Rectangular Sampling

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Lecture 4

Sampling

- Most signals are either explicitly or implicitly sampled
- Sampling is both similar and different in the 1-D and 2-D cases
- How a signal is sampled is often a design decision that is not fully understood or exploited, e.g. free parameter in computer vision and seismic processing
- Through sampling, continuous-time and discrete-time Fourier transforms are related

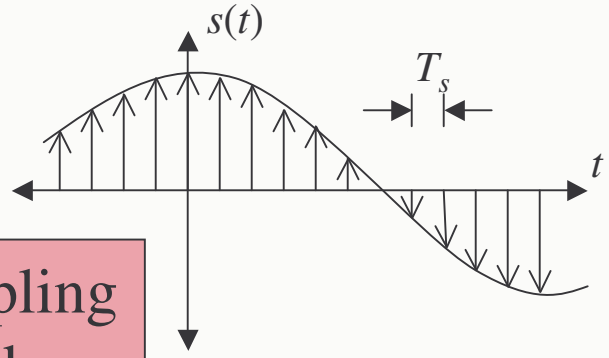
1-D Sampling: Time Domain

- **Many signals originate as continuous-time signals, e.g. conventional music or voice.**
- **By sampling a continuous-time signal at isolated, equally-spaced points in time, we obtain a sequence of numbers**

$$s[k] = s(k\,T_s) \quad k \in I$$

$$s_{sampled}(t) = s(t) \underbrace{\sum_{k=-\infty}^{\infty} \delta(t - k\,T_s)}_{\text{impulse train } \delta_{T_s}(t)}$$

Sampled analog waveform



T_s is sampling period

Under integration, e.g. a transform,

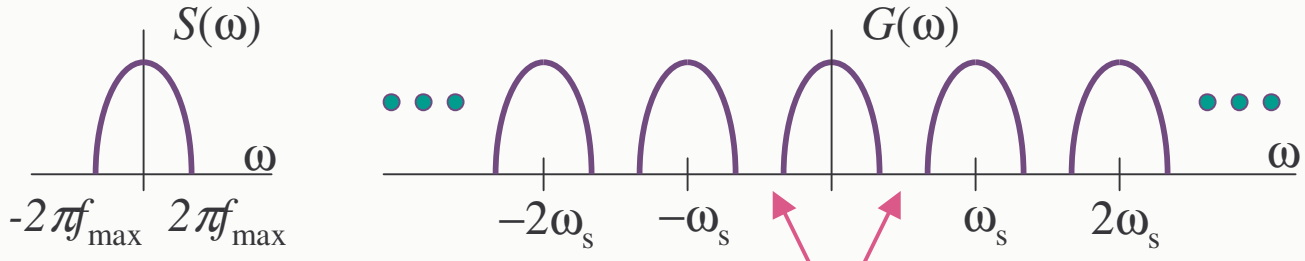
$$s_{sampled}(t) = \sum_{k=-\infty}^{\infty} \underbrace{s(k\,T_s)}_{s[k]} \delta(t - k\,T_s)$$

1-D Sampling: Frequency Domain

- **Sampling replicates spectrum of continuous-time signal at integer multiples of sampling frequency**
- **Fourier series of impulse train where $\omega_s = 2\pi f_s$**

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\,T_s) = \frac{1}{T_s}\left(1 + 2\cos(\omega_s\, t) + 2\cos(2\,\omega_s\, t) + \cdots\right)$$

$$g(t) = s(t)\,\delta_{T_s}(t) = \frac{1}{T_s}\left(s(t) + \underbrace{2s(t)\cos(\omega_s\, t)}_{\text{Modulation by } \cos(\omega_s\, t)} + \underbrace{2s(t)\cos(2\,\omega_s\, t)}_{\text{Modulation by } \cos(2\,\omega_s\, t)} + \cdots\right)$$



gap if and only if $2\pi f_{\max} < 2\pi f_s - 2\pi f_{\max} \Leftrightarrow f_s > 2 f_{\max}$

1-D Sampling: Shannon's Theorem

- **A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed from its samples $x[k] = x(k\,T_s)$ if the samples are taken at a rate f_s which is greater than $2\,f_{max}$**

  Nyquist rate = $2\,f_{max}$ *(proportional to bandwidth)*

  Nyquist frequency = $f_s/2$

- **What happens if $f_s = 2\,f_{max}$?**
- **Consider a sinusoid** $\sin(2\,\pi\,f_{max}\,t)$

  Use a sampling period of $T_s = 1/f_s = 1/(2f_{max})$

  Sketch: sinusoid with zeros at $t = 0,\ 1/(2\,f_{max}),\ 1/f_{max},\ \ldots$

1-D Sampling Theorem

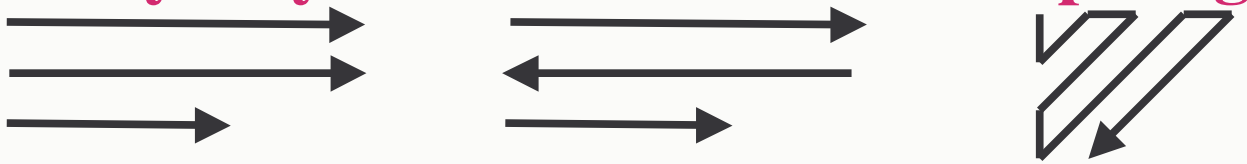
Assumption

- Continuous-time signal has no frequency content above f_{max}
- Sampling time is exactly the same between any two samples
- Sequence of numbers obtained by sampling is represented in exact precision
- Conversion of sequence to continuous-time is ideal

In Practice

Sampling 2-D Signals

- **Many possible sampling grids, e.g. rectangular and hexagonal**
- **Many ways to access data on sampled grid**



Raster scan *Serpentine scan* *Zig-zag scan*

- **Rectangular sampling** $x(n_1, n_2) = x_a(n_1 T_1, n_2 T_2)$
  - Continuous-time 2-D signal $x_a(t_1, t_2)$
  - 2-D sequence $x(n_1, n_2)$
  - Sampling intervals: horizontal T_1 vertical T_2

2-D Sampled Spectrum

- **2-D continuous-time Fourier transform**

$$X_a(\Omega_1,\Omega_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x_a(t_1,t_2)e^{-j\Omega_1 t_1}e^{-j\Omega_2 t_2}dt_1dt_2$$

$$x(t_1,t_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}X_a(\Omega_1,\Omega_2)e^{j\Omega_1 t_1}e^{j\Omega_2 t_2}d\Omega_1 d\Omega_2$$

- **Relevant properties**

$$x_a(t_1,t_2)**h_a(t_1,t_2)\iff X_a(\Omega_1,\Omega_2)H_a(\Omega_1,\Omega_2)$$

$$x_a(t_1,t_2)h_a(t_1,t_2)\iff\frac{1}{4\pi^2}X_a(\Omega_1,\Omega_2)**H_a(\Omega_1,\Omega_2)$$

- **Define 2-D impulse train (bed of nails): one impulses at each sampling location**

$$p_a(t_1,t_2)\triangleq\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\delta(t_1-T_1n_1,t_2-T_2n_2)$$

2-D Sampled Spectrum

- **The Fourier transform of a 2-D impulse train is another impulse train (as in 1-D):**

$$P_x(\Omega_1,\Omega_2)=\frac{(2\pi)^2}{T_1T_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta(\Omega_1-\frac{2\pi}{T_1}k_1,\Omega_2-\frac{2\pi}{T_2}k_2)$$

- **Define sampled version of**

$$\begin{aligned}
\hat{x}_x(t_1,t_2) &= x_x(t_1,t_2)p_x(t_1,t_2)\\
&= x_x(t_1,t_2)\sum_{n_1}\sum_{n_2}\delta(t_1-T_1n_1,t_2-T_2n_2)\\
&= \sum_{n_1}\sum_{n_2}x_x(t_1,t_2)\delta(t_1-T_1n_1,t_2-T_2n_2)\\
&= \sum_{n_1}\sum_{n_2}x_x(T_1n_1,T_2n_2)\delta(t_1-T_1n_1,t_2-T_2n_2)\\
\tilde{x}_x(t_1,t_2) &= \sum_{n_1}\sum_{n_2}x(n_1,n_2)\delta(t_1-T_1n_1,t_2-T_2n_2)
\end{aligned}$$

Only valid under integration (slide 4-10 and 4-12)

Depends only on current sample

2-D Sampled Spectrum

- **Equating transforms**

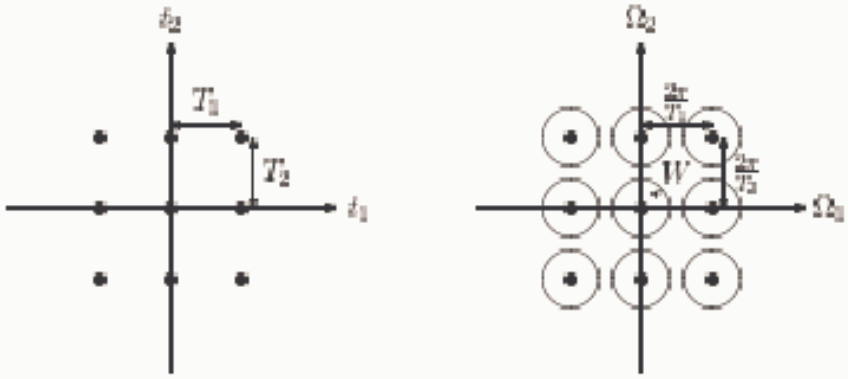
$$\tilde{X}_a(\Omega_1, \Omega_2) = \frac{1}{4\pi^2} X_a(\Omega_1, \Omega_2) ** \frac{4\pi^2}{T_1 T_2} \sum_{k_1} \sum_{k_2} \delta(\Omega_1 - \frac{2\pi}{T_1} k_1, \Omega_2 - \frac{2\pi}{T_2} k_2)$$

$$= \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a(\Omega_1 - \frac{2\pi}{T_1} k_1, \Omega_2 - \frac{2\pi}{T_2} k_2)$$

- **Continuous-time spectrum replicated in**
  - Horizontal frequency at multiples of $2\pi / T_1$
  - Vertical frequency at multiples of $2\pi / T_2$
- $\tilde{X}_a(\Omega_1, \Omega_2)$ **is related to** $X_a(\Omega_1, \Omega_2)$ **by**
  - Scaling in amplitude
  - Aliasing

2-D Sampled Spectrum

- **Lowpass filtering recovers bandlimited** $X_a(\Omega_1, \Omega_2)$
  - If $X_a(\Omega_1, \Omega_2) = 0$ for $\Omega_1^2 + \Omega_2^2 \geq W^2$, exact recovery requires
    $$T_1 < \frac{\pi}{W} \quad \text{and} \quad T_2 < \frac{\pi}{W}$$
  - If passband region were a square region of dimension $2W \times 2W$ region, then same condition would hold
- **For circular passband, optimal sampling grid for energy compaction is hexagonal**



Discrete vs. Continuous Spectra

- **2-D sampled spectrum**

$$\tilde{X}_a(\Omega_1,\Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n_1}\sum_{n_2} x(n_1,n_2)\delta(t_1-T_1n_1,t_2-T_2n_2)e^{-j\Omega_1 t_1}e^{-j\Omega_2 t_2}\,dt_1\,dt_2$$

$$= \sum_{n_1}\sum_{n_2} x(n_1,n_2)e^{-jn_1\Omega_1 T_1}e^{-jn_2\Omega_2 T_2}$$

- **2-D discrete-time Fourier transform**

$$X(\omega_1,\omega_2) = \sum_{n_1}\sum_{n_2} x(n_1,n_2)e^{-jn_1\omega_1}e^{-jn_2\omega_2}$$

$$X(\omega_1,\omega_2) = \tilde{X}_a\left(\frac{\omega_1}{T_1},\frac{\omega_2}{T_2}\right) = \frac{1}{T_1T_2}\sum_{k_1}\sum_{k2} X_a\left(\frac{\omega_1}{T_1}-\frac{2\pi}{T_1}k_1,\frac{\omega_2}{T_2}-\frac{2\pi}{T_2}k_2\right)$$

- **If $x(n_1,n_2) = x_a(n_1T_1,n_2T_2)$, $X(\omega_1,\omega_2)$ relates to $X_a(\Omega_1,\Omega_2)$ by**
  - Scaling in amplitude
  - Aliasing
  - Frequency normalization