General Multidimensional Sampling Part II

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Based on notes from Prof. Russell Mersereau (Georgia Tech)

1 1-D Case

$$X(\underline{\omega}) = \frac{1}{|\det \underline{V}|} \sum_{\underline{k}} X_a(\underline{V}^{-t}(\underline{\omega} - 2\pi \underline{k}))$$

$$X(\underline{\omega}) = \frac{1}{T} \sum_{k} X_a(\frac{\omega}{T} - \frac{2\pi k}{T})$$

Sampling Density = $\frac{1}{T}$ samples/sec.
Preventing aliasing requires $2\omega < \frac{2\pi}{T}$ *or* $\omega < \frac{\pi}{T}$

2 2-D Rectangular Case

$$x(n_1, n_2) = x_a(n_1 T_1, n_2 T_2)$$

$$\underline{V} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \quad \underline{U} = \begin{bmatrix} \frac{2\pi}{T_1} & 0 \\ 0 & \frac{2\pi}{T_2} \end{bmatrix}$$

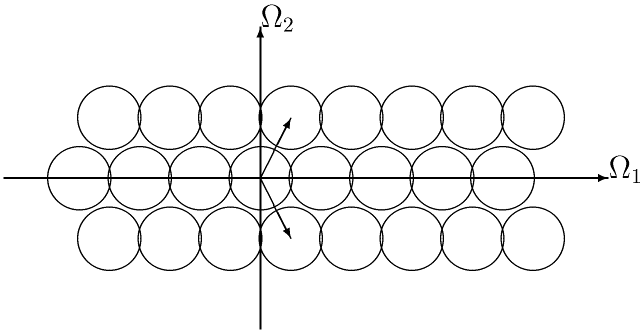
$$X(\underline{\omega}) = \frac{1}{|\det \underline{V}|} \sum_{\underline{k}} X_a(\underline{V}^{-t}(\underline{\omega} - 2\pi \underline{k}))$$

$$= \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a(\left[\begin{array}{cc} \frac{1}{T_1} & 0 \\ 0 & \frac{1}{T_2} \end{array} \right] \left[\begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] - 2\pi \left[\begin{array}{c} k_1 \\ k_2 \end{array} \right])$$

$$= \frac{1}{T_1 T_2} \sum_{k_1} \sum_{k_2} X_a(\frac{\omega_1}{T_1} - \frac{2\pi k_1}{T_1}, \frac{\omega_2}{T_2} - \frac{2\pi k_2}{T_2})$$

Sampling Density $= \frac{1}{|\det \underline{V}|} = \frac{1}{T_1 T_2}$ samples/m^2

3 2-D Hexagonal Sampling



$$\underline{U_1} = \left[\begin{array}{c} W \\ \sqrt{3}W \end{array} \right] \quad \underline{U_2} = \left[\begin{array}{c} W \\ -\sqrt{3}W \end{array} \right]$$

$$\underline{U} = \left[\begin{array}{cc} W & W \\ \sqrt{3}W & -\sqrt{3}W \end{array} \right] \quad \underline{V} = \left[\begin{array}{cc} \frac{\pi}{W} & \frac{\pi}{W} \\ \frac{\pi}{\sqrt{3}W} & \frac{\pi}{-\sqrt{3}W} \end{array} \right]$$

Sampling Density $= \frac{w^2\sqrt{3}}{2\pi^2}$

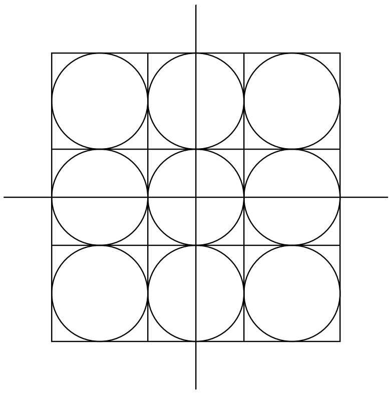


Figure 1: Sampling Density

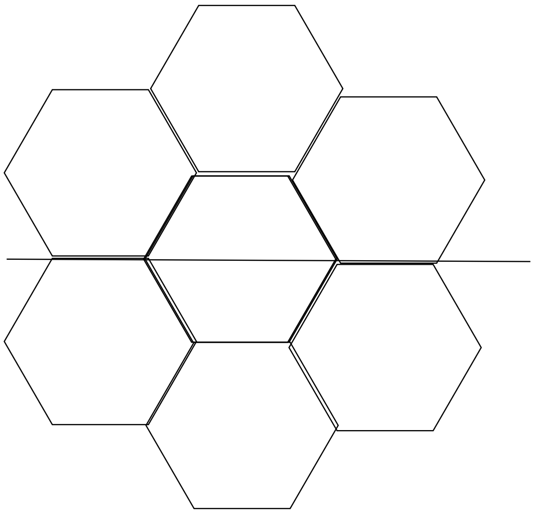


Figure 2: Sampling Density

4 The Nyquist Density

- For a given spectral support of the continuous-time analog signal, the *Nyquist Density* is that density resulting from maximally packed unaliased replication of the signal's spectrum.

$$\underline{V}_1 = \begin{vmatrix} \frac{\pi}{W} & 0 \\ 0 & \frac{\pi}{W} \end{vmatrix} \text{ and } \underline{V}_2 = \begin{vmatrix} \frac{1}{\sqrt{3}}\frac{\pi}{W} & \frac{1}{\sqrt{3}}\frac{\pi}{W} \\ \frac{\pi}{W} & -\frac{\pi}{W} \end{vmatrix}$$

- For circular spectral support, a hexagonal arrangement can accommodate 13.4% more bandwidth. Hexagonal is 13.4% more efficient. Spectral efficiency increases with dimension, see Table 2.1.

- Are there more efficient sampling grids for circularly bandlimited signals? YES!

- These involve periodically deleted samples, e.g. Figure 3.

- Signals on these grids are too difficult to analyze and process with a global view of the entire domain because

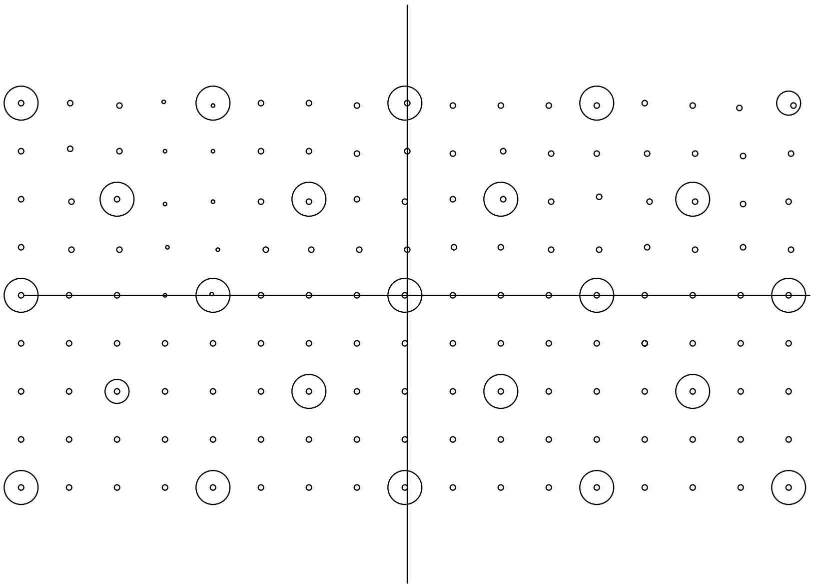


Figure 3: Example of periodically deleting samples for maximal packing of circularly bandlimited signals. The deleted samples are circled.

- DTFT is not defined.
- LTI systems are difficult to implement on this nonuniform grid.

However, periodic deletion of a sample in a block (frame) of samples happens in an communications receiver (such as an ADSL receiver) when the sampling device is sampling faster than the transmitter sampling rate. (The target sampling rate for both transmitter and receiver to use is set by the communication standard.) Similarly, when the sampling device in an communications receiver is sampling slower than the sampling rate being used in the transmitter, the sampling offset accumulates over the block of samples. The accumulation may be great enough so that one fewer sample than expected is in the frame over a fixed time interval. In this case, a sample may have to be inserted at the end of block. If the sampling device were a free-running crystal, then the deletion or insertion of a sample in each block would be periodic. This process of deleting or inserting a sample to align block boundaries in the receiver with that of the transmitter is known as *symbol synchronization*. Signal analysis and processing would occur one frame at a time.