Handout F: Fundamental Theorem of Linear Systems

Theorem: Let a linear time-invariant system g has an $e_f(t)$ denote the complex sinusoid $e^{j2\pi ft}$. Then, $g(e_f(.),t) = g(e_f(.),0)e_f(t) = c\, e_f(t)$.

Example: Analog RC Lowpass Filter

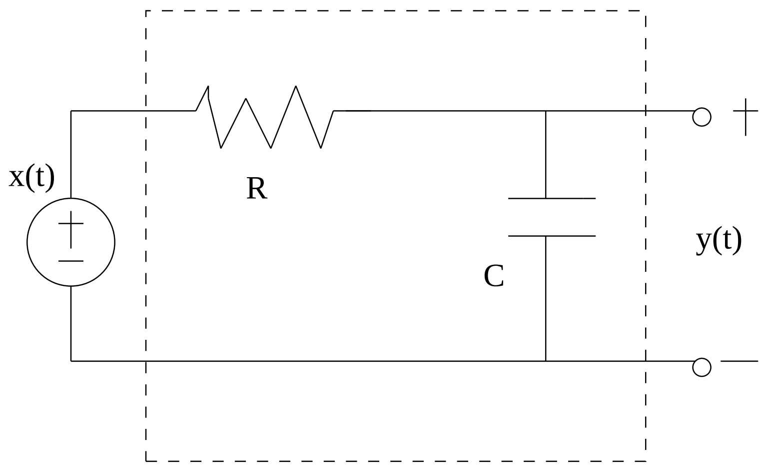

Figure 1: A First-Order Analog Lowpass Filter

The impulse response for the circuit in Fig. 1, i.e. the output measured at $y(t)$ when $x(t) = \delta(t)$, is

$$h(t) = \frac{1}{RC} e^{-\frac{1}{RC}t} u(t)$$

For a complex sinusoidal input, $x(t) = e_f(t) = e^{j2\pi ft}$,

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)\, d\lambda \\
&= \int_{-\infty}^{\infty} e^{j2\pi f(t-\lambda)} \frac{1}{RC} e^{-\frac{1}{RC}\lambda} u(\lambda)\, d\lambda \\
&= e^{j2\pi ft} \left[\frac{1}{RC} \int_{-\infty}^{\infty} e^{-j2\pi f\lambda} e^{-\frac{1}{RC}\lambda}\, d\lambda \right] \\
&= \left[\frac{\frac{1}{RC}}{j2\pi f + \frac{1}{RC}} \right] e^{j2\pi ft} \\
&= g(e_f(.),0)\, e_f(t)
\end{aligned}$$

So, $g(e_f(.),0) = H(f)$, which is the transfer function of the system.

Placeholder – please ignore.