Solution Set for Homework #1 *Version 1.0*

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PROBLEM 1

Prologue: This problem helps you to identify the points of interest in a sinusoidal signal and calculate the parameters of the waveform based on your observations. It relies on the definitions given in Sec. 2-3 of the *Signal Processing First* textbook.

Problem: Given a plot of a sinusoidal wave, determine the values for the amplitude (A), phase (φ) and frequency (ω_0) needed in the representation $x(t) = A\cos(\omega_0\, t + \varphi)$ where $\omega_0 = 2\,\pi\, f_0$.

Amplitude: To calculate A, consider the peak-to-valley amplitude (X_{pv}) of the waveform which is the difference between the maximum and minimum value:

$$X_{pp} = \text{Maximum} - \text{Minimum} = 6 - (-6) = 12. \text{ So, } \underline{A = X_{pv}/2 = 6}.$$

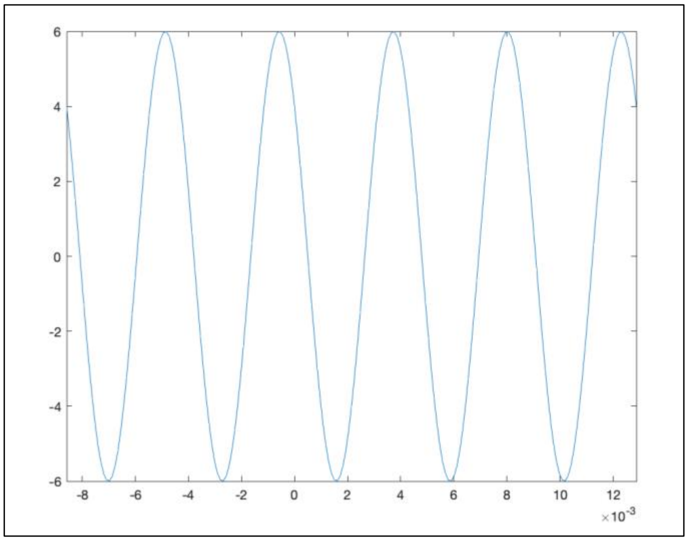
Phase: Estimate the phase φ. Set $t = 0$ and estimate the amplitude of the plot at $t = 0$:

$$x(0) = 6\cos(\varphi) = 4$$

So, $\varphi = \arccos\left(\frac{4}{6}\right) = 0.8411 = 0.2677\,\pi$.

Frequency: To estimate the frequency f_0, we first estimate the period and then invert the period. The plot shows five sinusoidal periods over a duration of approximately 21.5 ms, which is 4.3 ms per period. The associated frequency is $f_0 = \frac{1}{4.3\ ms} \approx 233$ Hz or $\omega_0 = 2\pi f_0 = 1461.2$ rad/s.

```
phi = 0.2677*pi;
f0 = 233;
T0 = 1/f0;
fs = 40*f0;
Ts = 1/fs;
tmin = -2/f0;
tmax = 3/f0;
t = tmin : Ts : tmax;
x = 6*cos(2*pi*f0*t + phi);
plot(t, x);
xlim( [tmin tmax] );
```



True values: $A = 6$, $f_0 = 233\ Hz$, $\varphi = 0.25\pi$

Epilogue: The reference points were chosen due to convenience, and the problem could have been solved by taking other points.

PROBLEM 2

Prologue: This problem asks you to calculate the summation of two sinusoids having the same amplitude and frequency, but different phases. By using the phasor addition rule, the answer can be derived.

Problem: Define $x(t) = 240\cos(\omega_0 t) + 240\cos(\omega_0 t + 60^o)$

Part (a). Express $x(t)$ in the form $x(t) = A\cos(\omega_0 t + \theta)$.

$$x(t) = 240\cos(\omega_0 t) + 240\cos(\omega_0 t + 60^\circ) = 240\cos(\omega_0 t) + 240\cos(\omega_0 t + \pi/3)$$

Using the phasor addition rule from section 2-6.2 of the textbook:

$$Re\left\{240\, e^{j\omega_0 t} + 240\, e^{j\left(\omega_0 t + \frac{\pi}{3}\right)}\right\} = Re\left\{240\, e^{j\omega_0 t} + 240\, e^{j\omega_0 t} e^{j\frac{\pi}{3}}\right\}$$

$$= Re\left\{240\left(1 + e^{j\frac{\pi}{3}}\right) e^{j\omega_0 t}\right\} = Re\{A\, e^{j\phi}\, e^{j\omega_0 t}\}$$

Using Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$240\, e^{j\frac{\pi}{3}} = 240\cos\left(\frac{\pi}{3}\right) + j\,240\sin\left(\frac{\pi}{3}\right) = 240\left(\frac{1}{2}\right) + j\,240\left(\frac{\sqrt{3}}{2}\right)$$

we obtain

$$A\, e^{j\phi} = 240\left(1 + e^{j\frac{\pi}{3}}\right) = 240 + 240\left(\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right)\right)$$

$$A\, e^{j\phi} = 240 + 240\left(\frac{1}{2}\right) + j\,240\left(\frac{\sqrt{3}}{2}\right) = 360 + j\,207.8461$$

So, $A = 415.69$ and $\phi = 0.5236$ rad

$$x(t) = Re\left\{415.69\, e^{j(\omega_0 t + 0.5236)}\right\} = 415.69\cos(\omega_0 t + 0.5236)$$

See the next page for the plot.

Part (b) Assume that $\omega_0 = 100\pi$ rad/s, which is 50 Hz, a common powerline main AC frequency worldwide. At a 50 Hz wall outlet, the voltage is commonly 240 V, which is the value of A here. Plot $x(t)$ over the range $-0.05s \le t \le 0.05s$ in MATLAB. Please write and submit your MATLAB code.

<u>MATLAB Code:</u>

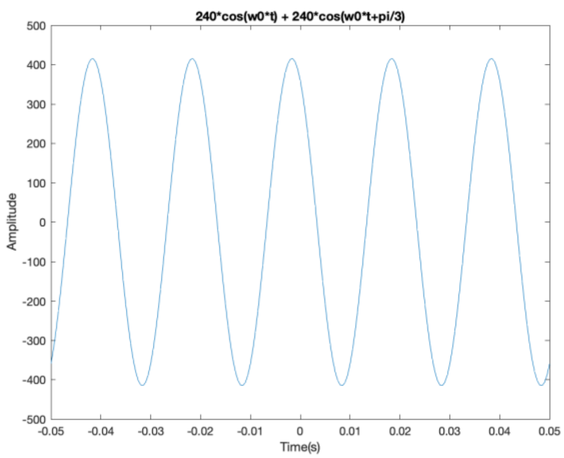
```
f0 = 50;                % Sinusoidal frequency in Hz
w0 = 2*pi*f0;
fs = 40*f0;              % Sampling frequency in Hz
Ts = 1/fs;               % Sampling time in seconds
```

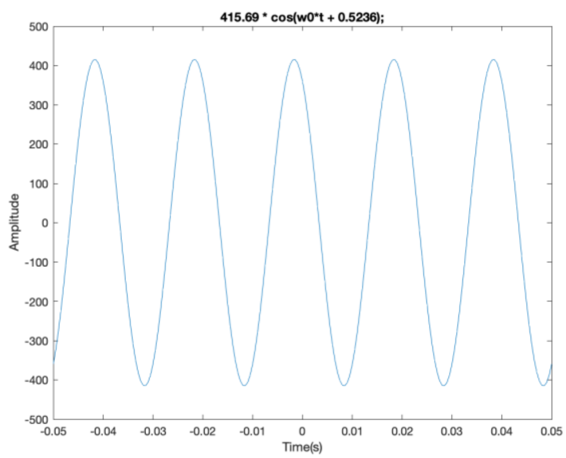
```
t = -0.05 : Ts : 0.05;
x = 240*cos(w0*t) + 240*cos(w0*t+pi/3);
plot(t,x);
xlabel('Time(s)');
ylabel('Amplitude');
title('240*cos(w0*t) + 240*cos(w0*t+pi/3)');
figure;
xParta = 415.69 * cos(w0*t + 0.5236);
plot(t, xParta);
xlabel('Time(s)');
ylabel('Amplitude');
title('415.69 * cos(w0*t + 0.5236);');
```

T_0 is the fundamental period of $x(t)$ where $T_0 = \frac{1}{f_0} = \frac{1}{50\text{ Hz}} = 20\ ms$

The range $-0.05 \leq t \leq 0.05$ covers $T_1 = 0.05 - (-0.05) = 0.1$s of the time, which corresponds to $\frac{T_1}{T_0} = \frac{0.1\ s}{0.02\ s} = 5$ sinusoidal periods in the plot.

The two plots are identical.





Part (c) Find a complex-valued signal $z(t)$ such that $x(t) = Re\{z(t)\}$.

The answer is calculated in part (a):

$$x(t) = Re\left\{415.69\ e^{j(\omega_0 t + 0.5236)}\right\} = 415.69 \cos(\omega_0 t + 0.5236)$$

$$z(t) = 415.69\ e^{j(\omega_0 t + 0.5236)}$$

PROBLEM 3:

Prologue: This problem shows that a time shift leads to a phase shift. This principle is used in ranging systems, such as ultrasound, sonar and radar, that try to localize objects in an environment. In these applications, the ranging system transmits a signal and goes stops transmitting to listen for the return signal. The signal will bounce off an object in the environment and return to the ranging system. The ranging system then computes the time delay between the transmitted and received signals t_{RTT} (roundtrip time) to estimate the distance to the object d by using the speed of propagation in the environment c via $d = \frac{1}{2}\, c\, t_{RTT}$. The ranging system can compute the time delay in the time domain or the phase delay in the frequency domain. A common time domain method uses correlation, which is covered in ECE 351K Probability and can be computed using convolution. We'll see convolution after midterm #1. A common frequency domain method uses Fourier transforms, which we'll see several times over the semester.

Problem: Here's the connection between time shift and phase shift:

$$x(t) = A\cos(2\pi f_0 t)$$
$$x(t - t_1) = A\cos\big(2\pi f_0 (t - t_1)\big) = A\cos(2\pi f_0 t - 2\pi f_0 t_1) = A\cos(2\pi f_0 t + \phi)$$

The phase shift is $\phi = -2\pi f_0 t_1$ corresponding to a time shift of t_1.

In the following parts, assume that the period of the sinusoidal wave is $T_0 = 8\ s$.

Part (a). When $t_1 = -2\ s$, the value of the phase is $\phi = \pi/2$. Explain whether it is True or False.

For $t_1 = -2\ s$, the phase shift is

$$\phi = -\frac{2\pi(-2)}{8} = \frac{\pi}{2}$$

True

Part (b) When $t_1 = 3\ s$, the value of the phase is $\phi = 3\pi/4$. Explain whether it is True or False.

For $t_1 = 3\ s$, the phase shift is

$$\phi = -\frac{2\pi(3)}{8} = -\frac{3\pi}{4}$$

False

Part (c) When $t_1 = 7\ s$, the value of the phase is $\phi = \pi/4$. Explain whether it is True or False.

For $t_1 = 7\ s$, the phase shift is

$$\phi = -\frac{2\pi(7)}{8} = -\frac{7\pi}{4}$$

Due to property $\cos(x + 2\pi) = \cos(x)$. Each multiple of 2π corresponds to picking a different peak.

$$\phi = -\frac{7\pi}{4} + 2\pi = \frac{\pi}{4}$$

True

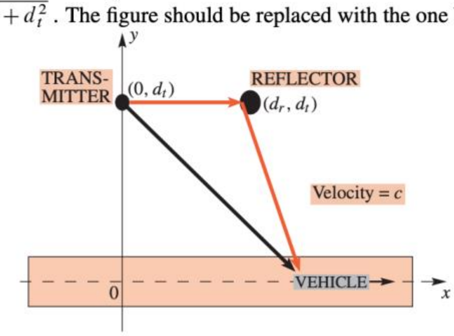
PROBLEM 4

Prologue: This problem shows how wireless reception can vary with the location of the receiving equipment (e.g. phone). In some locations, the receiving equipment will experience very weak reception, and strong reception in other locations. For the basic model of one reflector in this problem, the received amplitude strength is periodic with location on the horizontal axis. A takeaway is that moving the receiving equipment, e.g. walking while videostreaming on a cell phone, will provide a better connection on average than being stationary. When stationary, the phone could be experiencing little or no signal strength. When experiencing low data rates in Wi-Fi, it can help to move, esp. to have a line-of-sight to the access point and be closer to it.

Problem: In a mobile radio system (e.g. cell phones) there is one type of degradation that can be modeled easily with sinusitis. This is the case of *multipath fading* caused by reflections of the radio waves interfering destructively at some locations. Suppose a transmitting tower sends a sinusoidal signal, and mobile user receives not one but two copies of the transmitted signal: a direct-path transmission and a reflected-path signal (e.g., from a large building) as depicted below.

There's an error in the textbook figure. Here's the correct figure from the book errata:

3. page 34*, Figure 2-21, The diagram of the original figure does not correspond to the equations given in the problem. The general formula for the distance off the reflector, d_2, is $d_2 = d_r + \sqrt{(x-d_r)^2 + d_t^2}$. The figure should be replaced with the one below:



The received signals the sum of the two copies, and since they travel different distances they have different time delays. If the transmitted signal is $s(t)$, then the received signal is

$$r(t) = s(t - t_1) + s(t - t_2)$$

For simplicity, we were ignoring propagation losses. In a mobile phone scenario the distance between the mobile user and the transmission tower is always changing. Suppose that the direct path distance (in meters) is

$$d_1 = \sqrt{x^2 + d_t^2} = \sqrt{x^2 + 10^6}$$

where *x* is the position of a mobile user who is moving along x axis. Assume that the reflected-pat distance (in meters) is

$$d_2 = \sqrt{(x - d_r)^2 + d_t^2} + d_r = \sqrt{(x - 55)^2 + 10^6} + 55$$

Part (a) The amount of delay in seconds can be computed for both propagation paths, using the fact that the time tonight is the distance divided by the speed of light (3×10^8 m/s). Determine t_1 and t_2 as a function of the mobile's position.

$$t_1 = \frac{d_1}{c};\ d_1 = \sqrt{x^2 + d_t^2} = \sqrt{x^2 + 10^6}\ \text{m and } c = 3 \times 10^8\ \text{m/s}$$

$$t_1 = \frac{\sqrt{x^2 + d_t^2}}{c} = \frac{\sqrt{x^2 + 10^6}}{3 \times 10^8}$$

$$t_2 = \frac{d_2}{c}; d_2 = \sqrt{(x - d_r)^2 + d_t^2} + d_r = \sqrt{(x - 55)^2 + 10^6} + 55$$

$$t_2 = \frac{\sqrt{(x - d_r)^2 + d_t^2} + d_r}{c} = \frac{\sqrt{(x - 55)^2 + 10^6} + 55}{3 \times 10^8}$$

Part (b) Assume that the transmitted signal is $s(t) = \cos\,(\omega_0\, t)$ where $\omega_0 = 300 \times 10^6\, \pi$. Determine the received signal when $x = 0$. Prove that the received signal is a sinusoidal and find its amplitude, phase, and frequency when $x = 0$.

$$\begin{aligned}
t_1 &= \frac{\sqrt{10^6}}{3 \times 10^8} = 3.3333 \times 10^{-6}\ \text{s} \\
t_2 &= \frac{\sqrt{(-55)^2 + 10^6} + 55}{3 \times 10^8} = 3.5217 \times 10^{-6}\ \text{s} \\
s(t) &= \cos\,(300 \times 10^6 \pi t) \\
r(t) &= s(t - t_1) + s(t - t_2) \\
&= \cos\,\left(300 \times 10^6 \pi (t - 3.3333 \times 10^{-6})\right) + \cos\,\left(300 \times 10^6 \pi (t - 3.5217 \times 10^{-6})\right) \\
&= \cos\,(300 \times 10^6 \pi t - 1000\pi) + \cos\,(300 \times 10^6 \pi t - 1056.511356\pi)
\end{aligned}$$

Using the phasor addition rule:

$$\begin{aligned}
r(t) &= Re\left\{e^{j\left(300 \times 10^6 \pi t\right)} e^{j(-1000\pi)} + e^{j\left(300 \times 10^6 \pi t\right)} e^{j(-1056.511356\pi)}\right\} \\
&= Re\left\{e^{j\left(300 \times 10^6 \pi t\right)} \left[e^{j(-1000\pi)} + e^{j(-1056.511356\pi)}\right]\right\} \\
&= Re\left\{A\, e^{j\phi}\, e^{j\omega_0 t}\right\}
\end{aligned}$$

We convert $e^{j(-1000\pi)} + e^{j(-1056.511356\pi)}$ to polar form $A\, e^{j\phi}$:

```
w0 = (3*10^8)*pi;
dt = 10^3;
dr = 55;
c = 3*10^8;
t1 = dt / c;
t2 = (dr + sqrt(dt^2 + dr^2)) / c;
z = exp(-j*w0*t1) + exp(-j*w0*t2);
A = abs(z)
phi = angle(z)
```

which gives $A = 1.389$ and $\phi = -0.80324 = -0.2557\pi$. (Note that the Fall 2021 solutions had $A = 1.414$ and $\phi = -0.25\pi$ because -1056.511356π had been rounded to -1056.5π.)

$$r(t) = Re\left\{A\, e^{j\phi}\, e^{j\omega_0 t}\right\} = A \cos\,(\omega_0\, t\ +\ \phi)$$

Part (c) The amplitude of the received signal is a measure of its strength. Shows it as a mobile user moves, it is possible to find positions where the signal strength is zero. Find one such location.

$r(t) = s(t - t_1) + s(t - t_2)$
$= \cos\left(300 \times 10^6 \pi (t - t_1)\right) + \cos\left(300 \times 10^6 \pi (t - t_2)\right)$
A
$= \sqrt{[\cos(300 \times 10^6 \pi t_1) + \cos(300 \times 10^6 \pi t_2)]^2 + [(\sin(300 \times 10^6 \pi t_1) + \sin(300 \times 10^6 \pi t_2)]^2}$
when signal strength $= 0 \rightarrow A = 0$

$\omega_0 = 300 \times 10^6 \pi$

$[\cos(\omega_0 t_1) + \cos(\omega_0 t_2)]^2 + [\sin(\omega_0 t_1) + \sin(\omega_0 t_2)]^2 = 0$

$$\cos^2(\omega_0 t_1) + \cos^2(\omega_0 t_2) + 2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sin^2(\omega_0 t_1) + \sin^2(\omega_0 t_2) + 2\sin(\omega_0 t_1)\sin(\omega_0 t_2) = 0$$

$\cos^2(x) + \sin^2(x) = 1$

$2 + 2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + 2\sin(\omega_0 t_1)\sin(\omega_0 t_2) = 0$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$2 + 2\cos\left(\omega_0 (t_1 - t_2)\right) = 0.$
$\cos\left(300 \times 10^6 \pi (t_1 - t_2)\right) = -1$

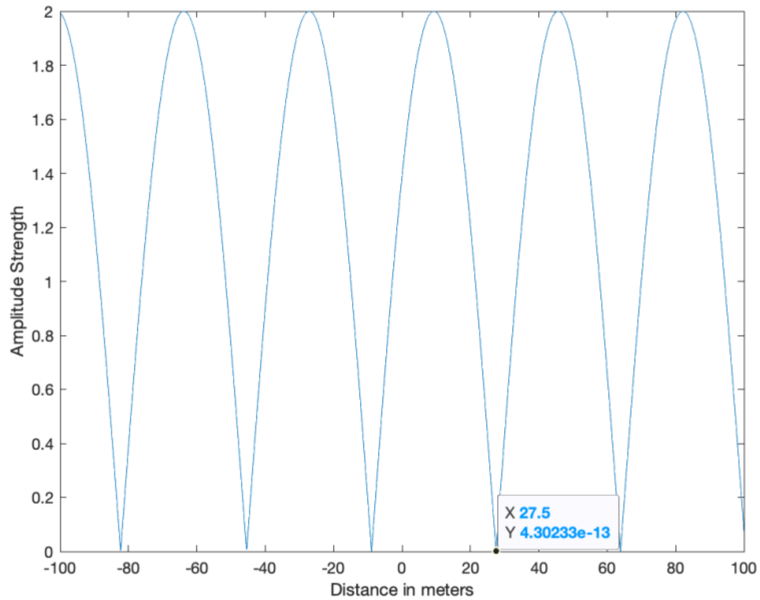
since the denominator of t_1 and t_2 is equal to 3×10^8

$$\cos\left(\pi\sqrt{x^2 + 10^6} - \pi\sqrt{(x - 55)^2 + 10^6} + 55\pi\right) = -1$$
$$\cos\left(\pi\sqrt{x^2 + 10^6} - \pi\sqrt{(x - 55)^2 + 10^6}\right) = 1$$
$$x^2 + 10^6 = (x - 55)^2 + y^6$$
$$x^2 = x^2 + 55^2 - 110x \rightarrow 110x = 55^2 \rightarrow x = 27.5 \text{ m}$$

Part (d) If you have access to MATLAB write a script that will plot signal strength versus position *x*, thus demonstrating that there are numerous locations where no signal is received. Use *x* in the range of $-100 \le x \le 100$.

```
x = -100 : 0.1 : 100;
c = 3e8;                % speed of light in m/s
dr = 55;
dt = 1e6;
w0 = 300*10^6*pi;       % carrier frequency (150 MHz)
t1 = sqrt(x.*x+dt)/c;
t2 = (sqrt((x.*x-2*dr*x+dr^2)+dt)+dr)/c;  % expanded sqrt((x-dr)^2 + dt) term
```

```
s1 = cos(w0*t1) + cos(w0*t2);
s2 = sin(w0*t1) + sin(w0*t2);
a = sqrt(s1.*s1 + s2.*s2);
plot(x, a);
xlabel('Distance in meters');
ylabel('Amplitude Strength');
```



In the Matlab plot, we've used the Data Cursor tool (under the Tools menu in the Matlab plot window) to show the value of *x* (distance) at 27.5 m where the value of *y* (Amplitude Strength) is zero. Please note that 4.3×10^{-13} is essentially zero. This number is not equal to zero due to the numerical accuracy in double-precision floating-point calculations.

References

James H. McClellan, Ronald W. Schafer & Mark A. Yoder, *Signal Processing First*, Prentice-Hall, ISBN 978-0130909992, 2003. Errata. On-line Companion.

Brian L. Evans and Firas Tabbara, "EE 313 Homework #1 Solution Set", The University of Texas at Austin, Fall 2021.