Solution Set for Homework #9

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1. Continuous-Time Fourier Transforms.

Compute the continuous-time Fourier transform $X(j\omega)$ for continuous-time signal $x(t)$ using the definition in Signal Processing First in equation (11.1)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

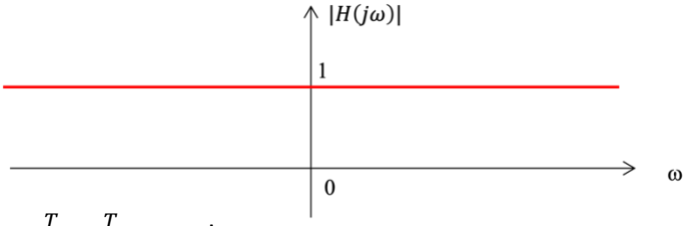
for the following time-domain signals $x(t)$. In addition, for each part, describe the frequency selectivity of the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

Solution:

a) $x(t) = \delta(t)$. We can use the sifting property for the Dirac delta:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big]_{t=0} = 1$$

Allpass filter. All frequencies pass through equally well. See Problem 8.1. Magnitude response plotted on the right.



b) Rectangular pulse of unit amplitude that lasts from $-\frac{T}{2}$ to $\frac{T}{2}$ seconds.

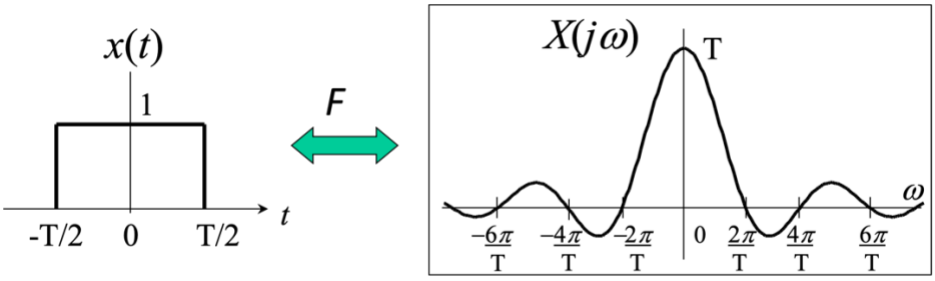
$$x(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)\right] e^{-j\omega t}dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t}\,dt$$

$$X(j\omega) = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{1}{j\omega}\left(e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}}\right) = \frac{2j\sin\left(\frac{T}{2}\omega\right)}{j\omega} = T\frac{\sin\left(\frac{T}{2}\omega\right)}{\frac{T}{2}\omega}$$

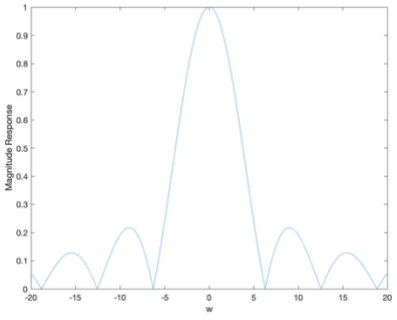
$$X(j\omega) = T\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) \text{ where } \mathrm{sinc(x)} = \frac{\sin(\pi x)}{\pi x}$$

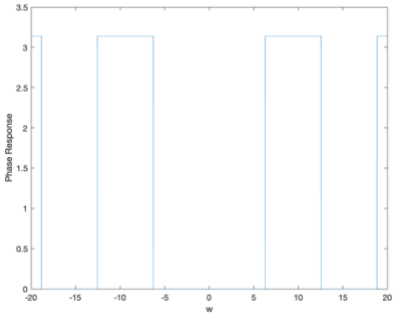
See also *Signal Processing First*, pp. 314-315. ***Lowpass filter.*** Lecture slide 15-6:



Magnitude and phase plots in MATLAB for $T = 1$ for $-20 < \omega < 20$ rad/s. In the magnitude response, the first zero to the right of zero frequency occurs at $\omega = \frac{2\pi}{T} = 2\pi$. The phase response is π whenever the sinc pulse goes negative.

```
T = 1;
w = -20 : 0.01 : 20;
X = T*sinc(w*T/(2*pi));
figure;
plot(w, abs(X));
xlabel('w');
ylabel('Magnitude Response');
figure;
plot(w, angle(X));
xlabel('w');
ylabel('Phase Response');
```



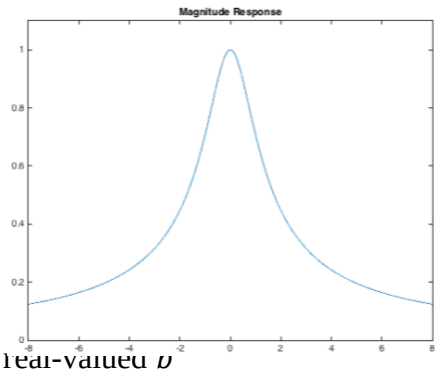


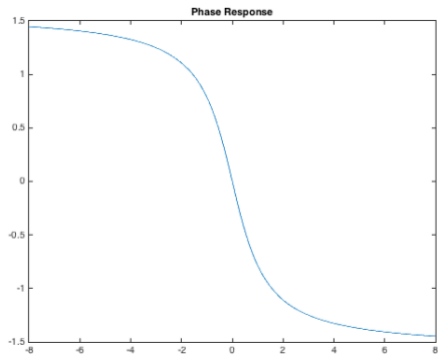
c) $x(t) = e^{-at}u(t)$ for positive and real-valued a

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = \frac{1}{a+j\omega}$$

See *Signal Processing First*, pp. 309 & 313. ***Lowpass filter.*** From Lecture Slide 15-5 for $a = 1$:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 + j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```



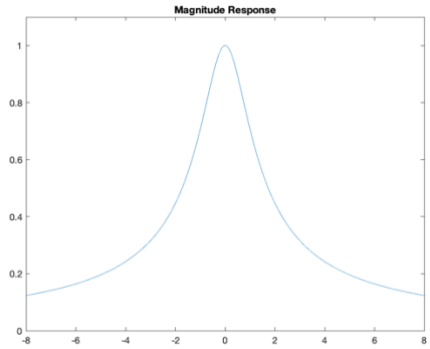


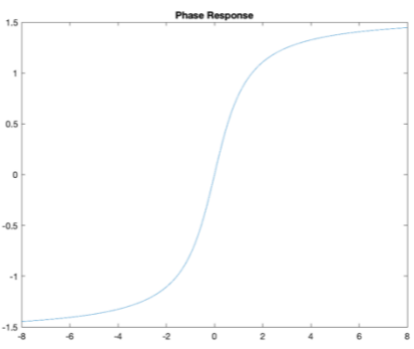
d) $x(t) = e^{bt}u(-t)$ for positive and real-valued b

$$X(j\omega) = \int_{-\infty}^{\infty} e^{bt}u(-t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(b-j\omega)t}dt = \frac{1}{b-j\omega}\left[e^{(b-j\omega)t}\right]_{-\infty}^{0} = \frac{1}{b-j\omega}$$

See *Signal Processing First*, pp. 314. ***Lowpass filter.*** Matlab plots for the magnitude and phase for $b = 1$. Magnitude response is the same as part (c) and phase response is flipped in frequency:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 - j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```



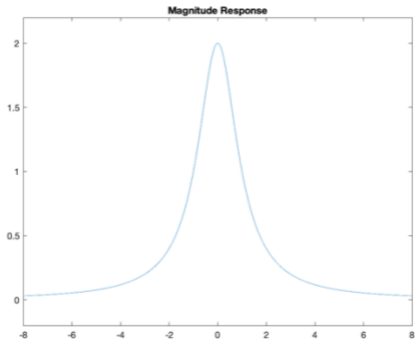


e) $x(t) = e^{-a|t|}$ for $-\infty < t < \infty$ for positive and real-valued a. When $t < 0$, $|t| = -t$. When $t > 0$, $|t| = t$. We can reuse the results from parts (c) and (d).

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{-a(-t)} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{a - j\omega + a + j\omega}{(a + j\omega)(a - j\omega)} = \frac{2a}{a^2 + \omega^2}$$



Lowpass filter. See magnitude response plot on the right for a = 1:

2. Continuous-Time Fourier Transforms Using Transform Properties and Pairs.

Signal Processing First, problem P-11.8, page 343. In the following, the Fourier transform $X(j\,\omega)$ is given. Using the tables of Fourier transforms and Fourier transform properties to determine the inverse Fourier transform for each case. You may give your answer either as an equation or a carefully labeled plot, whichever is most convenient.

(a) $X(j\omega) = \frac{e^{-j3\omega}}{2+j\omega}$

Solution for (a): We can rewrite the Fourier transform as $X(j\omega) = \frac{e^{-j3\omega}}{2+j\omega} = \left(\frac{1}{2+j\omega}\right) e^{-j3\omega}$

The Fourier transform pair related to the first term is

$$e^{-2\,t}\,u(t) \leftrightarrow \frac{1}{2 + j\omega}$$

The second term $e^{-j3\omega}$ relates to the time delay property, where $t_d = 3$:

$$x(t) = e^{-2(t-3)} u(t-3)$$

(b) $X(j\omega) = \frac{j\omega}{2+j\omega}$

Solution for (b) #1: (from a student's solution) We rearrange this expression as follows:

$$X(j\omega) = \frac{j\omega + 2 - 2}{2 + j\omega} = 1 - \frac{2}{2 + j\omega}$$

and then use the Fourier transform pairs

$$e^{-2\,t}\,u(t) \leftrightarrow \frac{1}{2 + j\omega} \quad \text{and} \quad \delta(t) \leftrightarrow 1$$

Using the linearity property of the Fourier transform

$$2\,e^{-2t}\,u(t) \leftrightarrow \frac{2}{2 + j\omega}$$

we obtain

$$x(t) = \delta(t) - 2\,e^{-2\,t}\,u(t)$$

Solution for (b) #2: We rearrange the Fourier transform expression as follows:

$$X(j\omega) = \frac{j\omega}{2 + j\omega} = \left(\frac{1}{2 + j\omega}\right)(j\omega)$$

Using the differentiation-in-time property of the Fourier transform, we obtain

$$x(t) = \frac{d}{dt}\left(2\, e^{-2\,t}\, u(t)\right) = e^{-2\,t}\, \delta(t) - 2\, e^{-2\,t}\, u(t)$$

Solutions for (b) #1 and #2 are identical expressions under integration using the sifting property of the Dirac delta. That is, the value of $e^{-2\,t}$ is 1 when $t = 0$.

(c) $X(j\omega) = \frac{j\omega}{2+j\omega} e^{-j3\omega}$

Solution: We build on the solution in part (b). We rearrange $X(j\omega)$ as

$$X(j\omega) = \frac{j\omega}{2 + j\omega} e^{-j3\omega} = \left(\frac{1}{2 + j\omega}\right)(j\omega)\left(e^{-j3\omega}\right)$$

We take the inverse Fourier transform of $\frac{1}{2+j\omega}$, differentiate the time-domain expression, and delay by 3s. That is, we take the solution in part (b) and delay it by 3s. From solution for (b) #1,

$$x(t) = \delta(t - 3) - 2\, e^{-2\,(t-3)}\, u(t - 3)$$

and from solution for (b) #2

$$x(t) = e^{-2\,(t-3)}\, \delta(t - 3) - 2e^{-2(t-3)} u(t - 3)$$

These are equivalent solutions (see part (b)).

(d) $X(j\omega) = \left(\frac{2\sin(\omega)}{\omega}\right)\left(\sum_{k=-\infty}^{\infty} \frac{\pi}{5} \delta\left(\omega - \frac{2\pi}{10} k\right)\right)$

Solution: Using the convolution-in-time property:

$$x_1(t) * x_2(t) \leftrightarrow X_1(j\omega)\, X_2(j\omega)$$

we have

$$X_1(j\omega) = \frac{2\sin(\omega)}{\omega}$$

$$u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \leftrightarrow \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega}{2}}$$

$$u(t + 1) - u(t - 1) \leftrightarrow \frac{2\sin(\omega)}{\omega}$$

And

$$X_2(j\omega) = \sum_{k=-\infty}^{\infty} \frac{\pi}{5}\, \delta\left(\omega - \frac{2\pi}{10} k\right) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{10}\, \delta\left(\omega - \frac{2\pi}{10} k\right)$$

$$\sum_{n=-\infty}^{\infty} \delta(t-10n) \leftrightarrow \frac{2\pi}{10}\sum_{k} \delta\left(\omega - \frac{2\pi}{10}k\right)$$

Therefore,

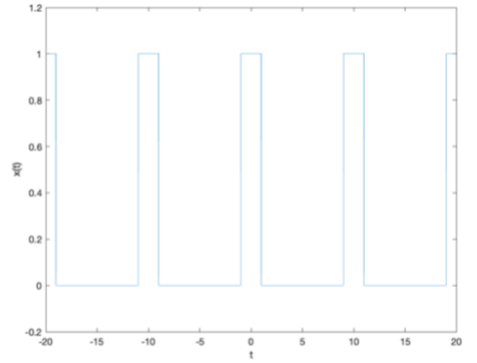
$$x(t) = x_1(t) * x_2(t) = \big(u(t+1) - u(t-1)\big) * \sum_{n=-\infty}^{\infty} \delta(t-10n)$$

The expression $u(t+1) - u(t-1)$ is a rectangular pulse of duration 2s centered at the origin, i.e. rect(t/2). We interchange the summation and convolution due to linearity of convolution. When convolving a signal $g(t)$ with a delayed Dirac delta $\delta(t-t_0)$, we obtain $g(t-t_0)$:

$$x(t) = \text{rect}\left(\frac{t}{2}\right) * \sum_{n=-\infty}^{\infty} \delta(t-10n) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t}{2}\right) * \delta(t-10n) = \sum_{n=-\infty}^{\infty} \text{rect}\left(\frac{t-10n}{2}\right)$$

This signal is a square wave with period 10s. The rectangular pulse in each period lasts for 2s.

```
t = -20 : 0.01 : 20;
x = rectpuls((t+20)/2) + rectpuls((t+10)/2)
+ rectpuls(t/2) + rectpuls((t-10)/2) +
rectpuls((t-20)/2);
plot(t, x);
ylim( [-0.2 1.2] );
xlabel('t');
ylabel('x(t)');
```



3. Transfer Functions in the Laplace Domain

Plot each signal in the time domain for -1 < t < 1, compute the Laplace transform including the region of convergence, and sketch the pole-zero plot and region of convergence for the following signals:

(a) $x(t) = \cos(20\,\pi\,t)\,u(t)$. *6 points*.

(b) $x(t) = e^{-8t}\,u(t)$. *6 points*.

(c) $x(t) = (1 - e^{-8t})\,u(t)$. *6 points*.

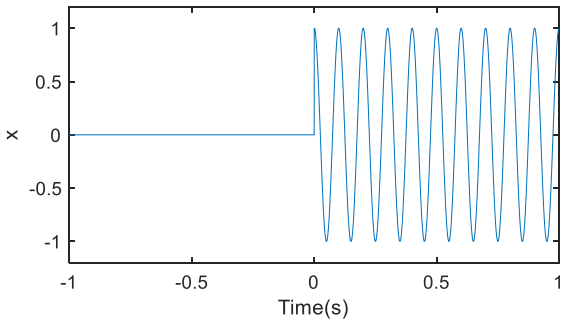
For each part, what is the frequency selectivity— lowpass, highpass, bandpass, bandstop, allpass or notch? *(Given at the end.)*

Solution:

(a) $x(t) = \cos(20\,\pi\,t)\,u(t)$

Plot in the time domain:

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = cos(20*pi*t).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```



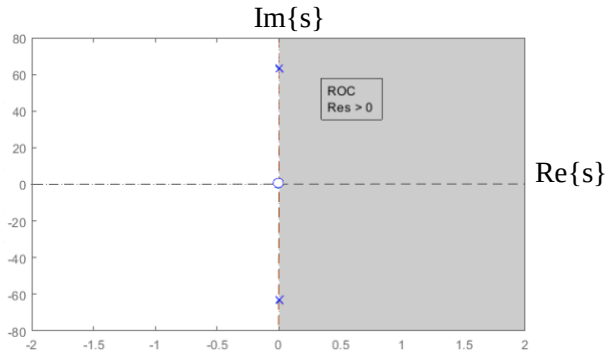
From $L\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2+\omega_0^2}$ for Re{s} >0.

$$X(s) = \frac{s}{s^2+(20\pi)^2} = \frac{s}{s^2+400\pi^2} \text{ for Re\{s\} > 0.}$$

Zeros are roots of numerator. Zero at $s = 0$.

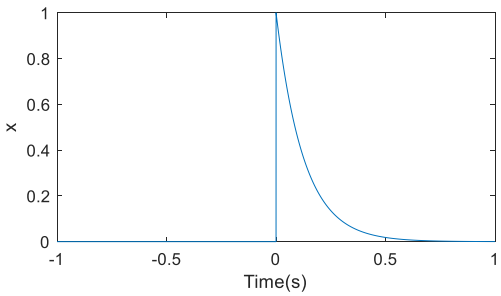
Poles are roots of denominator. Poles at $s = \pm j20\pi$

In the figure legend, Res means Re{s}.



(b) $x(t) = e^{-8t}\,u(t)$

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = exp(-8*t).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```

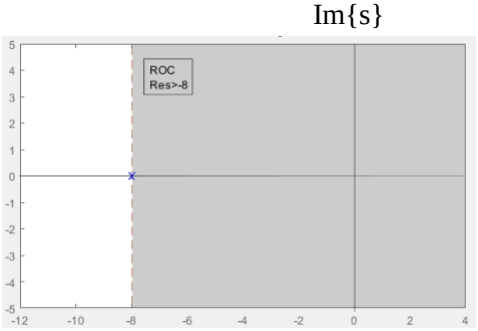


From $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ for Re{s} > -Re{a}.

$$X(s) = \frac{1}{s+8} \text{ for Re\{s\} } > -8$$

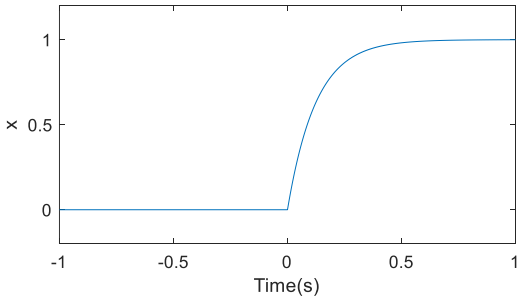
$pole: s+8=0 \Rightarrow s=-8$

In the figure legend, Res means Re{s}.



(c) $x(t) = (1-e^{-8t})\,u(t)$

```
t = -1:1/10000:1;
unitstep = zeros(size(t));
unitstep (t>= 0) = 1;
x = (1-exp(-8*t)).*unitstep;
plot(t,x)
xlabel('Time(s)')
ylabel('x')
```



$x(t) = (1-e^{-8t})u(t) = u(t) - e^{-8t}u(t)$

Im{s}

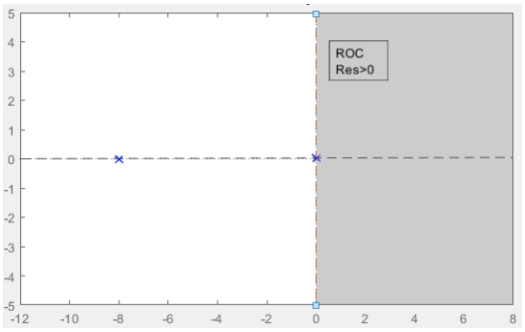
From $L\left\{e^{-at}u(t)\right\} = \frac{1}{s+a}$ for Re{s} > -Re{a}.

$$X(s) = \frac{1}{s} - \frac{1}{s+8} = \frac{8}{s\,(s+8)} \text{ for } Re\{s\} > 0$$

The region of convergence is the intersection of $Re\{s\} > 0$ and $Re\{s\} > -8$.

$poles: s(s+8) = 0 \Rightarrow s_1 = 0, s_2 = -8$

In the figure legend, Res means Re{s}.



Frequency Domain Representation

We have two ways to find the frequency-domain representation of a time-domain signal:

- convert the Laplace transform to the frequency domain by substituting $s = j\omega$ if the substitution is valid; i.e., the region of convergence for the Laplace transform includes the imaginary axis, or
- compute the continuous-time Fourier transform of the signal

Frequency Selectivity vs. Frequency Content.

- By "frequency selectivity," we seek to interpret the magnitude of the frequency response of an LTI system whose impulse response is $x(t)$. That is, we're trying to characterize the shape of the magnitude response as lowpass, highpass, bandpass, bandstop, allpass or notch, which indicates how input frequencies are amplified, passed or attenuated to the output.
- By "frequency content", we seek to characterize the frequency content in the signal $x(t)$.

Approach #1: Frequency Selectivity

The characterization into lowpass, highpass, bandpass, bandstop, allpass or notch is meaningful when the magnitude response is bounded for all frequencies; otherwise, if the input signal contains the frequency at which the LTI system has an unbounded response, then the output signal will have that frequency in infinite strength, which will make all the frequencies passed with bounded amplitude negligible. Once there is an infinite strength frequency component in the output signal, the bounded magnitude response for other frequencies won't matter.

A causal continuous-time LTI system is bounded-input bounded-output (BIBO) stable if the poles are in the left-hand side of the Laplace plane. An alternate condition for BIBO stability is when the region of convergence includes the imaginary axis (i.e. the $j\omega$ axis).

(a) The impulse response would correspond to an LTI system that is BIBO unstable because the region of convergence $Re\{s\} > 0$ does not include the imaginary axis (i.e. the $j\omega$ axis). The frequency responses is not bounded. **Selectivity not determined**.

(b) The frequency response can be obtained from the Laplace transform: $X(j\omega) = \frac{1}{j\omega+8}$. The magnitude response peaks at $\omega = 0$ and decreases by $\frac{1}{\omega}$ as $\omega \to \infty$. **Lowpass filter.**

(c) The impulse response would correspond to an LTI system that is BIBO unstable because the region of convergence $Re\{s\} > 0$ does not include the imaginary axis (i.e. the $j\omega$ axis). The frequency responses is not bounded. **Selectivity not determined.**

Approach #2: Frequency Content

(a) We take the Fourier transform of $x(t) = \cos(\omega_0\, t)\, u(t)$ where $\omega_0 = 20\pi$. We can use the fact that multiplication in the time domain is convolution in the frequency domain:

$$x_1(t)x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

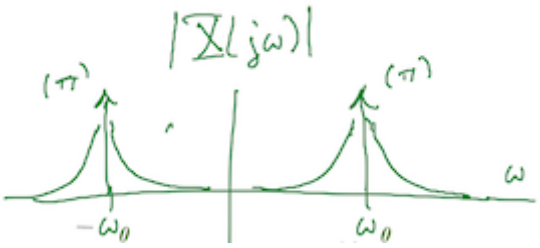
$$X_1(\omega) = \pi\, \delta(\omega + \omega_0) + \pi\, \delta(\omega - \omega_0)$$

$$X_2(\omega) = \pi\, \delta(\omega) + \frac{1}{j\omega}$$

$$\frac{1}{2\pi} X_1(\omega) * X_2(\omega) = \frac{1}{2\pi}\left(\pi\, \delta(\omega + \omega_0) + \pi\, \delta(\omega - \omega_0)\right) * \left(\pi\, \delta(\omega) + \frac{1}{j\omega}\right)$$

Convolving with Dirac delta: $\delta(\omega) * G(\omega) = G(\omega)$ and $\delta(\omega - \omega_0) * G(\omega) = G(\omega - \omega_0)$

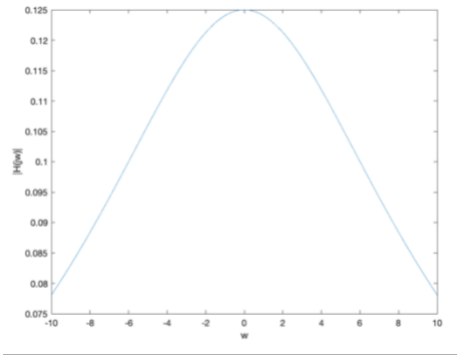
$$\frac{1}{2\pi} X_1(\omega) * X_2(\omega) = \frac{\pi}{2}\, \delta(\omega + \omega_0) + \frac{\pi}{2}\delta(\omega - \omega_0) + \frac{1}{j(\omega + \omega_0)} + \frac{1}{j(\omega - \omega_0)}$$



Infinite frequency components at $-\omega_0$ and ω_0. **Bandpass shape.**

(b) The frequency response can be obtained from the Laplace transform: $X(j\omega) = \frac{1}{j\omega + 8}$. The magnitude response peaks at $\omega = 0$ and decreases by $\frac{1}{\omega}$ as $\omega \to \infty$. **Lowpass shape.**

```
w = -10 : 0.01 : 10;
H = 1 ./ (j*w + 8);
plot(w, abs(H));
xlabel('w');
ylabel('|H(jw)|');
```

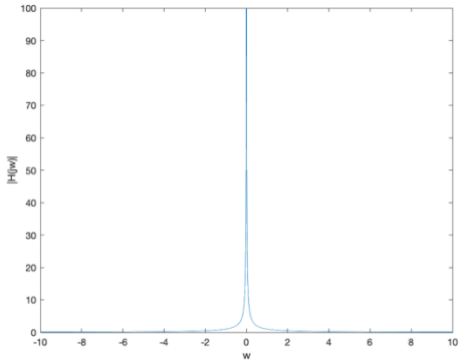


(c) We take the Fourier transform of $x(t) = (1 - e^{-8t})\, u(t) = u(t) - e^{-8t}\, u(t)$.

$$X(j\omega) = \left(\pi\, \delta(\omega) + \frac{1}{j\omega}\right) - \left(\frac{1}{j\omega + 8}\right)$$

Here's the magnitude plot of the terms $\frac{1}{j\omega} - \left(\frac{1}{j\omega+8}\right)$:

```
w = -10 : 0.01 : 10;
H = (1 ./ (j*w)) - (1 ./ (j*w + 8));
plot(w, abs(H));
xlabel('w');
ylabel('|H(jw)|');
```



Lowpass shape.

4. Transfer Function in the Laplace Domain (*not asked*)

A continuous-time system with input $x(t)$ and output $y(t)$ is described by the following linear constant coefficient differential equation for $t > 0^-$:

$$\frac{d}{dt}y(t) + 2y(t) = \frac{d}{dt}x(t)$$

Initial conditions are set to zero, i.e. $y(0^-) = 0$ and $x(0^-) = 0$, so the system will have linearity and time-invariant properties. (Notice the two different uses of "linear". In the case of linear constant different equation, "linear" refers to "affine" which is a line that does not necessarily go through the origin.)

(a) What is the transfer function $H(s)$ of the system in the Laplace domain including the region of convergence? *9 points.*

(b) What is the impulse response $h(t)$ of the system? *9 points.*

(c) Find the frequency response $H(j\omega)$ of the system from the transfer function. Why is the substitution $s = j\omega$ valid? *9 points.*

(d) From part (c), plot the magnitude response. What is the frequency selectivity— lowpass, highpass, bandpass, bandstop, allpass or notch? *9 points.*

(e) For $x(t) = u(t)$, find $X(s)$ and $Y(s)$. *9 points.*

(f) From part (e), find $y(t)$ by taking the inverse Laplace transform of $Y(s)$. *10 points.*

Solution:

Using the property: $L\left\{\frac{d}{dt}x(t)\right\} = sL\{x(t)\}$ for zero initial conditions, we get:

(a)

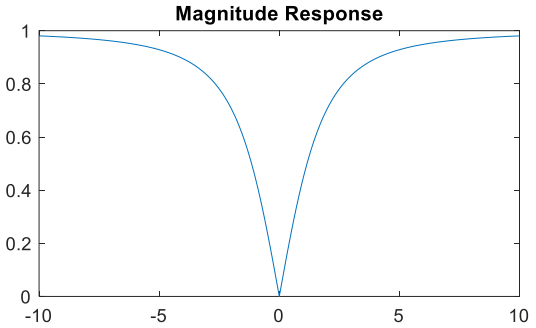
$$sY(s) + 2Y(s) = sX(s) => Y(s)(s+2) = sX(s) => H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s+2}$$

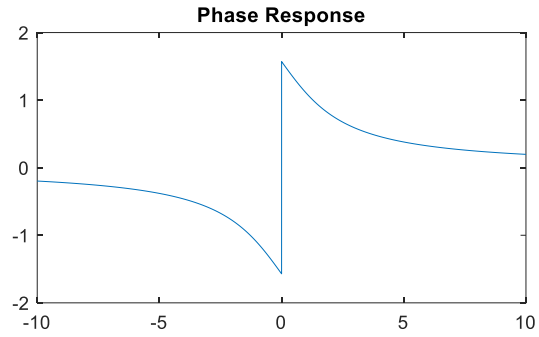
Because the system is causal, the region of convergence is Re{s} > -2.

(b) Using the Laplace transform pair $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ for Re{s} > -Re{a}, $L(\delta(t)) = 1$ for all s, we obtain

$$H(s)=\frac{s}{s+2}=\frac{s+2-2}{s+2}=1-\frac{2}{s+2}$$
$$h(t)=\delta(t)-2e^{-2t}u(t)$$

(c) $H(j\omega)=\frac{j\omega}{j\omega+2}$ by substituting $s=j\omega$ into $H(s)$ above. This substitution is valid because the imaginary axis lies within the region of convergence of Re$\{s\}$ > -2.





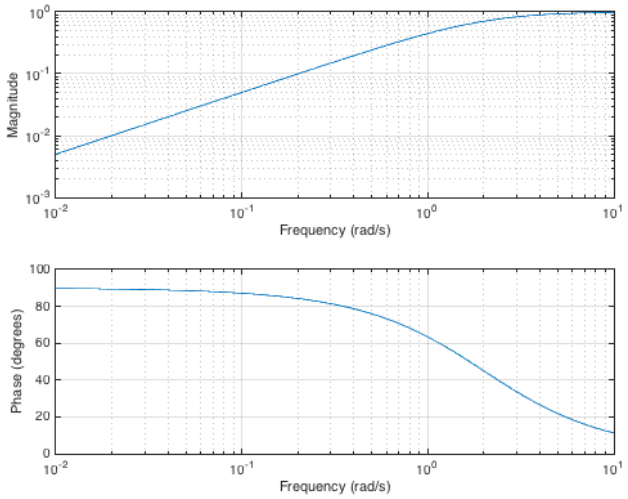
(d)

```
w = -10:1/10000:10;
H= j*w./(j*w+2);
Hmag=abs(H) ;
Hphase=angle(H);
plot(w,Hmag)
title('Magnitude Response');
figure
plot(w,Hphase)
title('Phase Response');
```

According to magnitude response, the filter notches out zero frequency. Hence, it is a notch filter. It could also be called a highpass filter, but a DC notch filter would be more descriptive and a better answer.

Here's the plot of the magnitude and phase using the `freqs` command in Matlab, which will plot the frequency responses on a log scale in frequency. The magnitude will also be on a log scale.

```
freqs( [1 0], [1 2] );
```



We see a highpass response over the frequencies plotted.

Please note `freqs( [1], [1 2] )` would mean $\frac{1}{s+2}$ for the transfer function instead of $\frac{s}{s+2}$.

(e) $x(t) = u(t) => X(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{1}{s}$ for Re{s} > 0.

We could have also obtained the transform by using $L\{e^{-at}u(t)\} = \frac{1}{s+a}$ and substituting a = 0.

$$Y(s) = H(s)X(s) = \frac{1}{s+2}$$

(f) Using the Laplace transform pair $L\{e^{-at}u(t)\} = \frac{1}{s+a}$, for we get $y(t) = e^{-2t}u(t)$