Basis Functions of 2-D Fourier Analysis

2-D Sinusoids are the Basis Functions of 2-D Fourier Analysis

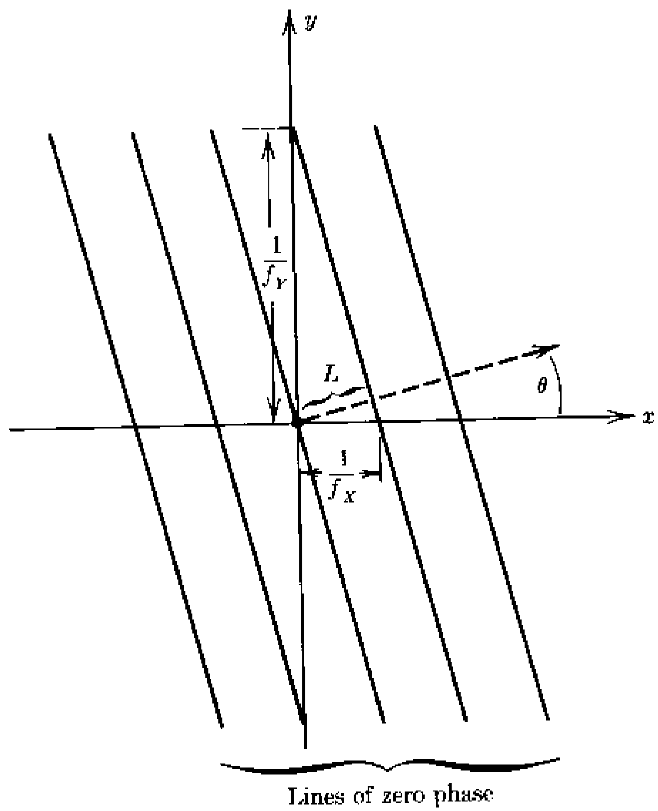


Figure 2-1 Lines of zero phase for the function $\exp[j2\pi(f_X x + f_Y y)]$.

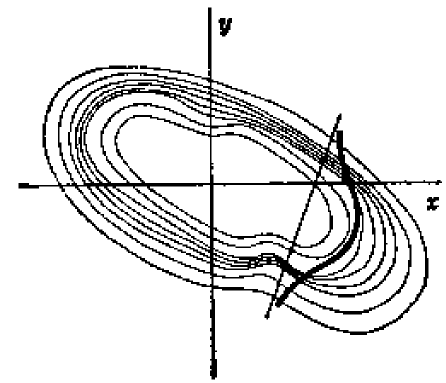
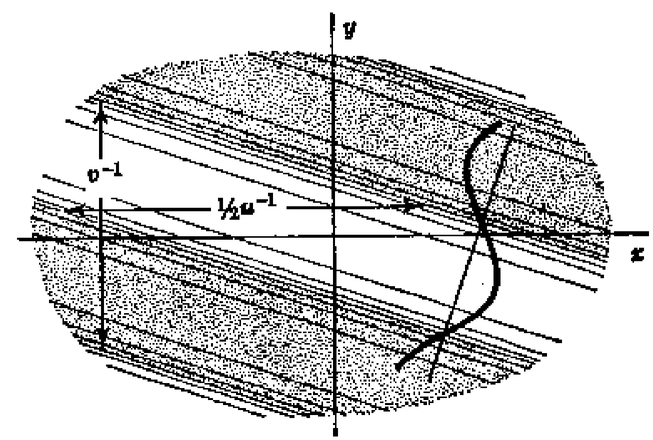


Fig. 12.1 A mountain



and a prominent Fourier component thereof.

Fourier Transform Guide

The 1-D Fourier transform integrals

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du \qquad F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

Basic theorems in 1-D

$f(x/a)$	scaling	$\lvert a \rvert F(a u)$
$f^*(x)$		$F^*(-u)$
$f(x - x_0)$	shift	$e^{-i2\pi u x_0} F(u)$
$e^{i2\pi u_0 x} f(x)$		$F(u - u_0)$
$f_1 \circledast f_2$	convolution	$F_1 F_2$
$f_1 f_2$		$F_1 \circledast F_2$
$f_1 \otimes f_2$	correlation	$F_1 F_2^*$
$f_1 + f_2$		$F_1 + F_2$

Fourier Guide – 1-D Functions

$\delta(x)$		1
1		$\delta(u)$
$\cos(2\pi u_0 x)$	sinusoids	$\frac{1}{2}[\delta(u-u_0)+\delta(u+u_0)]$
$\sin(2\pi u_0 x)$		$\frac{1}{2i}[\delta(u-u_0)-\delta(u+u_0)]$
$e^{i2\pi u_0 x}$		$\delta(u-u_0)$
$\exp(-\pi b^2 x^2)$	Gaussian	$\frac{1}{\lvert b\rvert}\exp\left[-\pi\left(\frac{u}{b}\right)^2\right]$
$\begin{Bmatrix} 1 & \lvert x\rvert \le x_0 \\ 0 & \lvert x\rvert > x_0 \end{Bmatrix}$	Rectangles	$2x_0\,\mathrm{sinc}(2ux_0)$
$Rect(x)$		$sinc(u) = \frac{\sin(\pi u)}{\pi u}$

Many EE books <u>*other than*</u> *Goodman or Bracewell use* $sinc(\pi u) = \frac{\sin(\pi u)}{\pi u}$

Fourier Guide – more 1-D Functions and Theorems

$Triangle(x)$		$sinc^2(u)$
$exp(-\pi x^2)$	Gaussian	$exp(-\pi u^2)$
$j \sin(2\pi x)$	sinusoids	$\frac{1}{2}\delta(u-1) - \frac{1}{2}\delta(u+1)$
$exp(j2\pi x)$		$\delta(u-1)$
$\cos(2\pi bx) f(x)$	modulation	$\frac{1}{2}[F(u-b) + F(u+b)]$
$\frac{df(x)}{dx}$		$j 2 \pi u F(u)$
$Comb(x/a)$ (δ areas = 1)		$(1/a)\, Comb(au)$ (δ areas = $1/a$) *This conserves energy per unit bandwidth.*

Fourier Guide in 2-D

The Fourier transform integrals

$$f(x, y) = \iint_{-\infty}^{\infty} F(u, v) e^{i2\pi(ux+vy)} du\, dv \qquad F(u, v) = \iint_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} dx\, dy$$

Theorems in 2-D

$f(ax, cy)$	scaling	$\frac{1}{\lvert a\rvert}\frac{1}{\lvert c\rvert} F\left(\frac{u}{a}, \frac{v}{c}\right)$
$f^*(x, y)$		$F^*(-u, -v)$
$f(x - x_0, y - y_0)$	shift	$e^{-i2\pi(ux_0 + vy_0)} F(u, v)$
$e^{i2\pi(u_0 x + v_0 y)} f(x, y)$		$F(u - u_0, v - v_0)$
$f_1 \circledast f_2$	convolution	$F_1 F_2$
$f_1 f_2$		$F_1 \circledast F_2$
$f_1 \otimes f_2$	correlation	$F_1 F_2^*$

Examples of 2-D Transforms

$Comb(x, y)$	$Comb(u, v)$
$Comb(x/a, y/b)$	$(1/ab) Comb(au, bv)$

$\delta(x)$ = *a single blade*

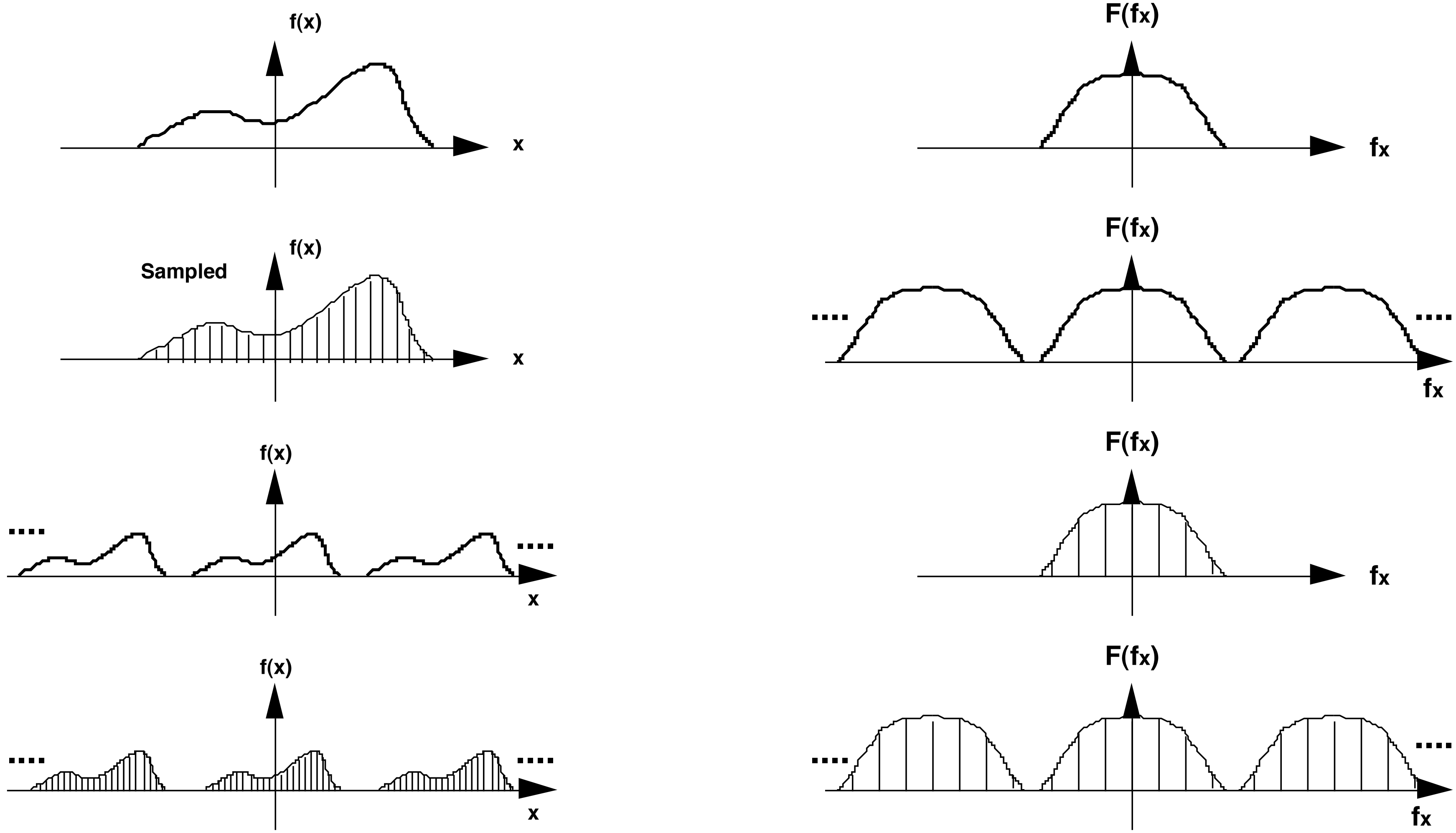
$Comb(x)$ = *a stack of blades*

$Comb(x)\,\delta(y)$ = *a 1-D comb of delta functions*

$Comb(x)\,Comb(y) = Comb(x)\,\delta(y) * Comb(y)\,\delta(x) = Comb(x, y)$
orchard or bed-of-nails function

Sampling and the Fourier Transform (DFT)

Space Domain   Spatial Frequency Domain



Diffraction Basics

- **Vector vs. Scalar diffraction models.**

- **What can you do with the Huygens idea about diffraction.**

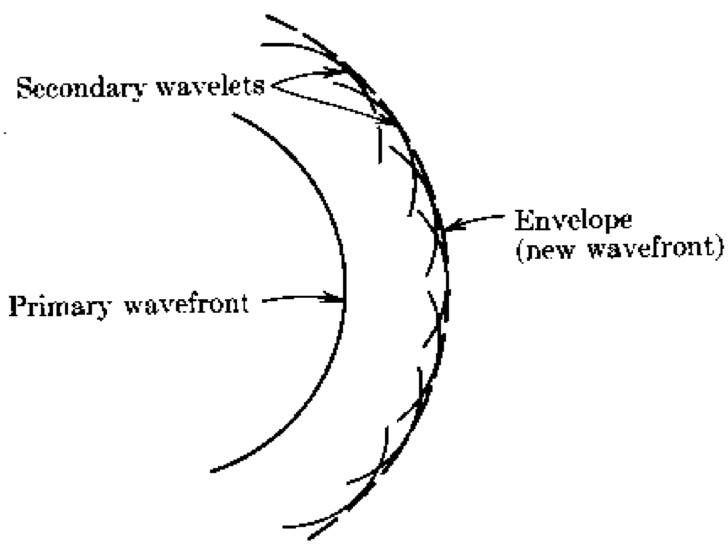


Figure 3-2 Huygens' envelope construction.

Scalar Diffraction

1. Picture of scalar diffraction.
2. Use of Green's Theorem and Green's functions.
3. Helmholtz and Kirchoff approach.

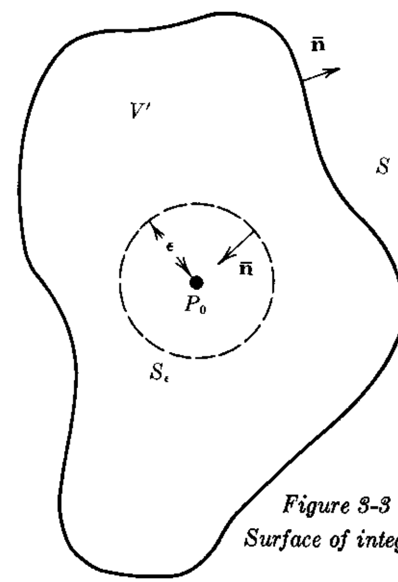


Figure 3-3
Surface of integration.

Theoretical Foundation of Scalar Diffraction

Select surface of integration

Define boundary conditions

- Kirchoff (first approach)
- Rayleigh (later approach)

Segment sphere into different regions

The sphere of integration is broken into several segments:

1. Inside the opening, Σ
2. Behind the screen, $\mathbf{S_1}$
3. On the sphere whose radius goes to infinity, $\mathbf{S_2}$

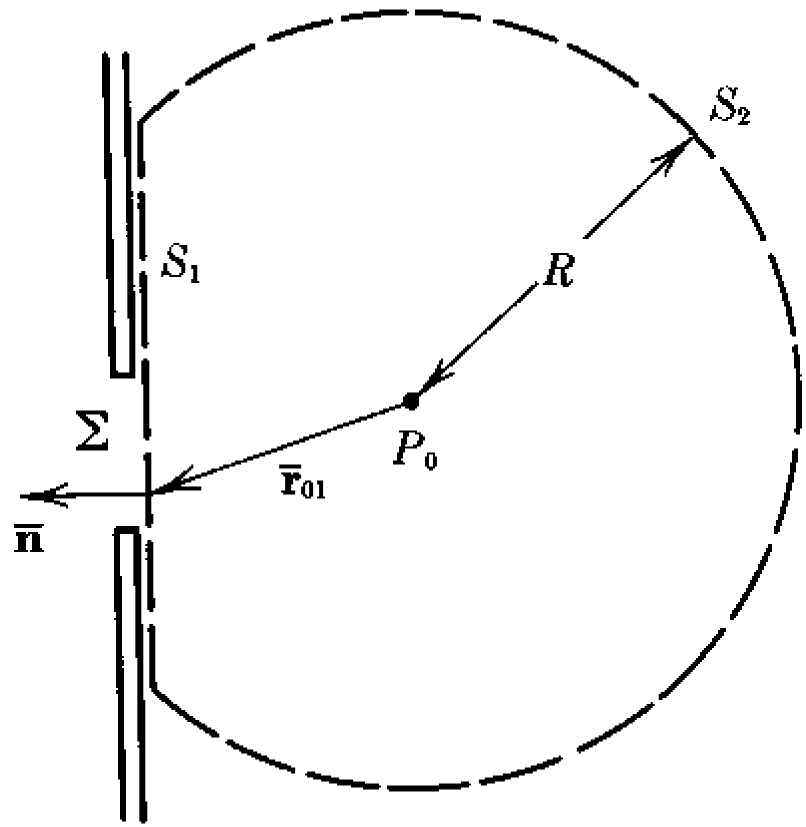


Figure 3-4 Kirchhoff formulation of diffraction by a plane screen.

Point sources and source Points for Green's Functions

Fresnel-Kirchhoff boundary cond.

- Illumination of the screen from a point source at P_2

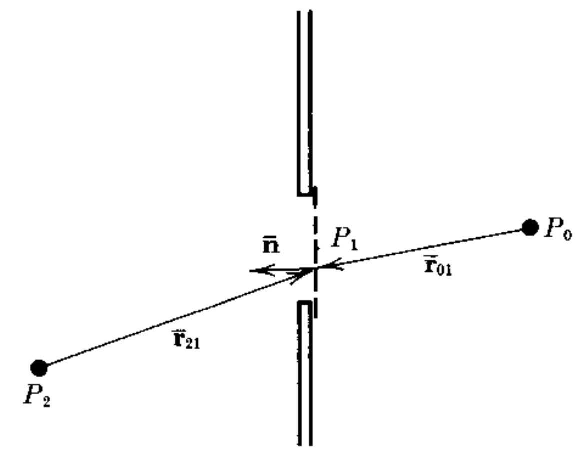


Figure 3-5 Point-source illumination of a plane screen.

Choice of Green's Functions in the Rayleigh-Sommerfled treatment of Diffraction

- In this case the Green's function source points are symmetric about the plane of the screen.

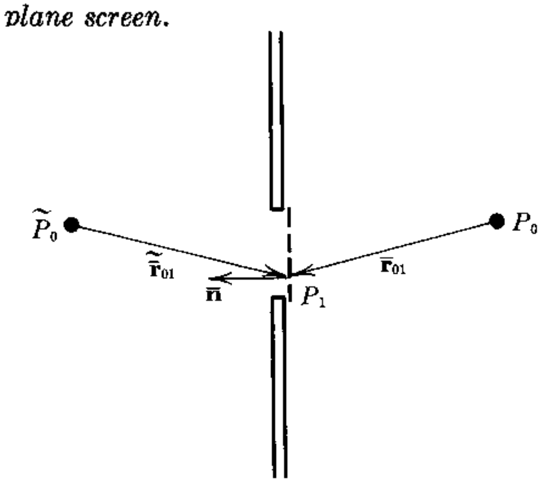


Figure 3-6 Rayleigh-Sommerfeld formulation of diffraction by a plane screen.

Fresnel Diffraction

- **Begin with the Rayleigh-Sommerfeld Diffraction Integral**

  (same as Huygens-Fesnel with correction)
- **Apply successive approximations:**

  paraxial and longer observer distances

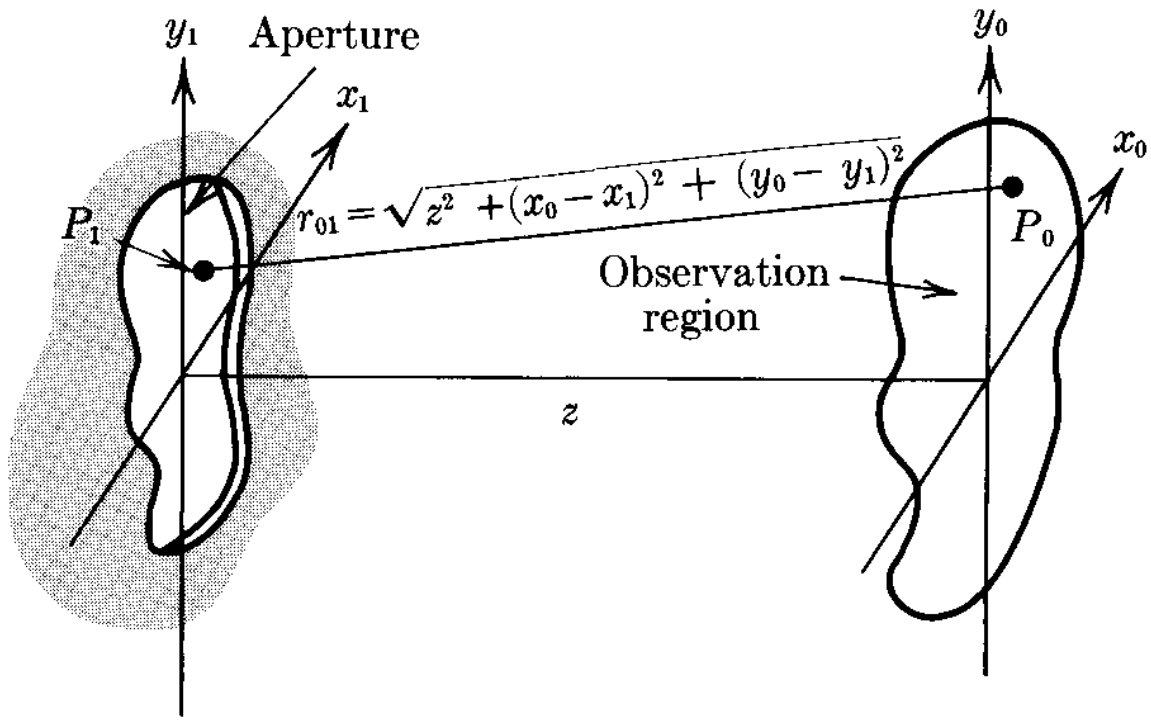


Figure 4-1 Diffraction geometry.

Phase of the Waves – Definitions

Choose the phase of the time phasor:

$$\exp[-j\,2\pi\nu t]$$

Positive direction plane wave:

$$\exp[\,j\,k \cdot z]$$

And for a diverging spherical wave:

$$\exp[\,j\,k \cdot r]$$

And for the quadratic (paraxial) approximation to the diverging spherical wave:

$$\exp[\,j\,k/2z\,(x^2 + y^2)]$$

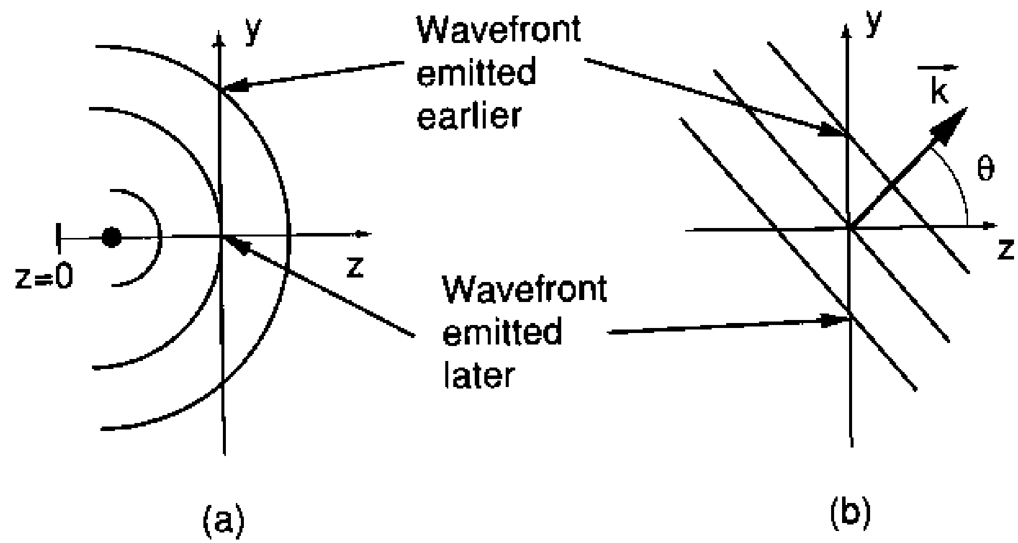


FIGURE 4.2
Determining the sign of the phases of exponential representations of (a) spherical waves and (b) plane waves.

Diffraction from Square and Circular Apertures

Distinguish between the "sinc-square" and the *Airy* diffraction patterns!!

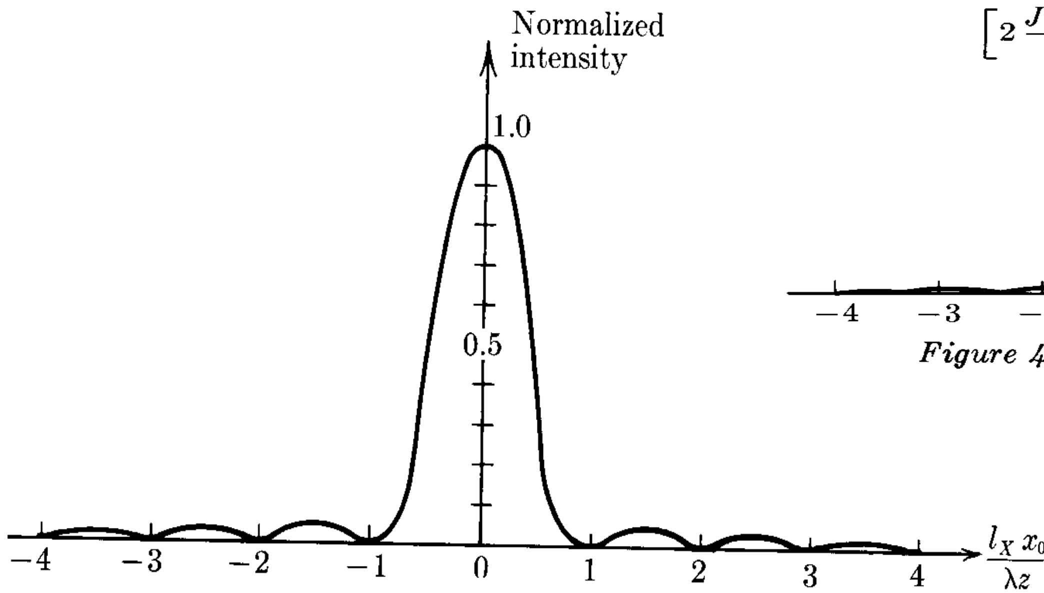


Figure 4-2 *Cross section of the Fraunhofer diffraction pattern of a rectangular aperture.*

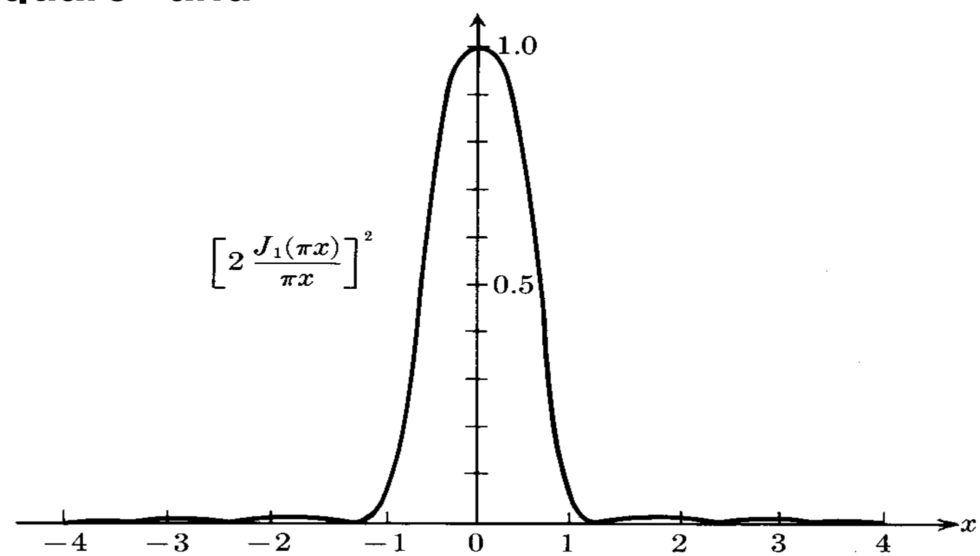


Figure 4-4 *Cross section of the Airy pattern.*