Solution Set for Homework #8

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PROBLEM 1: CONTINUOUS-TIME SYSTEM PROPERTIES. ***20 points.***

Signal Processing First, problem P-9.2, page 279.

In each of the following cases, state whether or not the continuous time system is (i) linear, (ii) time-invariant, (iii) stable, and (iv) causal. In each case, x(t) he represents the input and y(t) represents the corresponding output of the system. Provide a brief justification, either in the form of mathematical equations or statements in the form of complete, crack, push sentences. Remember, in order to show the system does not have the property, all you have to do is give an example and put up with is not satisfy the condition of the property.

(a) An exponential system: $y(t) = e^{x(t+2)}$. **Used in speech denoising and machine learning**.

(b) A phase modulator: $y(t) = \cos(\omega_c t + x(t))$. **Phase modulation is used for low-power transmission in IoT systems. The digital version, Phase Shift Keying, is used in RFID and Bluetooth and higher transmit power systems, such as Wi-Fi and cellular communications.**

(c) An amplitude modulator: $y(t) = (A + x(t)) \cos(\omega_c t)$. **Used in AM radio. Amplitude modulation (without the offset of *A*) is used in Wi-Fi, cellular and cable modems.**

(d) Take the even part of the input signal: $y(t) = \frac{x(t)+x(-t)}{2}$. **Primarily for theoretical analysis.**

***Solution:* Linearity**

When checking each system for linearity, we can use the quick test of input signal of 0 for all time, which is a by-product of the homogeneity property when the input signal is scaled by a = 0. If the output signal is not zero for all time, then the system is not linear. If the output is zero for all time, then we'll have to apply the mathematical definitions for homogeneity and additivity.

(a) $\boldsymbol{y(t) = e^{x(t+2)} = e^0 = 1}$. **Fails all-zero input test. Not linear.**

(b) $\boldsymbol{y(t) = \cos(\omega_c t + x(t)) = \cos(\omega_c t)}$. **Fails all-zero input test. Not linear.**

(c) $\boldsymbol{y(t) = (A + x(t)) \cos(\omega_c t) = A \cos(\omega_c t)}$. **Fails all-zero input test. Not linear.**

(d) $\boldsymbol{y(t) = \frac{x(t)+x(-t)}{2} = 0}$. **Passes all-zero input test. Check for homogeneity and additivity.**

- ***Homogeneity*. Input** $\boldsymbol{a\, x(t)}$. **Output** $\boldsymbol{y_{scaled}(t) = \frac{(a\, x(t))+(a\, x(-t))}{2} = a\, \frac{x(t)+x(-t)}{2} = a\, y(t)}$
- ***Additivity*. Input** $\boldsymbol{x_1(t) + x_2(t)}$. **Output**

$$\boldsymbol{y_{additive}(t) = \frac{(x_1(t)+x_2(t))+(x_1(-t)+x_2(-t))}{2} = \frac{x_1(t)+x_1(-t)}{2} + \frac{x_2(t)+x_2(-t)}{2} = y_1(t) + y_2(t)}$$

Yes, system (d) is linear.

***Solution:* Time-Invariance**

For time-invariant system, shift of the input signal by any real-valued t causes the same shift in output signal, i.e. ***x*(*t* - t) means *y*(*t* - t) for all t.**

(a) $y(t) = e^{x(t+2)}$. **Input** $x(t-\tau)$. **Output** $y_{shifted}(t) = e^{x((t-\tau)+2)} = e^{x(t-\tau+2)} = y(t-\tau)$. ***Time-invariant.***

(b) $y(t) = \cos(\omega_c t + x(t))$. **The signal** $\cos(\omega_c t)$ **is part of the system and does not shift in time when the input shifts in time.** ***Time-varying.***

(c) $y(t) = (A + x(t))\cos(\omega_c t)$. **The signal** $\cos(\omega_c t)$ **is part of the system and does not shift in time when the input shifts in time.** ***Time-varying.***

(d) $y(t) = \frac{x(t)+x(-t)}{2}$. **The copy of the input signal shifts in the same way that the input signal shifts. The copy that is reversed in time gives the negated shift.** ***Time-varying.***

Solution: **Stability**

A stable system will always produce a bounded amplitude output signal when given a bounded amplitude input signal. Let $|x(t)| < B < \infty$

(a) $|y(t)| = \left|e^{x(t+2)}\right| \le |e^{B}|$. **Bounded output.** ***Stable.***

(b) $y(t) = \cos(\omega_c t + x(t))$. **Output will always be in range [-1, 1] regardless of the value of** $x(t)$**. Bounded output.** ***Stable.***

(c) $|y(t)| = |(A + x(t))\cos(\omega_c t)| \le |A + x(t)|\,|\cos(\omega_c t)| \le |A + x(t)| \le |A| + |x(t)| \le |A| + B$. **Bounded output.** ***Stable.***

(d) $|y(t)| = \left|\frac{x(t)+x(-t)}{2}\right| \le \frac{|x(t)|}{2} + \frac{|x(-t)|}{2} \le B$. **Bounded output.** ***Stable.***

Solution: **Causality**

A causal system depends only on current and previous input values and/or previous output value to compute an output value.

(a) $y(t) = e^{x(t+2)}$. **Depends on input 2 seconds in the future.** ***Not Causal.***

(b) $y(t) = \cos(\omega_c t + x(t))$. **Only depends on the current input value** $x(t)$. ***Causal.***

(c) $y(t) = (A + x(t))\cos(\omega_c t)$. **Only depends on the current input value** $x(t)$. ***Causal.***

(d) $y(t) = \frac{x(t)+x(-t)}{2}$. **Check specific values of time** ***t*****.**

- **When** $t = 0$, $y(0) = (\, x(0) + x(0)\,) / 2 = x(0)$. **Causal.**
- **When** $t = 2$, $y(2) = (\, x(2) + x(-2)\,) / 2$. **Causal.**
- **When** $t = -2$, $y(-2) = (\, x(-2) + x(2)\,) / 2$. **Depends on future input** $x(2)$. ***Not Causal.***

PROBLEM 2: CONTINUOUS-TIME AVERAGING FILTERS. *32 points.*

For a **continuous**-time LTI system with input signal $x(t)$ and impulse response $h(t)$, the output signal $y(t)$ is the convolution of $h(t)$ and $x(t)$:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

(a) Compute the output $y(t)$ when the input $x(t)$ is a rectangular pulse of amplitude 1 for $t \in [0, T_x]$ and amplitude 0 otherwise and $x[n]$ is filtered by an LTI unnormalized averaging filter whose impulse response $h(t)$ is a rectangular pulse of amplitude 1 for $t \in [0, T_h]$ and amplitude 0 otherwise. Assume $T_x \neq T_h$.

i. Write an equation relating output $y(t)$ and input $x(t)$. *4 points*

Solution: **With $\boldsymbol{h(t) = 1}$ for $\boldsymbol{t \in [0, T_h]}$,**

$$\boldsymbol{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{0}^{T_h} x(t-\lambda)\, d\lambda}$$

We can apply the change of variables $\boldsymbol{u = t - \lambda}$. $\boldsymbol{du = -d\lambda}$. As $\boldsymbol{\lambda \to 0, u \to t}$. As $\boldsymbol{\lambda \to T_h, u \to t - T_h}$. This gives

$$\boldsymbol{y(t) = \int_{0}^{T_h} x(t-\lambda)\, d\lambda = \int_{t-T_h}^{t} x(u)\, du}$$

The averaging filter integrates the input signal over the previous $\boldsymbol{T_h}$ seconds.
Although not asked, this filter is stable unlike the integrator over all time in part (b).

ii. What is(are) the initial condition(s) and what value should it(they) be set to? *3 points*

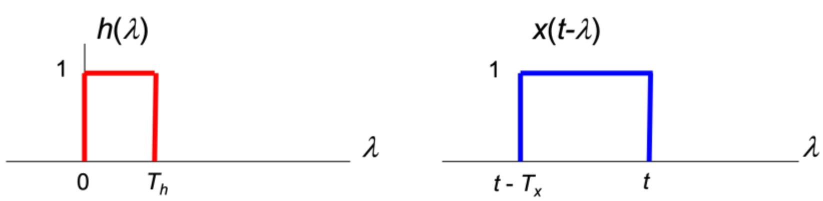
Solution: **The initial conditions are the memory of the previous $\boldsymbol{T_h}$ seconds of the input signal. This memory (signal buffer) would have to be initially zeroed out.**

iii. Develop a formula for $y(t) = h(t) * x(t)$ using the convolution definition in terms of T_x and T_h. Show the intermediate steps in computing the convolution. *6 points*

Solution: **Trapezoid of duration $\boldsymbol{T_y = T_h + T_x}$. Let $\boldsymbol{T_{min} = \min(T_h, T_x)}$ and $\boldsymbol{T_{max} = \max(T_h, T_x)}$. As we flip and slide one rectangular pulse against the other, partial overlap occurs from 0 to $\boldsymbol{T_{min}}$ seconds, complete overlap from $\boldsymbol{T_{min}}$ to $\boldsymbol{T_{max}}$ seconds, and partial overlap from $\boldsymbol{T_{max}}$ to $\boldsymbol{T_y}$ seconds.**

$$\boldsymbol{y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \le t < T_{min} \\ T_{min} & \text{for } T_{min} \le t < T_{max} \\ T_y - t & \text{for } T_{max} \le t < T_y \\ 0 & \text{for } t > T_y \end{cases}}$$

The details of the flip-and-slide are analogous to problem 1(a)iii for the discrete-time convolution of two rectangular pulses, and explained next.
We will hold $\boldsymbol{h(t)}$ in place and flip and slide $\boldsymbol{x(t)}$about $\boldsymbol{h(t)}$:



There are five cases to consider:

1. **No overlap. $\boldsymbol{t < 0}$. Amplitude is 0.**
2. **Partial overlap. $\boldsymbol{0 \le t < T_{min}}$. Amplitude is $\boldsymbol{t}$.**
   Initial overlap at the origin, and integration of a point is zero. Each shift by a time unit adds that much to the area.

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{0}^{t} d\lambda = t$$

3. **Complete overlap.** $T_{min} \le t < T_{max}$. **Amplitude is** T_{min}.
   Here, T_{min} **seconds overlap, and each amplitude has a value of one.**
4. **Partial overlap.** $T_{max} \le n < T_y$. **Amplitude is** $T_y - t$.
   Amplitude reduces by the same amount that t **is shifted.**

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{t-T_x}^{T_h} d\lambda = T_h - (t - T_x) = T_y - t$$

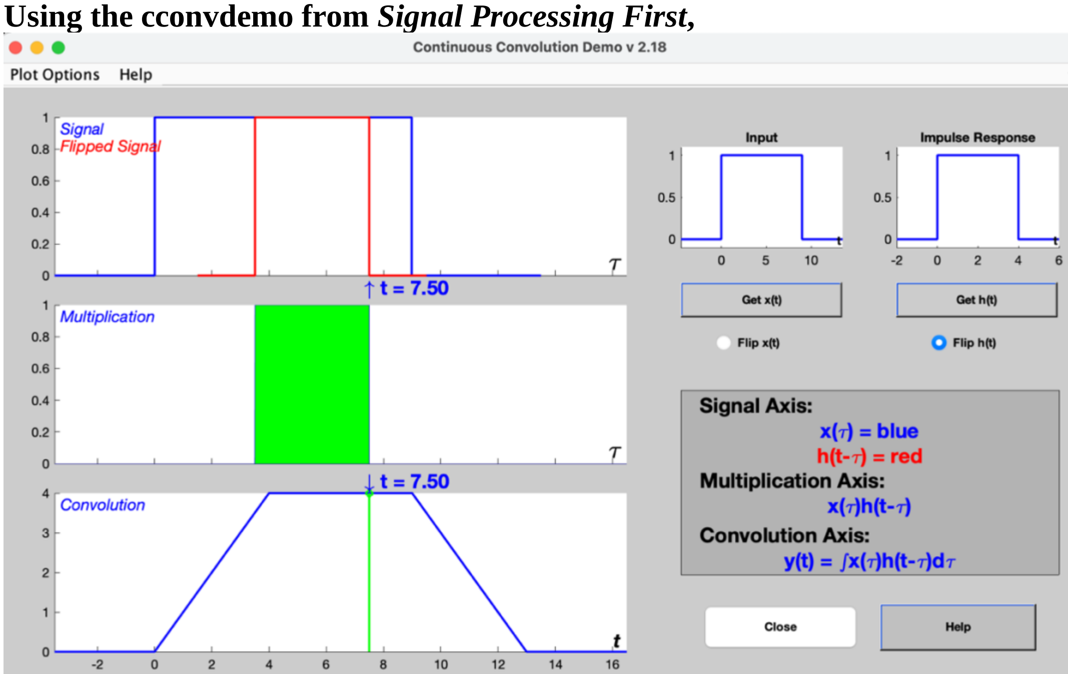
5. **No overlap.** $t \ge T_y$. **Amplitude is 0.**

iv. Validate the formula for $y(t)$ to compute the convolution for $T_x = 9$ seconds and $T_h = 4$ seconds. *3 points.*

***Solution*: Trapezoid of duration** $T_y = 4 + 9 = 13$ **seconds. Let** $T_{min} = \min(4, 9) = 4$ **seconds and** $T_{max} = \max(4, 9) = 9$ **seconds. As we flip and slide one rectangular pulse against the other, partial overlap occurs from 0 to** T_{min} **seconds, complete overlap from** T_{min} **to** T_{max} **seconds, and partial overlap from** T_{max} **to** T_y **seconds.**

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \le t < 4 \\ 4 & \text{for } 4 \le t < 9 \\ 13 - t & \text{for } 9 \le t < 13 \\ 0 & \text{for } t > 13 \end{cases}$$

Using the cconvdemo from *Signal Processing First*,



(b) When an input signal has an average value of zero, i.e. the DC component is zero, an LTI integrator can be used as an averaging filter. The differential equation governing the input-output relationship is

$$y(t) = \int_{0^-}^{t} x(\tau)\, d\tau \quad \text{for } t \geq 0$$

The integrator was operating before *t* = 0 seconds, but we weren't able to observe that. We are observing the system starting at *t* = 0 seconds.

Since we are starting the integration at *t* = 0 seconds, there is ambiguity as to whether Dirac delta signal would be included at time 0. We can use 0^- as the lower limit to indicate that integration starts at time 0 before the impulse occurs.

i. What is(are) the initial condition(s) and what value should it(they) be set to? *3 points*
 ***Solution*: The initial condition is the initial integration value *y*(0). It should be set to zero as a necessary condition for LTI properties to hold.**

ii. What is the impulse response? *3 points*
 ***Solution:* For input $\boldsymbol{x(t) = \delta(t)}$, the output is the impulse response**

$$\boldsymbol{h(t) = \int_{0^-}^{t} \delta(\tau)\, d\tau = u(t)}$$

iii. Develop a formula for $y(t) = h(t) * x(t)$ using the convolution definition when the input signal is $x(t) = u(t)$. Note that $x(t)$ has bounded amplitude. *9 points*
 ***Solution:* Using the convolution definition,**

$$\boldsymbol{y(t) = h(t) * x(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(\lambda)\, u(t-\lambda)\, d\lambda}$$

$\boldsymbol{u(\lambda)}$ is 1 for $\boldsymbol{\lambda \geq 0}$ and 0 otherwise, whereas $\boldsymbol{u(t-\lambda)}$ is 1 for $\boldsymbol{t - \lambda \geq 0}$ or $\boldsymbol{\lambda \leq t}$. Also, note that $\boldsymbol{t \geq 0}$ because $\boldsymbol{\lambda \geq 0}$:

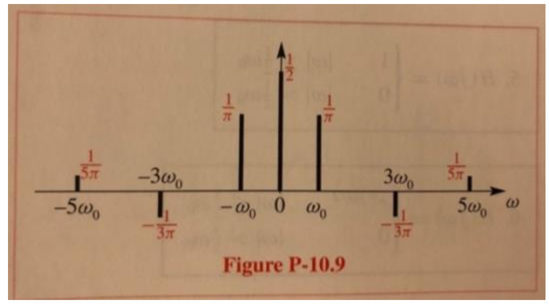
$$\boldsymbol{y(t) = \int_{0}^{t} d\lambda = t\, u(t)}$$

iv. Is the LTI integrator bounded-input bounded-output (BIBO) stable? Your work in part iii might be helpful. *3 points*
 ***Solution:* For input signal $\boldsymbol{x(t) = u(t)}$, whose amplitude is bounded in [0, 1], the output $\boldsymbol{y(t) = t\, u(t)}$ grows without bound as $\boldsymbol{t \to \infty}$. LTI integrator is not stable.**

<u>PROBLEM 3: CONTINUOUS TIME-FREQUENCY RESPONSE</u>. *48 points.*

Signal Processing First, problem P-10.9, page 305. In addition, for each of the seven filters given, describe the frequency selectivity in the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

Consider an LTI system whose frequency response $H(j\omega)$ is unknown. The system has a periodic input whose spectrum is shown in Fig. P-10.9.



For each part of this problem, the output of the system is given and the frequency response must be determined by selecting from the list numbered 1-7 below. Chose the frequency response $H(j\omega)$ of the system that could have produced the specific output when the input is the signal with the spectrum in Fig. P-10.9.

Solution: Fig. P-10.9 plots the following Fourier series coefficients

$$a_k = \begin{cases} \dfrac{1}{2}, & k = 0 \\ \dfrac{1}{\pi}, & k = \pm 1 \\ 0, & k = \pm 2 \\ -\dfrac{1}{3\pi}, & k = \pm 3 \\ 0, & k = \pm 4 \\ \dfrac{1}{5\pi}, & k = \pm 5 \end{cases}$$

which can be used in the Fourier series formula

$$x(t) = \sum_{k=-5}^{5} a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

Due to the system having linear and time-invariant properties, all the frequency components in the output signal had to be present in the input signal. That is, a linear time-invariant (LTI) system cannot create new frequencies.

Using LTI system properties, the output signal is simply the sum of the system's response to each frequency component of the input signal:

$$y(t) = \frac{1}{2}H(j0) + \frac{2}{\pi}\cos(\omega_0 t)H(j\omega_0) - \frac{2}{3\pi}\cos(3\omega_0 t)H(j3\omega_0) + \frac{2}{5\pi}\cos(5\omega_0 t)H(j5\omega_0)$$

We can write the frequency response into polar form as $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$:

$$\begin{aligned} y(t) = \frac{1}{2}H(j0) + \frac{2}{\pi}|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) - \\ \frac{2}{3\pi}|H(j3\omega_0)|\cos(3\omega_0 t + \angle H(j3\omega_0)) + \\ \frac{2}{5\pi}|H(j5\omega_0)|\cos(5\omega_0 t + \angle H(j5\omega_0)) \end{aligned}$$

Please see lecture slide 14-6 and *Signal Processing First* Section 10-2.

a) $y(t)=\frac{1}{2}$

The output of system can be obtained by the following formula

$$y(t)=\sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

Hence, we can find the value of $H(j\omega)$ for frequencies that are present in the input.

$$H(jk\omega_0)=\begin{cases}1, & k=0\\ 0, & k=\pm1,\pm3,\pm5\end{cases}$$

Filter 5 has similar response, so the input has passed through this lowpass filter.

$$H(j\omega)=\begin{bmatrix}1 & |\omega|\le\frac{1}{2}\omega_0\\ 0 & |\omega|>\frac{1}{2}\omega_0\end{bmatrix}$$

b) $y(t)=\frac{1}{2}+\frac{2}{\pi}\cos\left[\omega_0\left(t-\frac{1}{2}\right)\right]$

$$y(t)=\sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}=\frac{1}{2}+\frac{2}{\pi}\cos\left[\omega_0\left(t-\frac{1}{2}\right)\right]$$

$$H\left(jk\omega_0\right)=\begin{cases}1, & k=0\\ e^{-j\omega_0/2}, & k=\pm1\\ 0, & k=\pm3,\pm5\end{cases}$$

Filter 6 has this property and will give similar output. This lowpass filter removes frequencies above $\frac{3\omega_0}{2}$ and delays the input by ½ sample. We can obtain the delay by computing the group delay for the filter as follows: Group Delay$(\omega)=-\frac{d}{d\omega}\angle H(j\omega)=-\frac{d}{d\omega}\left(-\frac{\omega}{2}\right)=\frac{1}{2}$.

$$H(j\omega)=\begin{bmatrix}e^{-j\frac{\omega}{2}} & |\omega|\le\frac{3}{2}\omega_0\\ 0 & |\omega|>\frac{3}{2}\omega_0\end{bmatrix}$$

c)

$$y(t)=\frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$y(t)=\sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}=\frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$H\left(jk\omega_0\right)=\begin{cases}1, & k=\pm1\\ 0, & k=0,\pm3,\pm5\end{cases}$$

The original signal has passed through a bandpass filter which removes all frequencies present in the input signal except ω_0. Filter 7 shows a bandpass filter with this property.

$$H(j\omega)=\begin{cases}1, & \frac{1}{2}\omega_0<|\omega|<\frac{3}{2}\omega_0\\ 0, & |\omega|<\frac{1}{2}\omega_0 \; or \; |\omega|>\frac{3}{2}\omega_0\end{cases}$$

d)

$$y(t)=x(t)-\frac{1}{2}=\frac{2}{\pi}\cos(\omega_0 t)-\frac{2}{3\pi}\cos(3\omega_0 t)+\frac{2}{5\pi}\cos(5\omega_0 t)$$
$$y(t)=\sum_{k=-5}^{k=5}H(jk\omega_0)a_k e^{jk\omega_0 t}=\frac{2}{\pi}\cos(\omega_0 t)-\frac{2}{3\pi}\cos(3\omega_0 t)+\frac{2}{5\pi}\cos(5\omega_0 t)$$
$$H\left(jk\omega_0\right)=\begin{cases}0, & k=0\\ 1, & k=\pm1,\pm3,\pm5\end{cases}$$

This filter passes all frequencies except $\omega = 0$, therefore it acts as a highpass filter or a bandpass filter. Filter 1 is a highpass filter that can produce this output.

$$H(j\omega)=\begin{bmatrix}0 & |\omega|\le\frac{1}{2}\omega_0\\ 1 & |\omega|>\frac{1}{2}\omega_0\end{bmatrix}$$

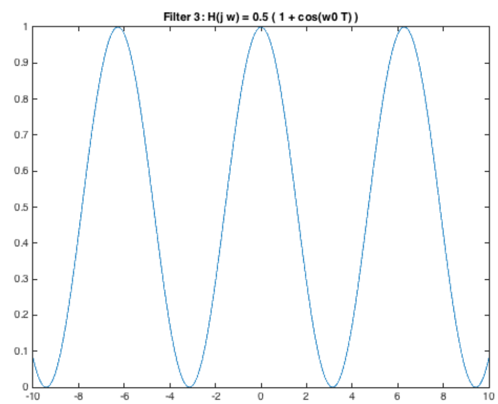
e)

$$y(t)=x(t-\frac{1}{2})$$
$$y(t)=\sum_{k=-5}^{k=5}H(jk\omega_0)a_k e^{jk\omega_0 t}$$
$$H\left(jk\omega_0\right)=e^{-jk\omega_0/2}$$

Filter 2 gives this response, which is allpass.

$$H(j\omega)=e^{-j\omega/2}$$

Filters 3 and 4 cannot produce any of the output signals.



<u>Filter 3:</u>

$$H(j\omega)=\frac{1}{2}\left[1+\cos(\omega T_0)\right]$$

$$for\;\omega=k\omega_0\rightarrow H(jk\omega_0)=\frac{1}{2}\left[1+\cos(k\omega_0 T_0)\right]=\frac{1}{2}\left[1+\cos(2\pi k)\right]=1$$

For $\omega = k\omega_0$, this filter passes all the harmonic frequencies; however, it rejects sub-harmonic frequencies at $\omega = \frac{k\omega_0}{2}$. This filter has a periodic magnitude response as shown above.

<u>Filter 4</u>: This filter removes frequencies above $\frac{3\omega_0}{2}$ and is a lowpass filter.

Epilogue: The LTI ideal delay is a building block in continuous-time systems.

An LTI system with a constant non-zero magnitude response such as $|H(j\omega)| = 1$ passes all frequencies through equally well. This is called an *allpass filter*.

From the phase response, we can determine the group delay in seconds through the LTI system for a particular frequency by taking the derivative of the phase response and negating it. For a phase response of $\angle H(j\omega) = -\frac{\omega}{2}$, the group delay would be $\frac{1}{2}$ seconds, which is the delay in the ideal delay system. See also problem 8.2(b) below.

If we could only observe the ideal delay for time $t \geq 0$, then we would have to set the initial conditions to zero as a necessary condition for the ideal delay to be LTI. Please see Handout U Property of Time-Invariance (Shift-Invariance) for a System Under Observation for an example of an ideal delay under observation for time $t \geq 0$.