Adding Random Variables and Connections with the Signals and Systems Pre-requisite

Problem

A key connection between a Linear Systems and Signals course and a Probability course is that when two independent random variables are added together, the resulting random variable has a probability density function (pdf) that is the convolution of the pdfs of the random variables being added together. That is, if X and Y are independent random variables and $Z = X + Y$, then $f_Z(z) = f_X(z) * f_Y(z)$ where $f_R(r)$ is the probability density function for random variable R and * is the convolution operation. This is true for continuous random variables and discrete random variables. *(An alternative to a probability density function is a probability mass function. They represent the same information but in different formats.)*

a) Consider two fair six-sided dice. Each die, when rolled, generates a number in the range of 1 to 6, inclusive, with each outcome having an equal probability. That is, each outcome is uniformly distributed. When adding the outcomes of a roll of these two six-sided dice, one would have a number between 2 and 12, inclusive.
   1) Tabulate the likelihood for each outcome from 2 to 12, inclusive.
   2) Compute the pdf of Z by convolving the pdfs of X and Y. Compare the result to the first part of this sub-problem (a)-(1).

b) Compute the pdf of continuous random variable Z where $Z = X + Y$ and X is a continuous random variable uniformly distributed on [0, 2] and Y is a continuous random variable uniformly distributed on [0, 4]. Assume that X and Y are independent.

c) A constant value C can be modeled as a pdf with only one non-zero entry. Recall that the pdf can only contain non-negative values and that the area under a continuous pdf (or equivalently the sum of a discrete pdf) must be 1.
   1) Plot the pdf of a discrete random variable X that is a constant of value C.
   2) Plot the pdf of a continuous random variable Y that is a constant of value C.
   3) Using convolution, determine the pdf of a continuous random variable Z where $Z = X + Y$. Here, X has a uniform distribution on [0, 3] and Y is a constant of value 2. Assume that X and Y are independent.

Solution

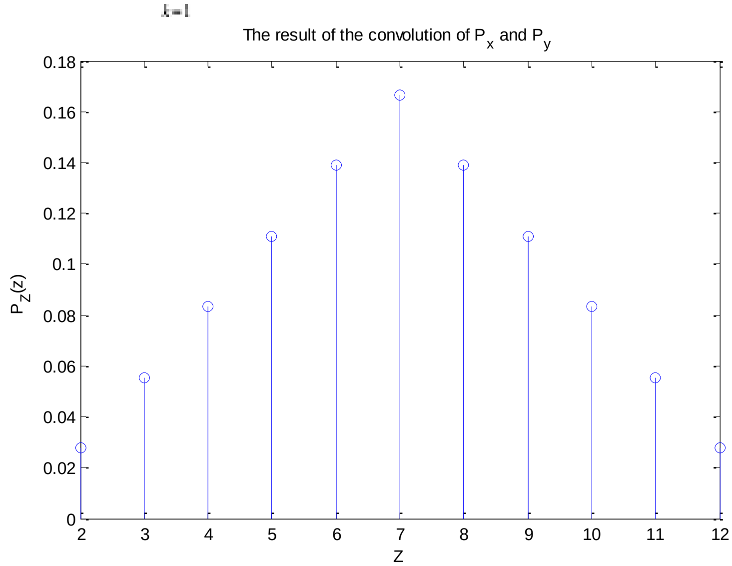
(a) (1) Likelihood for each outcome from 2 to 12
Let Xbe the number generated when the first die is rolled and Y be the number generated when the second die is rolled. Since each outcome is uniformly distributed for each die, $P(X = x) = 1/6$ where $x \in \{1, 2, 3, 4, 5, 6\}$ and $P(Y = y) = 1/6$ where $y \in \{1, 2, 3, 4, 5, 6\}$:

Z	P(z)
2	1/36
3	2/36

4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

(2) Adding the two random variables results in another random variable $Z = X + Y$ which takes on values between 2 and 12, inclusive. Since the dice are rolled independently, the numbers generated are independent.

$$p_Z(z) = p_{X+Y}(z) = p_X(z) * p_Y(z) = \sum_{k=1}^{6} p_X(k) p_Y(z-k).$$



The convolution of two rectangular pulses of the same length N samples gives a triangular pulse of length $2N - 1$ samples. Example calculations:

$$p_z(2) = p_x(1)p_y(1) = \frac{1}{36}$$

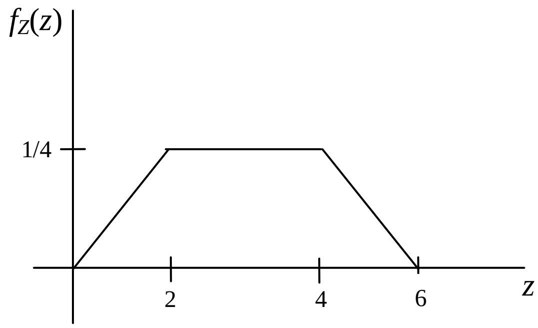
$$p_z(3) = p_x(1)p_y(2) + p_x(2)p_y(1) = \frac{2}{36}$$

$$p_z(4) = p_x(1)p_y(3) + p_x(2)p_y(2) + p_x(3)p_y(1) = \frac{3}{36}$$

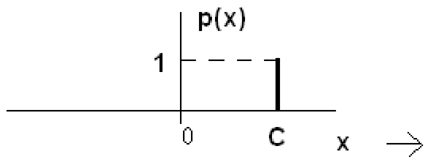
Evaluating the above convolution, we get the same pdf as obtained in the table. The output of the Matlab simulation of the convolution is displayed in the above graph. The *conv* method was used for the convolution. The *stem* method was used for plotting.

(b) X is uniformly distributed on [0, 2]. Therefore $f_X(x) = \frac{1}{2}$ for all $x \in$ [0,2]. Similarly, since Y is uniformly distributed on [0, 4], $f_Y(y) = \frac{1}{4}$ for all $y \in$ [0, 4].

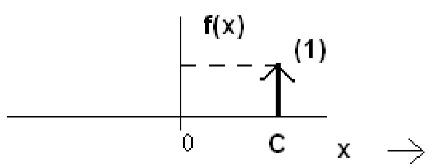
$$f_{X+Y}(z) = f_X(z) * f_Y(z) = \begin{cases} \int_0^z f_X(\lambda) - f_Y(z-\lambda)d\lambda = \frac{z}{8} & 0 \le z \le 2 \\ \int_0^2 f_X(\lambda) - f_Y(z-\lambda)d\lambda = \frac{1}{4} & 2 \le z \le 4 \\ \int_{z-4}^2 f_X(\lambda) - f_Y(z-\lambda)d\lambda = \frac{6}{8} - \frac{z}{8} & 0 \le z \le 2 \end{cases}$$



(c) (1)The answer is a Kronecker (discrete-time) impulse located at $x = C$.



(2) For a continuous random variable we require that $\int_{-\infty}^{\infty} f_X(x)dx = 1$ and this is satisfied by an continuous impulse (Dirac delta functional) at C. Mathematically, $\int_{-\infty}^{\infty} \delta(x-C)dx = 1$



(3) X is uniformly distributed on [0, 3]. Therefore $f_X(x) = \frac{1}{3}$ for all $x \in$ [0, 3]. Y has a constant value of 2 and hence $f_Y(y) = \delta(y-2)$. Since X and Y are independent, $Z = X + Y$ implies that

$$f_{X+Y}(z) = f_X(z) * \delta(z-2) = \begin{cases} \frac{1}{3} & 2 \le z \le 5 \\ 0 & otherwise \end{cases}$$

This follows from the fact that convolution by $\delta(z-2)$ shifts $f_X(z)$ by 2.