[10:30am] Capacity for transmission over additive noise channel

A student asked about a result derived by Claude Shannon in 1948 about how fast bits could in theory be communicated over a noisy channel, which is called channel capacity.

Channel capacity C provides a theoretical upper bound on the bit rate in bits/s

$$C = B\frac{D}{2}\log_2(1 + \text{SNR})$$

where B *is* Bandwidth (in Hz) and D *is* Dimension (1 for PAM or 2 for QAM).

The term $\frac{D}{2}\log_2(1 + \text{SNR})$ is the spectral efficiency in bits/s/Hz where

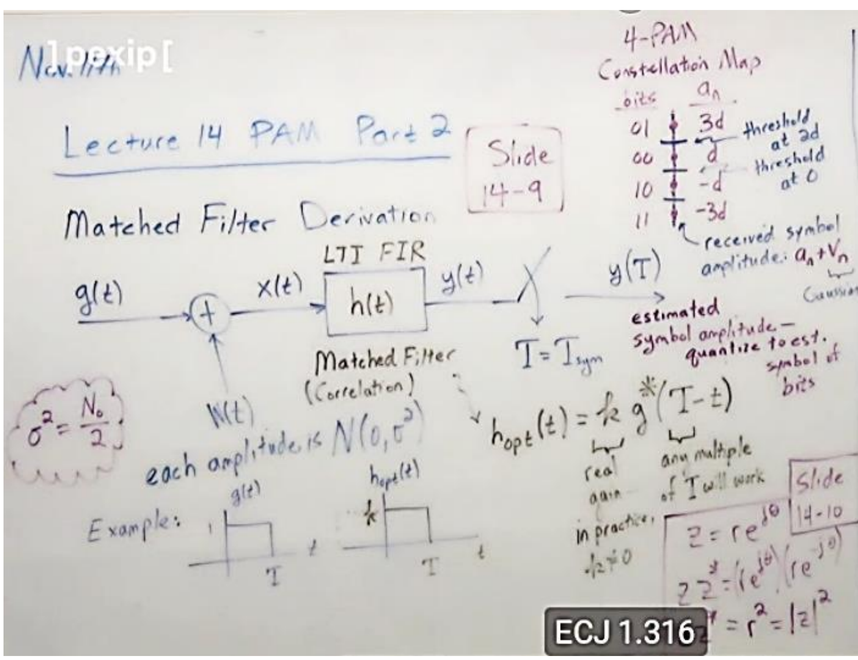
$$\text{SNR} = \frac{\text{Signal Power}}{\text{Noise Power}} \text{ (in linear units)}$$

The channel capacity formula, which is for a single transmitter, single receiver system, gives an upper bound but does explain how to reach the bound. It does give some insights:

- The largest increase in channel capacity comes from increasing bandwidth.
- A secondary increase can come from improving spectral efficiency, such as by
  - increasing the effective received SNR through matched filtering and other means
  - using more transmit and receive antennas as used in Wi-Fi and LTE systems

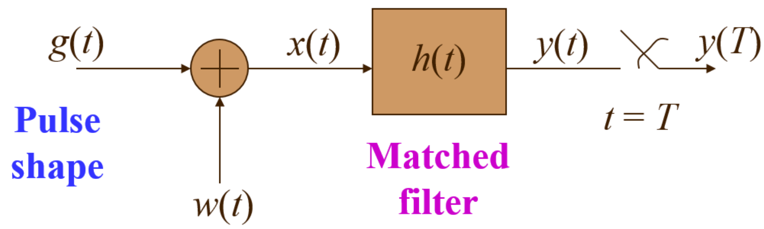
Communication system designers have developed different modulation, error control coding, multiple antenna, and multiple carrier methods in an attempt to reach the bound. Multicarrier approaches divide the bandwidth into parallel narrowband subchannels, as is done in orthogonal frequency division multiplexing (OFDM) in Wi-Fi and LTE systems.

[10:35am] Matched filter (Lecture 14 Part 2)



Presentation is based on the following sections of *Communication Systems* (3rd ed) by Simon Haykin:

- Section 4.6 Random Processes
- Section 4.9 Ergodic Processes
- Section 4.10 Transmission of a Random Process Through a Linear Filter
- Section 4.11 Power Spectral Density
- Section 4.12 Gaussian Processes



- $g(t)$ is the pulse shaping filter
- $h(t)$ is the matched filter impulse response
- $w(t)$ is additive noise; each amplitude value follows a Gaussian distribution $N(0, \sigma^2)$
- $y(T)$ is the received symbol amplitude
- Result from derivation of the optimal matched filter from later today:

$$h_{\text{opt}}(t) = \underbrace{k}_{\text{gain}} \; g^*(T - t)$$

4-PAM constellation map

bits	a_n
01	$3d$
00	d
10	$-d$
11	$-3d$

- $g^*(\cdot)$ denotes complex conjugate of $g(\cdot)$
- Received symbol amplitude: $\underbrace{a_n}_{\substack{\text{transmitted}\\\text{symbol}\\\text{amplitude}}} + \underbrace{v_n}_{\substack{\text{Gaussian}\\\text{random}\\\text{variable}}}$
- Quantize received symbol amplitude to nearest transmitted symbol amplitude to estimate th symbol of bits

[10:50] Matched filter derivation

- Goal: maximize output signal power $g_0(t) = g(t) * h(t)$ at $t = T$ where $T = T_{\text{sym}}$
- Goal: minimize output noise power $n(t) = w(t) * h(t)$
- Equivalently, maximize η, the peak pulse SNR

$$\eta = \frac{\text{Peak Signal Power}}{\text{Average Noise Power}} = \frac{|g_0(t)|^2}{E\{n^2(t)\}}$$

- $E\{n^2(t)\}$ means to find the expected (average) instantaneous noise power $n^2(t)$
- We don't know the amplitude values of $n(t)$, but we can find its power spectrum
  - Power spectrum for signal $x(t)$ w/ Fourier transform $X(f)$ is $P_x(f) = |X(f)|^2$
  - Autocorrelation of $x(t)$ is $R_x(\tau) = x(\tau) * x^*(-\tau)$
  - Autocorrelation is symmetric: $R_x(\tau) = R_x(-\tau)$
  - Multiplication in Fourier domain is convolution in time domain
  - Conjugation in Fourier domain is reversal an conjugation in time
- Result: power spectrum is Fourier transform of autocorrelation:

$$\begin{aligned} P_x(f) &= |X(f)|^2 \\ &= X(f)X^*(f) \\ &= \mathcal{F}\{x(\tau) * x^*(-\tau)\} \\ &= \mathcal{F}\{R_x(\tau)\} \end{aligned}$$

- For a two-sided random signal $n(t)$, the Fourier transform may not exist

- Power spectrum does in fact exist for $n(t)$: $P_n(f) = \mathcal{F}\{R_n(\tau)\}$
- For zero-mean Gaussian random process $n(t)$ with variance σ^2:

$$R_n(\tau) = E\{n(t) * n(t+\tau)\} = \sigma^2 \delta(\tau)$$
$$P_n(f) = \mathcal{F}\{R_n(\tau)\} = \sigma^2$$

- Noise at output of matched filter is $n(t) = w(t) * h(t)$ which has power spectrum

$$S_N(f) = S_W(f) S_H(f) = \frac{N_0}{2} |H(f)|^2$$

- Optimal matched filter result (see slides 14-12, 14-13, 14-14 for full derivation):

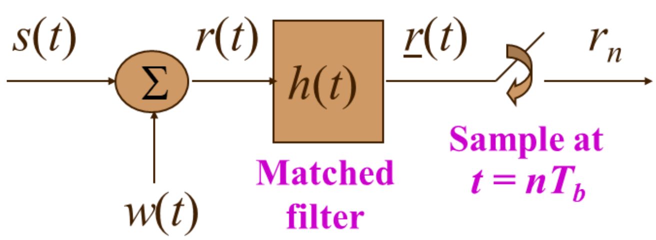
$$h_{opt} = k\, g^*(T - \tau)$$

  - Impulse response is dependent on the symbol period
  - Scaled, conjugated, time-reversed and shifted version of the pulse shape $g(t)$
  - Maximizes peak pulse SNR

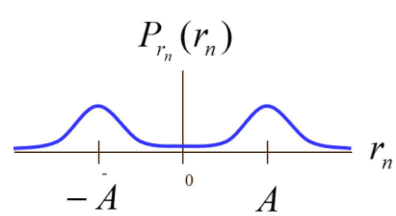
[11:30] Matched filter for rectangular pulse

- Convolve received input signal with rectangular pulse of duration T sec
- Equivalent to "integrate and dump"
- $g_0(t) = g(t) * h(t)$ becomes a triangular pulse

[11:35] Symbol error probability



- $w(t)$ is spectrally flat noise that follows a gaussian distribution $N(0, \sigma^2)$
- Lowpass filtering Gaussian process produces a Gaussian process
  - Variance (noise power) scaled by twice filter's bandwidth
  - Noise power at output is σ^2/T (see slide 14-19)
- For symbol amplitudes of $\pm A$, Received signal r_n is $\pm A + v_n$
- Threshold to estimate transmitted amplitude
- Probability of error will be area under tail of Gaussian:



$$P(\text{error}|s(nT_b) = -A) = P(-A + v_n > 0) = P(v_n > A) = P\left(\frac{v_n}{\sigma} > \frac{A}{\sigma}\right)$$

- Can be expressed in terms of Q function: $P(\text{error}) = Q\left(\sqrt{\frac{A^2}{\sigma^2}}\right) = Q\left(\sqrt{SNR}\right)$
  - $Q(x) = 1 - \Phi(x)$, where $\Phi(x)$ is the CDF for a Gaussian
  - Q function can also be related to error function in Matlab