ell||v_m|B_{\ell m} - \tfrac{1}{2}(|v_\ell|^2 - |v_m|^2)G_{\ell m}, \\ &= |v_\ell||v_m|B_{\ell m} - \tfrac{1}{2}(|v_\ell|^2 - |v_m|^2)5G^\star_{\ell m}. \end{aligned}$$

By lemma 4, $\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\overline{P}, 5G^\star) = 0$. We assume that G and P satisfy the conditions on $G^\star$ and $P^\star$ in corollary 3. We obtain that:

$$\begin{aligned} \begin{bmatrix} \overline{P} \\ 5G^\star \end{bmatrix}^\dagger E \begin{bmatrix} \overline{P} \\ 5G^\star \end{bmatrix} &\approx \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\overline{P}, 5G^\star) - \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}), \\ &\qquad \text{by (4) for } P = \overline{P} \text{ and } G = 5G^\star, \\ &= 0 - \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}). \end{aligned}$$

Dividing both sides by $5^2 = 25$ yields:

$$\begin{bmatrix} \frac{1}{5}\overline{P} \\ G^\star \end{bmatrix}^\dagger E \begin{bmatrix} \frac{1}{5}\overline{P} \\ G^\star \end{bmatrix} = -\frac{1}{25}\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}).$$

But evaluating (4) for $P = \frac{1}{5}\overline{P}$ and $G = G^\star$, we obtain:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\frac{1}{5}\overline{P}, G^\star) - \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}) \approx \begin{bmatrix} \frac{1}{5}\overline{P} \\ G^\star \end{bmatrix}^\dagger E \begin{bmatrix} \frac{1}{5}\overline{P} \\ G^\star \end{bmatrix}.$$

Combining these yields:

$$\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\frac{1}{5}\overline{P}, G^\star) - \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}) \approx -\frac{1}{25}\frac{\partial \hat{p}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}).$$

That is, for a vector of net injections equal to $\frac{1}{5}\overline{P}$ the error between the incremental and DC PTDFs is only about $\frac{1}{25}$-th, or around 5%, of the value of the DC PTDF itself.

Now recall that given a vector of net injections equal to $\overline{P}$ the flow on ℓm was equal to the steady-state stability limit $\overline{p}_{\ell m}$. Therefore, for a vector of net injections only one-fifth as large, the flow on ℓm will be roughly only one-fifth as large. In summary, for conditions such that the flow on line ℓm is no more than about 20% of the steady-state stability limit, the error between the incremental and DC PTDFs will be no more than about 5% of the value of the DC PTDF.

That is, for a thermally limited line with thermal limits that bind at no more than 20% of $\overline{p}_{\ell m}$, the error in the PTDF at all feasible operating conditions would be no more than 5% of the DC PTDF. Using the data in [13, Table 3.1], this condition would be satisfied for a 138 kV line that is no more than about 25 miles long; for a 345 kV line that is no more than about 65 miles long; and for a 765 line that is no more than about 130 miles long.

In contrast, for a stability limited line with flows that are greater than 20% of $\overline{p}_{\ell m}$ or if the assumption of constant voltage is not maintained, then the error in the PTDF may become large. For example, if there are binding voltage constraints because of a lack of reactive power support, the PTDFs may vary significantly. It is not appropriate to rely on the constancy of PTDFs under these circumstances. We note that some errors in the PTDFs are positive while others are negative since, as discussed in [3, §3.3], the PTDFs from a point of injection and withdrawal to each of the lines in a cutset of a lossless power system must always sum to one.

V. Power to current distribution factors

Although PTDFs are often discussed in relation to thermally limited lines, in fact it is the heating due to current flowing on the line that determines the thermal limit. Instead of considering the effect of the change of power injection on the power flowing down a line, a more direct measure of the effect on a thermal constraint is the effect of a change in injection on the magnitude of the current flowing down the line. (In practice, the magnitude of the complex flow is often used as a proxy to the magnitude of the current.)

We again maintain the assumption that the voltage magnitude at each bus is constant. However, this assumption is insufficient to yield a result similar to corollary 3. In order for the power to current magnitude distribution factor from k to line ℓm to be approximately constant, we will see that we must additionally require that $|v_\ell| = |v_m|$. The reason for this is that if these voltages are different then there will be reactive power flowing along the line and the PTDFs will change more rapidly with flow.

Moreover, since we are interested in current magnitudes but the current magnitude is not differentiable at the condition of zero current, we will define a “directed” current magnitude that is differentiable and captures the relevant behavior of the current magnitude. We will assume that the power to current distribution factor of interest is relevant to a thermal limit that corresponds to power flowing from bus ℓ to bus m, so that $\Theta_\ell > \Theta_m$ at the operating point.

Ignoring the current flowing in the shunt capacitance of the line, the square of the current magnitude is given by:

$$\begin{aligned} &(G_{\ell m}^2 + B_{lm}^2)(|v_\ell|^2 + |v_m|^2 - 2|v_\ell||v_m|\cos(\Theta_\ell - \Theta_m)) \\ &\quad= (G_{\ell m}^2 + B_{lm}^2) \\ &\qquad \times ([|v_\ell| - |v_m|]^2 + 2|v_\ell||v_m|[1-\cos(\Theta_\ell - \Theta_m)]), \\ &\quad= 2(G_{\ell m}^2 + B_{lm}^2)|v_\ell||v_m|[1-\cos(\Theta_\ell - \Theta_m)], \end{aligned}$$

if $|v_\ell| = |v_m|$.

Paralleling the development in section III, we define the function $\tilde{i}_{\ell m} : \mathbb{R}^{L+n} \times \mathbb{R}^n \to \mathbb{R}$ by:

$$\forall G \in \mathbb{R}^{L+n}, \forall \Theta \in \mathbb{R}^n, \tilde{i}_{\ell m}(G, \Theta) = \begin{cases} \sqrt{G_{\ell m}^2 + B_{lm}^2}\sqrt{2|v_\ell||v_m|[1-\cos(\Theta_\ell - \Theta_m)]}, & \text{if } \Theta_\ell > \Theta_m, \\ -\sqrt{G_{\ell m}^2 + B_{lm}^2}\sqrt{2|v_\ell||v_m|[1-\cos(\Theta_\ell - \Theta_m)]}, & \text{if } \Theta_\ell < \Theta_m, \\ 0, & \text{if } \Theta_\ell = \Theta_m. \end{cases}$$

The function $\tilde{i}_{\ell m}$ is twice partially differentiable and its absolute value is the magnitude of the current on the line ℓm. (Strictly speaking, in the presence of shunt capacitance, this function is equal to the current only at the mid-point of the line between the busses ℓ and m.) To relate the current to the net injections, we define the function $\hat{i}_{\ell m} : \mathbb{R}^n \times \mathbb{R}^{L+n} \to \mathbb{R}$ by:

$$\forall (P, G) \in \mathcal{N}, \hat{i}_{\ell m}(P, G) = \tilde{i}_{\ell m}(G, \theta(P, G)).$$

The (incremental) power to current magnitude distribution factor (PIDF) from injection at bus k to current magnitude on the line ℓm is the sensitivity:

$$\frac{\partial \hat{i}_{\ell m}}{\partial P_k}(P^\star, G^\star) = \frac{\partial \tilde{i}_{\ell m}}{\partial \Theta}(G^\star, \Theta^\star)\frac{\partial \theta}{\partial P_k}(P^\star, G^\star).$$

For brevity, we call this sensitivity "the PIDF from k to line ℓm."

As in corollary 3 of section IV, we have:

$$\frac{\partial \hat{i}_{\ell m}}{\partial P_k}(P^\star, G^\star) = \frac{\partial \hat{i}_{\ell m}}{\partial P_k}(\mathbf{0}, \mathbf{0}) + o(P^\star, G^\star),$$

so that the incremental PIDFs at the operating point $P^\star$ differ from the PIDFs calculated from the DC load flow by an error that is quadratic and higher order in $(P^\star, G^\star)$.

Note that under the assumption of constant voltages, the PTDFs and the PIDFs calculated from the DC power flow are proportional to each other. At other operating points, however, the PIDFs can be expected to change more rapidly with flows than the PTDFs unless the condition $|v_\ell| = |v_m|$ is maintained.

The implication is that in a thermally limited system, for PIDFs to be constant, voltage support must be provided on the constrained lines to make the voltages constant and equal at both ends of each flowgate. If these conditions are not satisfied, then the PIDFs (and indeed the thermal capacity) will vary with loading. In particular,

- the PIDFs will vary as $|v_\ell| - |v_m|$ varies and
- the capacity to transmit real power will fall as the voltages at the sending or receiving end fall, since the thermal capacity is the product of the current carrying capability and the operating voltage.

In the extreme, if voltage constraints are binding then we cannot rely on constancy of PTDFs [1], [15]. In section VI, we will mention a refinement of the calculation to help alleviate this limitation.

VI. Refinements for flowgate applications

One important application of PTDFs is in flowgate transmission rights mechanisms. In this application, the most critical condition is when a line, for example line ℓm, is congested. If the injections in the system for this condition were known, we could directly use the incremental PTDF evaluated at this operating point. However, because many system conditions can give rise to congestion on a line ℓm, we cannot determine these conditions precisely in advance. The results we have developed show that in some cases the DC PTDF is a reasonable approximation that applies over a range of conditions.

However, we can refine the estimate of the incremental PTDF by utilizing the conditions for line ℓm to be congested. In particular, suppose that a thermally limited line ℓm is congested when the angle difference across the line is given by $\Theta_\ell - \Theta_m = \overline{\Delta\Theta}_{\ell m}$. Define $\overline{\Theta} \in \mathbb{R}^n$ to be a vector of all zeros except that in the ℓ-th place there is the entry $\overline{\Delta\Theta}_{\ell m}$. Then we can approximate the incremental PTDF when line ℓm is congested by:

$$\begin{aligned} \frac{\partial \hat{p}_{\ell m}}{\partial P_k}(P^\star, G^\star) &= \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G^\star, \overline{\Theta})\frac{\partial \theta}{\partial P_k}(P^\star, G^\star), \\ &\approx \frac{\partial \tilde{p}_{\ell m}}{\partial \Theta}(G^\star, \overline{\Theta})\frac{\partial \theta}{\partial P_k}(\mathbf{0}, \mathbf{0}), \end{aligned}$$

where we note that $\dfrac{\partial \tilde{p}_{\ell m}}{\partial P_k}$ depends only on $G_{\ell m}$ and $\Theta_\ell - \Theta_m$. That is, we use the actual congestion conditions to evaluate the sensitivity of the flow on line ℓm to angle, but then approximate the sensitivity of angle on injections using the DC approximation. A similar approach applies for PIDFs.

This approximation could potentially be applied even for voltage limited lines. In particular, we could consider the conditions under which voltage constraints are binding on the flow along a line and use this to estimate $\dfrac{\partial \tilde{p}_{\ell m}}{\partial P_k}$ or $\dfrac{\partial \tilde{i}_{\ell m}}{\partial P_k}$ at the conditions where voltage constraints are binding.

VII. Results for Electric Reliability Council of Texas System

In this section, we empirically validate the theoretical results using the Electric Reliability Council of Texas (ERCOT) system. Figure 1 shows DC PTDFs versus the incremental PTDFs

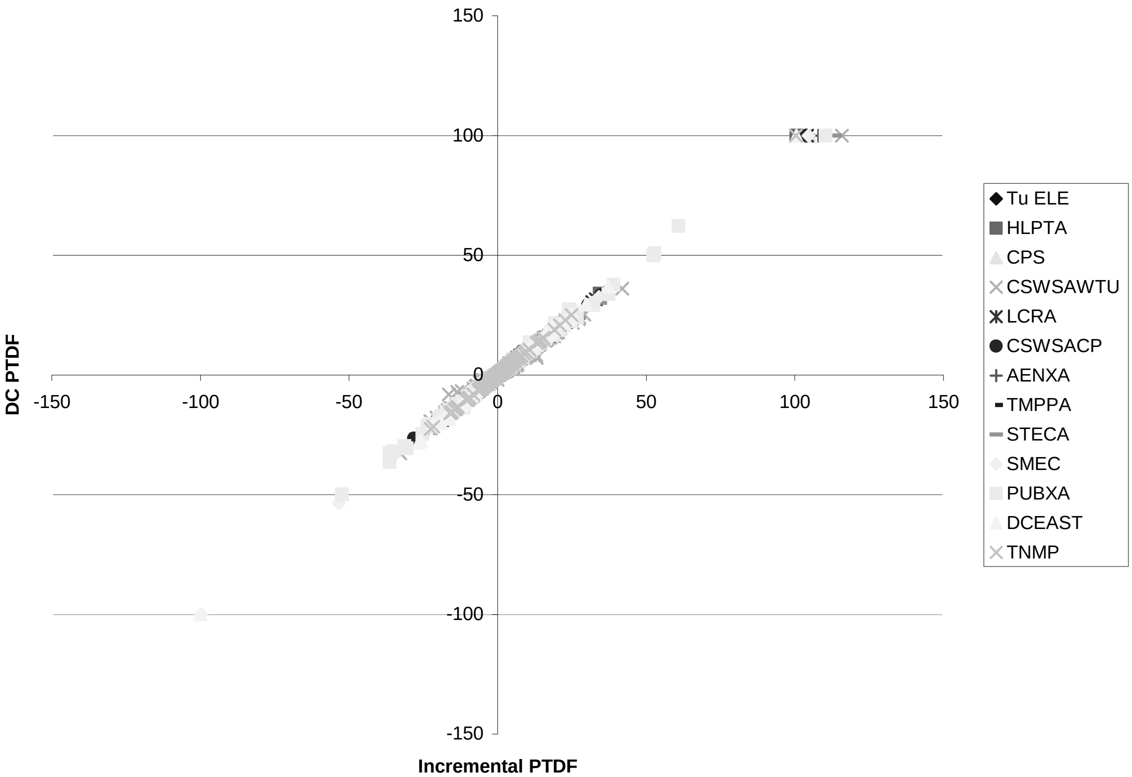

Fig. 1. DC PTDFs versus incremental PTDFs at 2002 Summer peak conditions for ERCOT system.

calculated for a 2002 Summer peak study case for thirteen different points of injection distributed across ERCOT and for 5989 transmission lines. Each point of injection is illustrated with a different symbol in figure 1. For each PTDF, the point of withdrawal is the reference bus. That is, there are approximately 78,000 DC PTDFs and incremental PTDFs represented in figure 1. The PTDFs are shown as percentages and essentially all of them fall on a line with slope equal to one and intercept equal to zero.

Figure 1 shows that for all points of injection and almost all lines, the DC PTDF and incremental PTDF are essentially the same. The only exception are the 13 PTDFs calculated for the line that joins the reference bus to the rest of the system. All power flowing to the reference bus flows through this line and consequently the incremental affect on losses throughout the system is reflected in this line.

It is important to note that the bus voltages for the Summer peak case are not all held constant. That is, corollary 3 seems to hold approximately even when the assumption of constant voltage is not completely met.

VIII. Conclusion

In this paper we presented conditions for PTDFs to be approximately independent of the injections and withdrawals in an electric power system. We showed theoretically that for systems with losses and arbitrary topology, multiple points of injection and withdrawal, and losses, the PTDFs are relatively independent of injections and withdrawals if topology is fixed, voltages are held constant, and the flows on lines are sufficiently less than the steady-state stability limit.

We also analyzed power to current magnitude distribution factors PIDFs. For relative constancy of the PIDFs from k to a line ℓm, we found that we must assume that $|v_\ell| = |v_m|$ in addition to the assumptions for relative constancy of PTDFs. That is, we must assume that there is adequate voltage support as a condition for the effectiveness of flowgate rights schemes.

In the context of a contingency limited system, this requires that controllable voltage support must also be available under contingency conditions. The conditions for the relative constancy of the distribution factors are stringent and may not hold in typical transmission systems.

We showed that the theoretical predictions are well corroborated for the ERCOT system, even when voltages are not held exactly constant. In future work, an extensive numerical study of the Eastern, Western, and ERCOT Interconnections will be undertaken to complement and further validate the theoretical results in this paper.

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