Infinite Impulse Response Filters

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Introduction

- Background
- Bounded Input, Bounded Output (BIBO) stability of quarter-plane IIR filters
- Not all output masks are recursively computable: Figure 1.

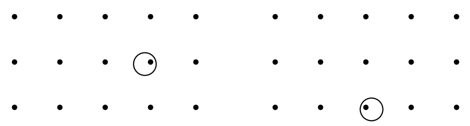

Figure 1: Two examples of not recursive computable masks

Background

- 2-D difference equations

$$\begin{aligned} y(n_1,n_2) &= \sum_{r_1}\sum_{r_2} a(n_1-r_1,n_2-r_2)\ x(r_1,r_2) \\ &\quad - \sum_{k_1\neq n_1}\sum_{k_2\neq n_2} b(n_1-k_1,n_2-k_2)\ y(k_1,k_2) \end{aligned}$$

- Input and output masks
  - $a(n_1,n_2)$ is called the support of the input mask
  - $b(n_1,n_2)$ is called the support of the output mask
  - Angle of support β: minimum angle enclosing support of mask
  - $y(n_1,n_2)$ is computed as in Figure 2

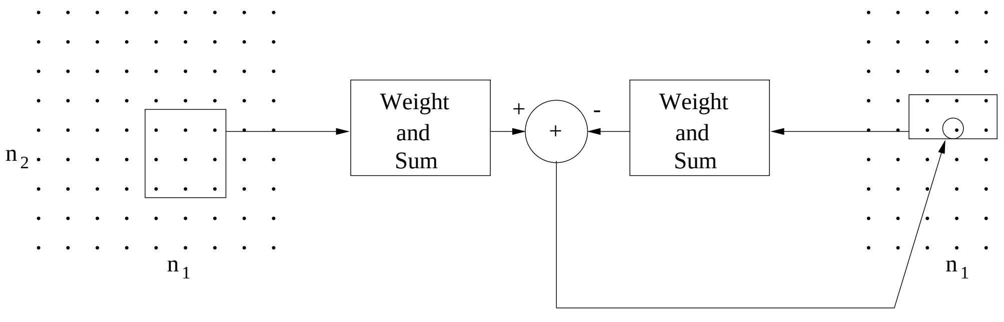

Figure 2: Recursive computation.

  - What degrees of freedom exist for moving the output mask?
  - What must the boundary conditions be?

Recursive Computation

- Consider a four-tap all-pole IIR filter to be passed over an $N \times N$ image in a raster scan fashion, e.g. in Floyd-Steinberg error diffusion halftoning.
- What are the boundary conditions?
- How many rows of the image do we need to keep in memory at one time?
- What parallelism exists if any?
- What is the tradeoff between the amount of parallelism exploited and memory?

Recursive Computability

- Recursive computability: Computing the difference equation using known values of the shifted output samples and initial conditions.
  - Support of output mask
  - Initial conditions
    * Initial conditions must be 0, and must lie outside the support of the output sequence, for the filter to be Linear Shift Invariant (LSI).
  - Order of recursion
- Types of recursively computable masks
  - Quarter-plane masks: supported in a quadrant in $\Re^2$
  - Non-Symmetric Half Plane (NSHP): supported in a half-plane in $\Re^2$
- Not all possible orderings are equivalent
  - Amount of storage
  - Degree of parallelism

Boundary Conditions

- For a recursive system, how do you choose the boundary conditions.
  - If the system is to be LTI, the initial conditions must be zero outside the support of the filter.
- Example: $y[n_1, n_2] = y[n_1 - 1, n_2] + y[n_1 + 1, n_2 - 1] + x[n_1, n_2]$
  - Now assume a different set of boundary conditions: Figure 3.

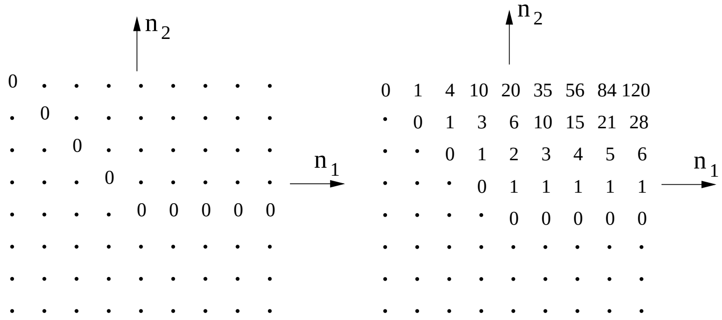

Figure 3: Boundary conditions (left) Response of $y[n_1, n_2]$ to $\delta[n_1, n_2]$ (right)

  - Response to $\delta[n_1, n_2]$: Figure 3
  - Response to $\delta[n_1 - 1, n_2 - 1]$: Figure 4
  - This is a shifted version of the above: consistent with support of the impulse response
  - Let's begin guessing at some boundary conditions: Figure 4
  - Calculated response to $\delta[n_1, n_2]$: Figure 5

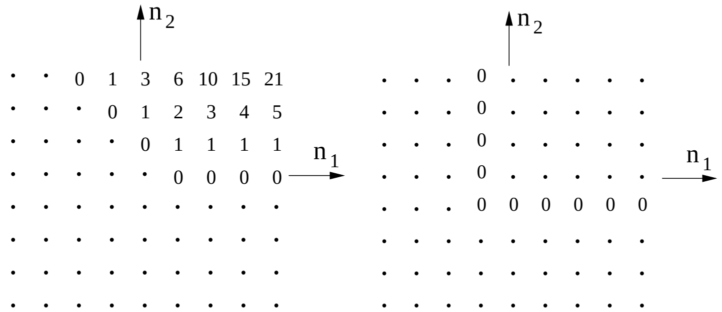

Figure 4: Response of $y[n_1, n_2]$ to $\delta[n_1 - 1, n_2 - 1]$ (left) Another boundary condition (right)

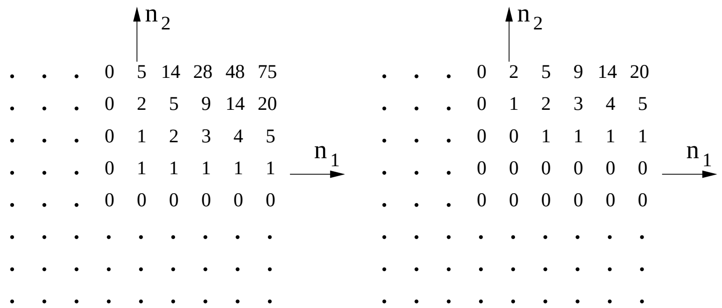

Figure 5: Response of $y[n_1, n_2]$ to $\delta[n_1, n_2]$ (left) Response of $y[n_1, n_2]$ to $\delta[n_1 - 1, n_2 - 1]$ (right)

- Response to $\delta[n_1 - 1, n_2 - 1]$: Figure 5
- Note: This is not a shifted version of the response $\delta[n_1, n_2]$: These boundary conditions do not lead to an LTI system.

The Transfer Function

$$H(z_1, z_2) = \sum_{n_1}\sum_{n_2} h[n_1, n_2]\, z_1^{-n_1} z_2^{-n_2}$$

- The z-transform will not converge for all values of (z_1, z_2)
- If it converges for $z_1 = e^{j\omega_1}$, $z_2 = e^{j\omega_2}$, then the Discrete-Time Fourier Transform exists and the system is stable.

$$|z_1| = 1 \text{ and } |z_2| = 1 \Rightarrow \text{Unit Bicircle}$$

Properties of the z-transform

- Separable Signals: $v[n_1]\, w[n_2] \Longleftrightarrow V(z_1)\, W(z_2)$
- Linearity
- Shift
- Convolution
- Linear mapping (look familiar?)

$$\begin{aligned}
x[n_1, n_2] &= \begin{cases} w[m_1, m_2] & n_1 = Im_1 + Jm_2, n_2 = Km_1 + Lm_2 \\ 0 & \text{otherwise} \end{cases} \\
IL - JK &\neq 0 \\
X(z_1, z_2) &= W(z_1^I\, z_2^K, z_1^J\, z_2^L);
\end{aligned}$$

Z-transform of Linear Mappings

- Linear mapping

$$
\begin{aligned}
x[n_1, n_2] &= \begin{cases} w[m_1, m_2] & n_1 = Im_1 + Jm_2, n_2 = Km_1 + Lm_2 \\ 0 & \text{otherwise} \end{cases} \\
IL - JK &\neq 0 \\
X(z_1, z_2) &= W(z_1^I \, z_2^K, z_1^J \, z_2^L)
\end{aligned}
$$

- Notice the regular insertion of zeros

- This is upsampling by upsampling matrix

$$\mathbf{R} = \begin{bmatrix} I & J \\ K & L \end{bmatrix}$$

where $\det \mathbf{R} = IL - JK \neq 0$

- Frequency response $X(\omega)$ of an upsampler to input $W(\omega)$

$$X_u(\omega) = X(\mathbf{R}^t \omega)$$

- Let $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$

$$
\begin{aligned}
X(z_1, z_2) &= W(z_1^I \, z_2^K, z_1^J \, z_2^L) \\
X(e^{j\omega_1}, e^{j\omega_2}) &= W(e^{jI\omega_1} e^{jK\omega_2}, e^{jJ\omega_1} e^{jL\omega_2}) \\
\mathbf{R}^t &= \begin{bmatrix} I & K \\ J & L \end{bmatrix}
\end{aligned}
$$

Inverse z-transform

$$
\begin{aligned}
x[n_1,n_2] &= \frac{1}{(2\pi j)^2}\oint_{c_2}\oint_{c_1} X(z_1,z_2) z_1^{n_1-1} z_2^{n_2-1} dz_1 dz_2 \\
X(z_1,z_2) &= \frac{1}{1-az_1^{-1}-bz_2^{-1}} \quad |a|+|b| \leq 1
\end{aligned}
$$

- Choose ROC that includes the unit bicircle.

$$
\begin{aligned}
x[n_1,n_2] &= \frac{1}{(2\pi j)^2}\oint_{c_2}\oint_{c_1} \frac{z_1^{n_1} z_2^{n_2}}{z_1 z_2 - az_2 - bz_1} dz_1 dz_2 \\
&= \frac{1}{(2\pi j)^2}\oint_{c_2}\oint_{c_1} \frac{\frac{1}{z_2-b} z_1^{n_1} z_2^{n_2}}{z_1 - \frac{az_2}{z_2-b}} dz_1 dz_2 \\
&= \frac{1}{(2\pi j)^2}\oint_{c_2} \frac{1}{z_2-b} z_2^{n_2} \oint_{c_1} \frac{z_1^{n_1}}{z_1 - \frac{az_2}{z_2-b}} dz_1 dz_2
\end{aligned}
$$

- The inner integral (with respect to z_1) is the inverse z-transform of a first-order system with a pole at $\dfrac{az_2}{z_2-b}$

$$
\begin{aligned}
x[n_1,n_2] &= \frac{1}{2\pi j}\oint_{c_2} \frac{z_2^{n_2}}{z_2-b}\left(\frac{az_2}{z_2-b}\right)^{n_1} u(n_1) dz_2 \\
&= a^{n_1} u[n_1] \frac{1}{2\pi j}\oint_{c_2} \frac{z_2^{n_1+n_2}}{(z_2-b)^{n_1+1}} dz_2 \\
&= \frac{(n_1+n_2)!}{n_1! n_2!} a^{n_1} b^{n_2} u[n_1,n_2]
\end{aligned}
$$

- What does this tell us about other examples? very little

- Consider an example:

$$
\begin{aligned}
H(z_1,z_2) &= \frac{1}{1-az_1^{-1}-bz_2^{-1}-cz_1^{-1}z_2^{-1}} \\
B(e^{j\omega_1},z_2) &= 1-ae^{-j\omega_1}-bz_2^{-1}-ce^{-j\omega_1}z_2^{-1}
\end{aligned}
$$

Setting $B(e^{j\omega_1}, z_2) = 0$ and solving gives

$$z_2 = \frac{b + ce^{-j\omega_1}}{1 - ae^{-j\omega_1}}$$

This is a bilinear transformation which must map circles into circles.

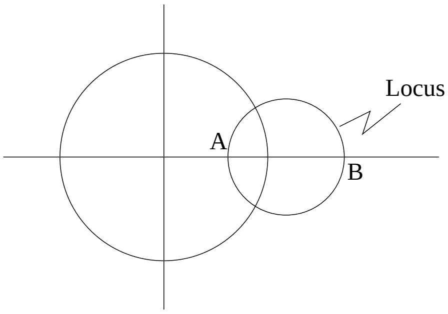

Figure 6: Bilinear transform

Point A $\quad \omega_1 = \pi \quad z_2 = \frac{b-c}{1+a}$

Point B $\quad \omega_1 = 0 \quad z_2 = \frac{b+c}{1-a}$

$$\left|\frac{b-c}{1+a}\right| < 1 \quad \left|\frac{b+c}{1-a}\right| < 1$$

From the other part of the locus

$$\left|\frac{a-c}{1+b}\right| < 1 \quad \left|\frac{a+b}{1-b}\right| < 1$$