


Prolog for the homework set. A tale of two transforms.

In engineering, the two common versions of the continuous-time Fourier transform either use ω in rad/s or f in Hz as the frequency variable, where $\omega = 2 \pi f$.

Applications. Many applications use frequencies in Hz instead of rad/s. In audio, notes are assigned frequencies in Hz. We would specify ranges of sub-woofer, woofer and tweeter frequencies in Hz, e.g. 20-200 Hz, 200-2,000 Hz, and 2,000-20,000 Hz, respectively. We talk about Bluetooth and Wi-Fi operating over the 2.4-2.499 GHz band, which is 100 MHz wide. (Wi-Fi works over other bands as well.) Frequency in units of rad/s happens at times in circuit design and signal processing theory. Also, rad/s can also be used as an angular frequency.

Visual transform tables. Here is a visual continuous-time Fourier transform table for both transforms from [Roberts Fundamentals of Signals & Systems Appendix D](#). 

Forward transforms. Here is the forward continuous-time Fourier transform in rad/s:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Let $\omega = 2 \pi f$:

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt$$

Note that both forward continuous-time Fourier transforms look similar.

Inverse transforms. Here is the inverse continuous-time Fourier transform in rad/s:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

Let $\omega = 2 \pi f$: As ω goes to infinity, so does f . Same for negative infinity. Also $d\omega = 2 \pi df$:

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(f)e^{j2\pi ft} df$$

Note that the inverse continuous-time Fourier transform in Hz does not have a scaling factor.

Example #1. Let $X(\omega) = \delta(\omega - \omega_c)$ which is a Dirac delta shifted to the right in the frequency domain in rad/s by ω_c :

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_c)e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_c t}$$

Example #2. Let's convert $\delta(\omega - \omega_c)$ into frequencies in Hz. Let $\omega = 2 \pi f$ and $\omega_c = 2 \pi f_c$:

$$\delta(2\pi f - 2\pi f_c) = \delta(2\pi(f - f_c)) = \frac{1}{2\pi} \delta(f - f_c)$$

The last step is valid under integration as explained in the handout [SamplingUnitStep.pdf](#). The expression in Hz is a Dirac delta shifted to the right in the frequency domain in Hz by f_c .

Let's check the inverse transform in Hz of $\frac{1}{2\pi} \delta(f - f_c)$ to make sure:

$$x(t) = \int_{-\infty}^{\infty} \hat{X}(f) e^{j2\pi f t} df = \int_{-\infty}^{\infty} \frac{1}{2\pi} \delta(f - f_c) e^{j2\pi f t} df = \frac{1}{2\pi} e^{j2\pi f_c t} = \frac{1}{2\pi} e^{j\omega_c t}$$

Problem 0.1. Continuous-Time Sinusoidal Generation.

In practice, we cannot generate a two-sided sinusoid $\sin(2\pi f_c t)$, nor can we wait until the end of time to observe a one-sided sinusoid $\sin(2\pi f_c t) u(t)$ where $u(t)$ is the unit step function.

In the lab, we can turn on a signal generator for a short time and observe the output in the time domain on an oscilloscope or in the frequency domain using a spectrum analyzer. In simulation, we can only observe a signal for finite amount of time.

Consider a finite-duration sine that is on from 0 sec to 1 sec given by the equation

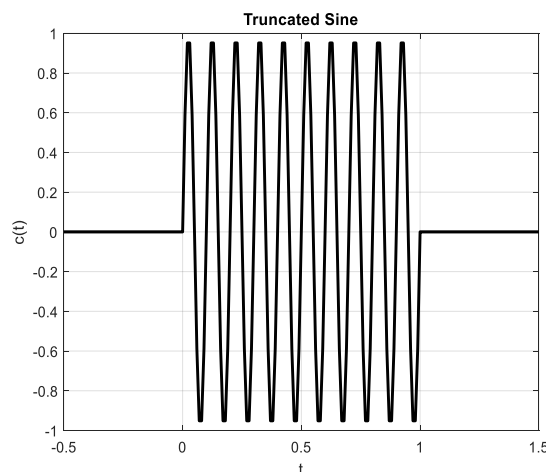
$$c(t) = \sin(2\pi f_c t) \text{rect}(t - 1/2)$$

where f_c is the carrier frequency (in Hz).

Part (a). Using MATLAB or Python, plot $c(t)$ for $-0.5 < t < 1.5$ for $f_c = 10$ Hz. Turn in your code and plot. You may find the `rectpuls` command useful. Give a formula for the Fourier transform of $c(t)$ for a general value of f_c .

Solution: Use MATLAB to plot the signal $c(t) = \sin(2\pi f_c t) \text{rect}(t - 1/2)$ for $-0.5 < t < 1.5$. We'll need to pick a sampling rate f_s so that $f_s > 2 f_{max}$ to obey the sampling theorem. The value of f_{max} could be computed using the answer in part (d) but we know from the modulation property of the Fourier transform that f_{max} is f_c plus the bandwidth of $\text{rect}(t - 1/2)$. Let's oversample by using

$f_s = 100$ Hz.



MATLAB code

```
fs = 100;
Ts = 1/fs;
t = -0.5 : Ts : 1.5;
fc = 10;
x = sin(2*pi*fc*t);
h = rectpuls(t-0.5);
c = x .* h;
figure
plot(t,c,'k','LineWidth',2)
grid
title('Truncated Sine')
xlabel('t')
ylabel('c(t)')
```

Note that the time-domain plot has 10 cycles of the sinusoid over 1sec (since $f_c = 10$ cycles/s).

Also note that the maximum is less than + 1 and minimum value is larger than - 1 due to insufficient sampling. With $\omega_c = 2 \pi f_c$, we take the Fourier transform of $c(t)$ and obtain

$$C(\omega) = \mathcal{F} \left\{ \sin(\omega_c t) \text{rect} \left(t - \frac{1}{2} \right) \right\} = \frac{1}{2\pi} \mathcal{F} \{ \sin(\omega_c t) \} * \mathcal{F} \left\{ \text{rect} \left(t - \frac{1}{2} \right) \right\} \quad \Big| \text{ by the Fourier transform}$$

property that Multiplication in the Time Domain \Leftrightarrow Convolution in the Frequency Domain.

We can lookup or derive the Fourier transforms of cosine and rectangular pulse

$$C(\omega) = \frac{1}{2\pi} (j\pi \delta(\omega + \omega_c) - j\pi \delta(\omega - \omega_c)) * \left(e^{-j\omega/2} \text{sinc} \left(\frac{\omega}{2\pi} \right) \right)$$

where $\text{sinc}(x) = \sin(\pi x) / (\pi x)$. Note that sinc has two different definitions. Then,

$$C(\omega) = \frac{j}{2} (\delta(\omega + \omega_c) - \delta(\omega - \omega_c)) * \left(e^{-j\omega/2} \text{sinc} \left(\frac{\omega}{2\pi} \right) \right)$$

$$C(\omega) = \frac{j}{2} \delta(\omega + \omega_c) * \left(e^{-j\omega/2} \text{sinc} \left(\frac{\omega}{2\pi} \right) \right) - \frac{j}{2} \delta(\omega - \omega_c) * \left(e^{-j\omega/2} \text{sinc} \left(\frac{\omega}{2\pi} \right) \right)$$

The sifting property for the Dirac delta applied to convolution gives $\delta(x + x_0) * g(x) = g(x + x_0)$:

$$C(\omega) = \frac{j}{2} e^{-j(\omega+\omega_c)/2} \text{sinc} \left(\frac{\omega + \omega_c}{2\pi} \right) - \frac{j}{2} e^{-j(\omega-\omega_c)/2} \text{sinc} \left(\frac{\omega - \omega_c}{2\pi} \right)$$

The first term is centered at $f_c = -\omega_c/2\pi$, and the second is centered at $f_c = \omega_c/2\pi$.

Part (b). Sketch by hand the magnitude of the Fourier transform of $c(t)$ for a general value of f_c .

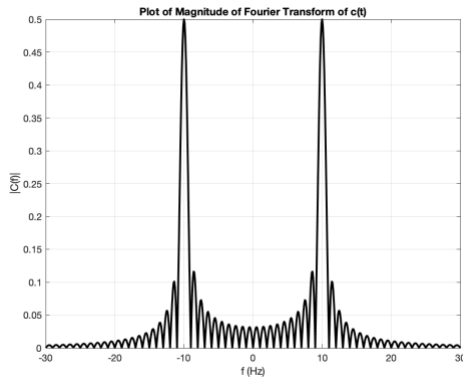
Using MATLAB or Python, plot the magnitude of the Fourier transform of $c(t)$ for $f_c = 5$ Hz. Turn in your code and plot.

Solution: We take the absolute value of $C(\omega)$ to calculate the magnitude response:

$$|C(\omega)| = \left| \frac{j}{2} e^{-j(\omega+\omega_c)/2} \text{sinc} \left(\frac{\omega + \omega_c}{2\pi} \right) - \frac{j}{2} e^{-j(\omega-\omega_c)/2} \text{sinc} \left(\frac{\omega - \omega_c}{2\pi} \right) \right|,$$

When plotting by hand, we can use the identity $|f(x) + g(x)| \leq |f(x)| + |g(x)|$:

$$|C(\omega)| \leq \left| \frac{j}{2} e^{-j(\omega+\omega_c)/2} \text{sinc} \left(\frac{\omega + \omega_c}{2\pi} \right) \right| + \left| -\frac{j}{2} e^{-j(\omega-\omega_c)/2} \text{sinc} \left(\frac{\omega - \omega_c}{2\pi} \right) \right| = \frac{1}{2} \left| \text{sinc} \left(\frac{\omega + \omega_c}{2\pi} \right) \right| + \frac{1}{2} \left| \text{sinc} \left(\frac{\omega - \omega_c}{2\pi} \right) \right|$$



MATLAB code

```

fc=10;
f=[-30:0.1:30];
C = 0.5*j*exp(-j*pi*(f+fc)).*sinc(f+fc)
- 0.5*j*exp(-j*pi*(f-fc)).*sinc(f-fc);
plot(f,abs(C),'k','LineWidth',2)
grid
title('Plot of Magnitude of Fourier
Transform of c(t)')
xlabel('f (Hz)')
ylabel('|C(f)|')

```

In Matlab, we plot the magnitude of the full formula for the Fourier transform $C(\omega)$ directly.

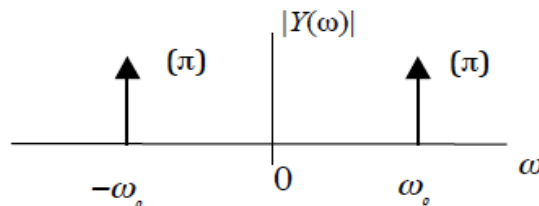
Center frequencies of the main lobes are at $f_c = 10$ Hz and $f_c = -10$ Hz.

A spectrum analyzer would display the above magnitude spectrum plus noise.

Part (c). Describe the differences between the magnitude of the Fourier transforms of $c(t)$ and a two-sided sine of the same frequency. What is the bandwidth of each signal?

Solution: Bandwidth is the non-zero extent in positive frequencies of the signal's spectrum.

Fourier transform of the two-sided sine $y(t) = \sin(2\pi f_0 t)$ is $Y(\omega) = j\pi (\delta(\omega + \omega_0) - \delta(\omega - \omega_0))$:



Each Dirac delta has width of zero and area of π . Bandwidth is zero. Same holds for a two-sided cosine $x(t) = \cos(2\pi f_0 t)$, which has Fourier transform $X(\omega) = \pi (\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$.

From the plot in part (b), the bandwidth of $C(\omega)$ is not zero. This is because of the multiplication of the two-sided sine by $\text{rect}(t - 1/2)$ in the time domain.

Here are some of the valid methods to estimate the bandwidth of bandpass signal $C(\omega)$:

- Set amplitude threshold of magnitude spectrum at an arbitrary point below which amplitudes are treated as if zero. If we use 0.15 as the threshold for the plot in part (b), then we can eyeball the estimated bandwidth of $C(\omega)$ to be 1.5 Hz.
- Measure the width of the main lobe in the magnitude spectrum between the two zero crossings on either side of the carrier frequency. We eyeball the bandwidth to be 2 Hz.
- Compute the power bandwidth that captures 90% of the total energy.

For power bandwidth, we define a function called myfun in file myfun.m to compute $|C(f)|^2$:

```
function Cpower = myfun(f)
fc = 10;
C = 0.5*j*exp(-j*pi*(f+fc)).*sinc(f+fc) - 0.5*j*exp(-j*pi*(f-fc)).*sinc(f-fc);
Cpower = abs(C) .^ 2;
end
```

To estimate total energy, we numerically integrate $|C(f)|^2$ by replacing ∞ with 1000 Hz:

```
totalEnergy = integral(@myfun,0,1000);    %% Gives 0.25 for fc = 10 Hz
```

A bandwidth of 1.6 Hz (i.e. from 9.2 to 10.8 Hz for $f_c = 10$ Hz) gives 90% of total energy:

```
areal = integral(@myfun,9.2,10.8);    %% Gives 0.2242 for fc = 5 Hz
```

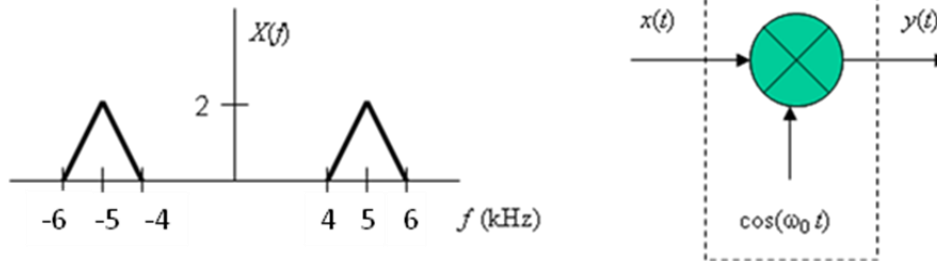
Matlab R2021a was used to execute the above code.

Epilog. Observing a signal in practice for a finite amount of time alters the signal's spectrum.

Estimating bandwidth via visual inspection can be very accurate.

Problem 0.2. Downconversion.

A signal $x(t)$ is input to a mixer to produce the output $y(t)$ where $y(t) = x(t) \cos(\omega_0 t)$ where $\omega_0 = 2\pi f_0$ and $f_0 = 5$ kHz. A block diagram of the mixer is shown below on the right. The Fourier transform of $x(t)$ is shown below on the left.



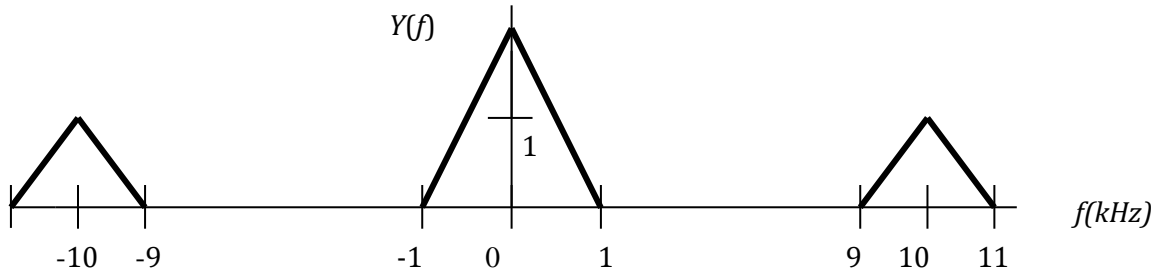
Part (a). Using Fourier transform properties, derive an expression for $Y(f)$ in terms of $X(f)$.

Solution: Fourier transform of $y(t) = x(t) \cos(\omega_0 t)$ is $Y(\omega) = \frac{1}{2\pi} \mathfrak{F} [\cos(\omega_0 t)] * X(\omega)$

Since $\cos(\omega_0 t) \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$, $Y(\omega) = \frac{1}{2}(X(\omega + \omega_0) + X(\omega - \omega_0))$

In a similar way, $Y(f) = \frac{1}{2}(\delta(f + f_0) + \delta(f - f_0)) * X(f) = \frac{1}{2}(X(f + f_0) + X(f - f_0))$

Part (b). Sketch $Y(f)$ vs. f . Label all important points on the horizontal and vertical axes.



Part (c). What operation would you apply to the signal $y(t)$ in part (b) to obtain a baseband signal? The process of extracting the baseband signal from a bandpass signal is known as *downconversion*.

Solution: Baseband signal may be obtained by applying a lowpass filter to $y(t)$ that passes frequencies in $[-1, 1]$ kHz and attenuates frequencies at and above 9 kHz in absolute value.

Epilog: In analog continuous-time circuits, downconversion is often implemented by sampling at f_0 and lowpass filtering. The lowpass filter would pass $[-1, 1]$ kHz and attenuate frequencies at and above 4 kHz in absolute value. See lecture slides 4-3 and 4-12.

Problem 0.3. Sampling in Continuous Time.

Sampling the amplitude of an analog, continuous-time signal $f(t)$ every T_s seconds can be modeled in continuous time as $y(t) = f(t) p(t)$ where $p(t)$ is the impulse train defined by

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

T_s is known as the sampling duration. The Fourier series expansion of the impulse train is

$$p(t) = \frac{1}{T_s} (1 + 2 \cos(\omega_s t) + 2 \cos(2 \omega_s t) + \dots)$$

where $\omega_s = 2 \pi / T_s$ is the sampling rate in units of radians per second. (See the Addendum at the end of the solution to this problem for the derivation of the Fourier series expansion.)

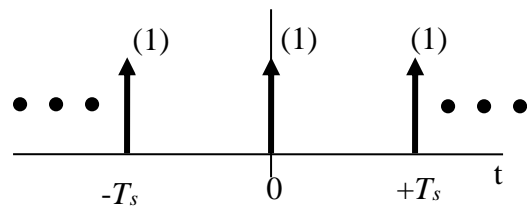
Part (a). Plot the impulse train $p(t)$.

Solution: Shown on the right is the plot of the impulse

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$

The notation (1) means that the area under the Dirac delta is 1. This is important because the amplitude of

the Dirac delta at the origin is undefined or could be interpreted as infinity.



Part (b). Note that in part (a), $p(t)$ is periodic. What is the period?

Solution: A signal $x(t)$ is periodic with period T if $x(t) = x(t - T)$ for all t . Fundamental period refers to the smallest positive value of T for which $x(t)$ is periodic. The impulse train $p(t)$ is periodic with fundamental period T_s . The fundamental period is used in the Fourier series

$$p(t) = \frac{1}{T_s} (1 + 2\cos(\omega_s t) + 2\cos(2\omega_s t) + \dots)$$

expansion of the impulse train where the sampling rate in rad/s is $\omega_s = 2\pi / T_s$.

Part (c). Using the Fourier series representation of $p(t)$ given above, please give a formula for $P(\omega)$, which is the Fourier transform of $p(t)$. Express your answer for $P(\omega)$ as an impulse train in the Fourier domain.

Solution: We apply the Fourier transform to $p(t)$ given immediately above to compute

$$P(\omega) = \frac{2\pi}{T_s} (\delta(\omega) + \delta(\omega - \omega_s) + \delta(\omega + \omega_s) + \delta(\omega - 2\omega_s) + \delta(\omega + 2\omega_s) + \dots)$$

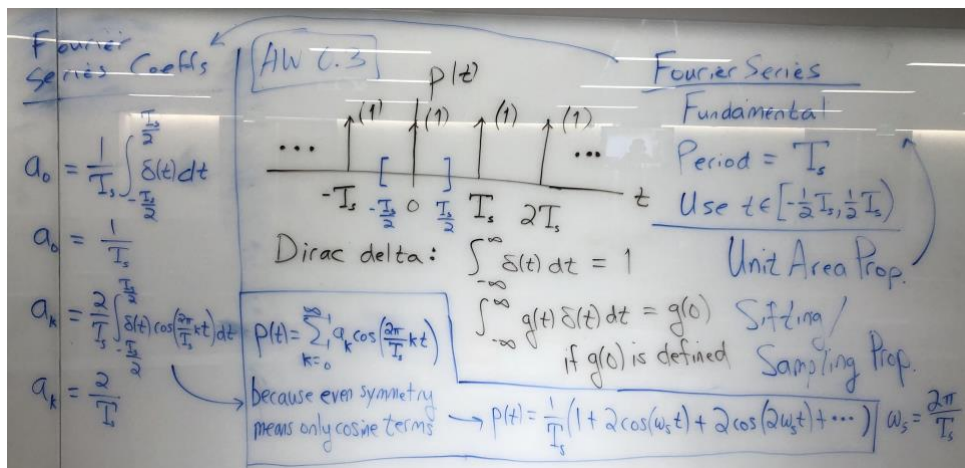
by using the transform pairs $\cos(\omega_s t) \leftrightarrow \pi[\delta(\omega - \omega_s) + \delta(\omega + \omega_s)]$ and $1 \leftrightarrow 2\pi\delta(\omega)$

Thus $P(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$

Part (d). What is the spacing of adjacent impulses in the impulse train in $P(\omega)$ with respect to frequency ω in rad/s?

Solution: Spacing is the distance between adjacent impulses in the Fourier domain, ω_s .

Addendum: Derivation for the Fourier series of an impulse train in the time domain is given by solution to Problem 2 on the [EE 313 Fall 2021 Final Exam](#) as well as below:



Problem 0.4. Discrete-Time Sinusoidal Generation.

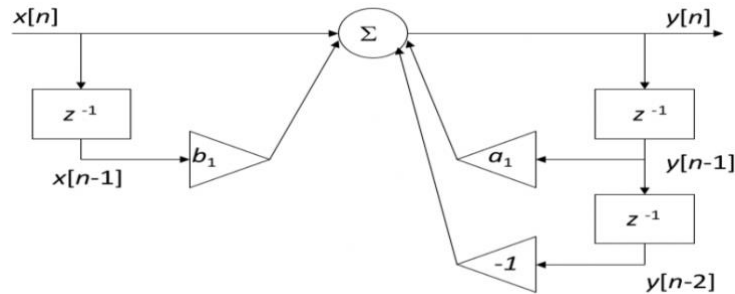
Consider a causal discrete-time linear time-invariant system with input $x[n]$ and output $y[n]$ being governed by the following difference equation:

$$y[n] = (2 \cos \omega_0) y[n-1] - y[n-2] + x[n] - (\cos \omega_0) x[n-1]$$

The impulse response of the above system is a *causal sinusoidal signal* with discrete-time frequency ω_0 in units of rad/sample. Normally, ω_0 would be in the interval $[-\pi, \pi)$. In lab #2, you'll implement the difference equation.

Part (a). Draw the block diagram for this system using add (or summation), multiplication (or gain), and delay blocks. Please label delay blocks with the text z^{-M} to denote a delay of M samples. Use arrowheads to indicate direction of the flow of signals.

Solution: Block diagram of the given filter with input $x[n]$ and output $y[n]$ and with coefficients $b_1 = -\cos \omega_0$ and $a_1 = 2 \cos \omega_0$:



Part (b). Please state all initial conditions. Please give values for the initial conditions to satisfy the stated system properties.

Solution: Initial conditions can be found by calculating the first few output values:

$$y[0] = (2 \cos \omega_0) y[-1] - y[-2] + x[0] - (\cos \omega_0) x[-1]$$

$$y[1] = (2 \cos \omega_0) y[0] - y[-1] + x[1] - (\cos \omega_0) x[0]$$

Hence, the initial conditions are given by $y[-1]$, $x[-1]$ and $y[-2]$. These values correspond to the initial values in the delay-by-one-sample blocks, which are denoted by z^{-1} .

For a system to satisfy the linearity property, the system must be at rest; i.e., the initial conditions must be zero. Same goes for the system property of time-invariance. We will show this on [lecture slides 3-12 and 3-13](#) and Handout U [“Time-Invariance Property for a System Under Observation”](#).

Note about Causality. We're concerned here observing a system starting at a particular time-- we don't know the input or output prior to the starting time. As a *system* property, causality would mean that the output of the system would depend on previous outputs and current and previous inputs, and not on future inputs, current output or future outputs. A *signal* is causal if all amplitude values are zero left of the origin. A *causal LTI system* must have an impulse response that is a causal signal and being LTI would already mean that the initial conditions are zero. An LTI system could be causal or anti-causal. (Anti-causal would mean processing an input signal from the starting time in a negative time direction towards negative infinity.) It is uncertain to whether a system property of causality by itself would mean that a system under observation would have as a necessary condition that the system be at rest. When the system property of causality is coupled with LTI properties, the LTI would already require that the system under observation be at rest.

Part (c). Find the equation for the transfer function in the z -domain, including the region of convergence.

Solution: $y[n] = (2 \cos \omega_0) y[n-1] - y[n-2] + x[n] - (\cos \omega_0) x[n-1]$

Taking z -transform of both sides, we get

$$Y(z) = (2 \cos \omega_0) z^{-1} Y(z) - z^{-2} Y(z) + X(z) - (\cos \omega_0) z^{-1} X(z)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 - (\cos \omega_0) z^{-1}}{1 - 2(\cos \omega_0) z^{-1} + z^{-2}} \Rightarrow H(z) = \frac{1 - (\cos \omega_0) z^{-1}}{1 - 2(\cos \omega_0) z^{-1} + z^{-2}}$$

To find the region of convergence, we need to find the poles of this transfer function:

$$1 - (2 \cos \omega_0) z^{-1} + z^{-2} = 0$$

By multiplying each side by z^2 (assuming that $z \neq 0$):

$$z^2 - (2 \cos \omega_0) z + 1 = 0$$

Roots are located at $\frac{1}{2} (-b \pm \sqrt{\Delta})$. Here,

$$b = -2 \cos \omega_0$$

$$\Delta = 4 \cos^2 \omega_0 - 4 = -4 \sin^2 \omega_0$$

Since $\Delta < 0$, there are two complex roots:

$$x_1 = \frac{2 \cos \omega_0 + 2j \sin \omega_0}{2} = \cos \omega_0 + j \sin \omega_0; x_2 = \frac{2 \cos \omega_0 - 2j \sin \omega_0}{2} = \cos \omega_0 - j \sin \omega_0$$

Both poles have a magnitude of 1. As a result, they lie on the unit circle in the z -plane. The largest pole magnitude is equal to 1, i.e. $\max\{|x_i|\} = 1$. With a causal system, the region of convergence

refers to a region outside of a circle that has the radius of the largest pole magnitude, i.e. the ROC is $|z| > \max\{|x_i|\} = 1$.

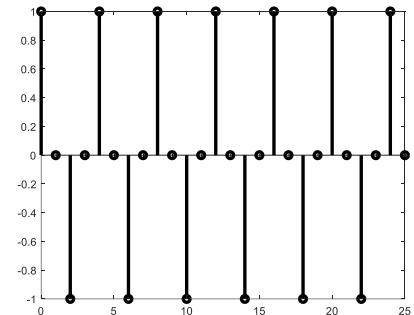
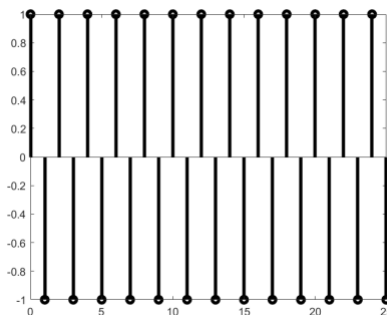
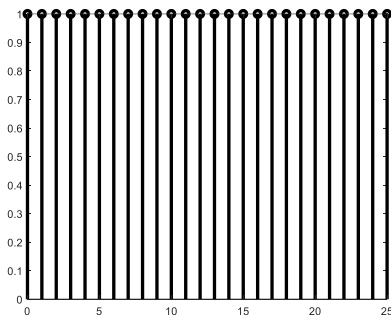
The system is *not* bounded-input bounded-output (BIBO) stable because the region of convergence does not include the unit circle. A BIBO stable, causal system, must have all its poles inside the unit circle. We'll see this in [lecture slides 6-13 and 6-14](#).

Part (d). Compute the inverse z-transform of the transfer function in part (c) to find the impulse response of the system.

Solution: By using inverse z-transform tables, the above $H(z)$ with the given region of convergence will have an impulse response of $h[n] = \cos(\omega_0 n) u[n]$.

Part (e). Using MATLAB or Python, plot the impulse response obtained in part (d) for ω_0 equal to 0, π , and a value in the interval $(0, \pi)$ of your choosing. Turn in your code and plots.

Solution: ω_0 is 0 rad/sample (left), π rad/sample (center) and $\pi/2$ rad/sample (right):



The sample MATLAB code for $\omega_0 = \pi/2$:

```
w0 = pi/2;  
n = [0:25];  
u = n >= 0; % implements stepfun  
x = cos(w0*n);  
h = x.*u;  
figure, stem(n,h, 'k', 'LineWidth', 3);
```