Two-Dimensional Linear Shift-Invariant Systems

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Figure 1

Review of Linearity

A system is linear if it is homogeneous and additive.

1. Homogeneity: scaling the input results in scaling the output by the same amount for all possible scaling values

   $T[\ a\ \mathrm{x}(n_1, n_2)\ ] = a\ T[\ \mathrm{x}(n_1, n_2)\ ]$ for all a

2. Additivity: the response of two signals added together at the input is equal to the sum of the individual responses.

   $T[\ \mathrm{x}_1(n_1, n_2) + \mathrm{x}_2(n_1, n_2)\ ] = T[\ \mathrm{x}_1(n_1, n_2)\ ] + T[\ \mathrm{x}_2(n_1, n_2)\ ]$

Shortcut for testing whether or not a system is linear:

1. input a signal that is identically zero for all coordinates (n_1, n_2), and
2. if the output signal is not identically zero for all coordinates (n_1, n_2), then it is not linear

What about the following systems? Input is $\mathrm{x}(n_1, n_2)$, and output is $\mathrm{y}(n_1, n_2)$.

1. $y(n_1, n_2) = x(n_1, n_2)$ — identity
2. $y(n_1, n_2) = x(n_1, n_2) + 1$ — add DC offset
3. $y(n_1, n_2) = x^2(n_1, n_2)$ — squarer
4. $y(n_1, n_2) = c(n_1, n_2)\ x(n_1, n_2)$ — spatially-dependent gain $c(n_1, n_2)$

Review of Shift Invariance

A system is shift-invariant if a shift in the spatial index results in the same shift on the output, for all possible shifts. For a discrete-time signal, the index is integer-valued.

Shift Invariance: $T[\ x(n_1 - m_1, n_2 - m_2)\ ] = T[\ x(n_1, n_2)\ ]\ \big|_{n_1 = n_1 - m_1,\ n_2 = n_2 - m_2}$

1. $y(n_1, n_2) = x(n_1, n_2)$
2. $y(n_1, n_2) = x(n_1, n_2) + 1$
3. $y(n_1, n_2) = x^2(n_1, n_2)$
4. $y(n_1, n_2) = c(n_1, n_2)\ x(n_1, n_2)$

2-D Linear Shift-Invariant (LSI) Sequences

Advantages

1. Mathematics are tractable and rich.
2. LSI systems are uniquely represented by their 2-D impulse response.

3. Linear transforms. Complex exponentials are eigenfunctions LSI systems, and Laplace and z transforms are based on complex exponential kernels.
4. Additive decompositions are useful.

But:

Superpositions are not quite as useful for images, e.g. occlusion of objects in a scene.

2-D Convolution

- An arbitrary 2-D sequence can be decomposed into a linear combination of *shifted impulses*.

$$x(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)$$

- Applying a linear shift-invariant system T that operates on spatial indices (n_1, n_2) to the input signal $x(n_1, n_2)$

$$y(n_1, n_2) = T\left[\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)\right]$$

Applying addivity,

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} T\left[x(k_1, k_2)\delta(n_1 - k_1, n_2 - k_2)\right]$$

Applying homogeneity with respect to (n_1, n_2),

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) T\left[\delta(n_1 - k_1, n_2 - k_2)\right]$$

The term $h(n_1, n_2) = T[\delta(n_1, n_2)]$ is the impulse response of the system.

A linear shift-invariant system is uniquely characterized by its impulse response.

Substituting $h(n_1, n_2)$, we obtain the two-dimensional linear convolution formula:

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2)$$

- Two-dimensional linear convolution denoted with two asterisks: $y = x ** h$
  - Conceptually: same as 1-D
  - Mechanically: a lot more work

Example 2-D linear convolution:

$$h(n_1, n_2) = u(n_1, n_2) = u(n_1)u(n_2) \qquad x(n_1, n_2) = \begin{cases} 1 \text{ if } 0 \le n_1 \le N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1 \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

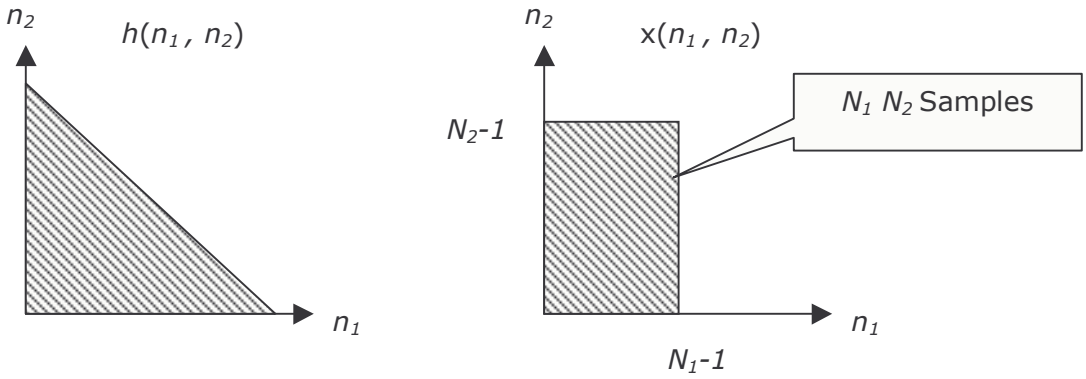


Figure 2:
$h(n_1,n_2)$ fills quadrant

$$y(n_1,n_2)=\sum_{k_1}\sum_{k_2}x(k_1,k_2)h(n_1-k_1,n_2-k_2)$$

$h(n_1-k_1,n_2-k_2)$ is a rotated and shifted version of $h(k_1,k_2)$

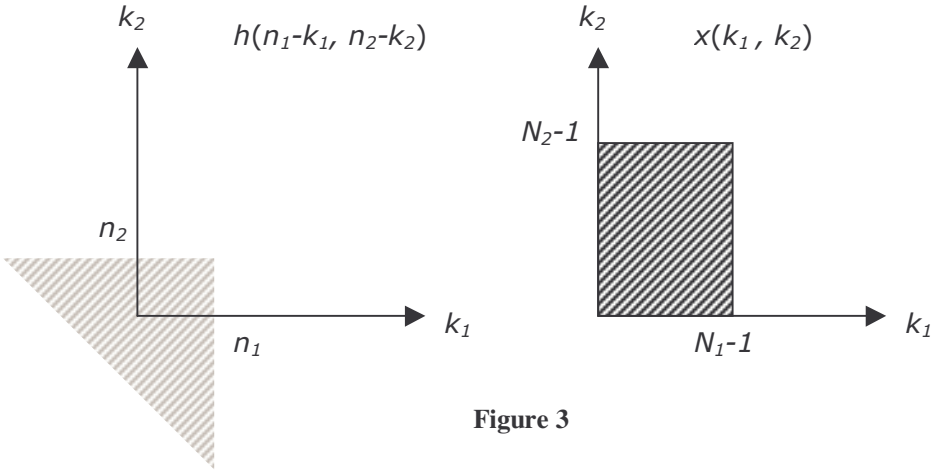


Figure 3

Shape of output sequence can assume one of five different forms depending on value of (n_1,n_2).

Case #1

$$n_1<0 \text{ or } n_2<0$$
$$x(k_1,k_2)h(n_1-k_1,n_2-k_2)=0,\forall k_1,k_2$$
$$y(n_1,n_2)=0$$

Case #2

$$0 \le n_1 \le N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1$$
$$y(n_1, n_2) = \sum (\text{nonzero samples of } x(k_1, k_2))\, h(n_1 - k_1, n_2 - k_2) = (n_1 + 1)(n_2 + 1)$$

Case #3

$$n_1 \ge N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1$$
$$y(n_1, n_2) = N_1(n_2 + 1)$$

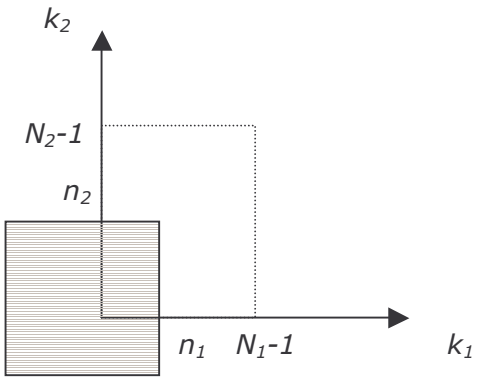


Figure 4

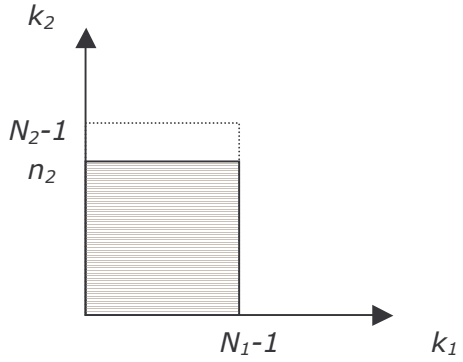


Figure 5

Case #4

$$n_2 \ge N_2 - 1 \text{ and } 0 \le n_1 \le N_1 - 1$$
$$y(n_1, n_2) = N_2(n_1 + 1)$$

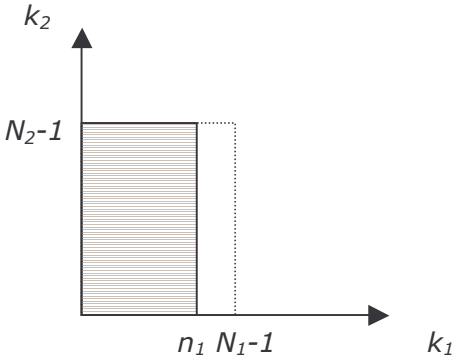


Figure 6

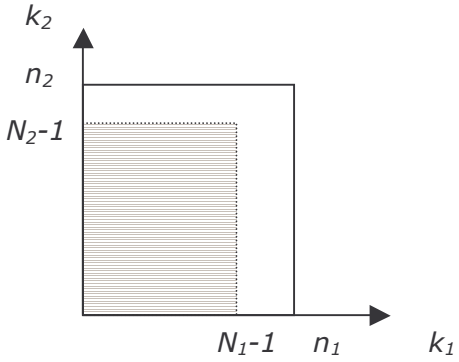


Figure 7

Case #5

$$n_1 \ge N_1 - 1 \text{ and } n_2 \ge N_2 - 1$$
$$y(n_1, n\ ) = N_1 N_2$$

$$y(n_1, n_2) = \begin{cases} 0 & \text{for } n_1 < 0 \text{ or } n_2 < 0 \\ (n_1 + 1)(n_2 + 1) & \text{for } 0 \le n_1 \le N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1 \\ N_1(n_2 + 1) & \text{for } n_1 \ge N_1 - 1 \text{ and } 0 \le n_2 \le N_2 - 1 \\ N_2(n_1 + 1) & \text{for } 0 \le n_1 \le N_1 - 1 \text{ and } n_2 \ge N_2 - 1 \\ N_1 N_2 & \text{for } n_1 \ge N_1 - 1 \text{ and } n_2 \ge N_2 - 1 \end{cases}$$

Validate your answer by checking the value of $y(n_1, n_2)$ at the endpoints of each interval

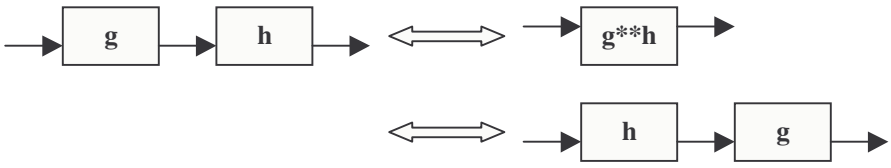
Note that $y(n_1, n_2)$ is separable.

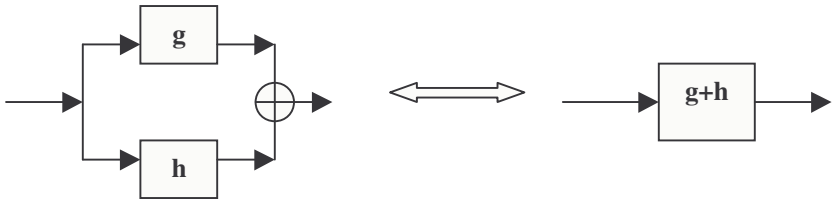
Theorem:

if $x(n_1, n_2) = x_1(n_1)x_2(n_2)$ and $h(n_1, n_2) = h_1(n_1)h_2(n_2)$
then $y(n_1, n_2) = y_1(n_1)y_2(n_2)$
where $y_1(n_1) = x_1(n_1) * h_1(n_1)$
$y_2(n_2) = x_2(n_2) * h_2(n_2)$
$x(n_1, n_2)$ is separable- x is a productof two1 - D rectangular pulses.

Additional Theorems:

1. $x **h = h **x$

2. 

3. 

What is the implicit assumption for these theorems to hold?

Separable Systems:

A system is separable if its impulse response is separable sequence.

$$h(n_1, n_2) = h_1(n_1)h_2(n_2)$$

Separable systems can be implemented faster than non-separable ones by using more memory to store intermediate computations.

$$y(n_1, n_2) = \sum \sum x(n_1 - k_1, n_2 - k_2)h_1(k_1)h_2(k_2)$$

$$y(n_1, n_2) = \sum_{k_1} h_1(k_1)\left[\sum_{k_2} h_2(k_2)x(n_1 - k_1, n_2 - k_2)\right]$$

Then

$$g(n_1, n_2) = \sum x(n_1, n_2 - k_2) h_2(k_2)$$

$$y(n_1, n_2) = \sum_{k_1} g(n_1 - k_1, n_2) h_1(k_1)$$

a. To compute $g(n_1, n_2)$, convolve each column of x with h_2.

b. To compute $y(n_1, n_2)$, convolve each row of g with h_1

Computational Savings:

x is $M \times M$, h is $N \times N$

1. 1 - D x with 1 - D h: $M \times N$ multiplies
2. 2 - D non - separable x with non - separable h: $M^2 N^2$ multiplies
3. 2 - D non - separable x with separable h: $M(MN) + (M + N - 1)MN \approx 2M^2 N$ if $M >> N$
4. 2 - D separable x with separable h: $2MN + (M + N - 1)^2 \approx M^2$ if $M >> N$

   $2MN$ to compute the separable convolution in the two dimensions
   $(M+N-1)^2$ to multiply out the separable result