Solution Set for Homework #6

By Prof. Brian Evans and Mr. Firas Tabbara

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PROBLEM 1: FREQUENCY AND STEP RESPONSES

For each of the following linear time-invariant (LTI) systems, determine the impulse response, step response, and frequency response. Plot the magnitude and phase of the frequency response using `freqz`:

a) First-order unnormalized averaging filter (lowpass filter): $y[n] = x[n] + x[n-1]$ for $n \geq 0$ and the initial condition $x[-1] = 0$ to satisfy LTI properties.

b) First-order difference filter (highpass filter): $y[n] = x[n] - x[n-1]$ for $n \geq 0$ and the initial condition $x[-1] = 0$ to satisfy LTI properties

c) Second-order difference filter (highpass filter): $y[n] = x[n] - 2\,x[n-1] + x[n-2]$ for $n \geq 0$ and the initial condition $x[-1] = 0$ and $x[-2] = 0$ to satisfy LTI properties

Solution for part (a) subpart 1: The impulse response is the output (response) when the input is the discrete-time impulse $\delta[n]$. When $x[n] = \delta[n]$, the output is
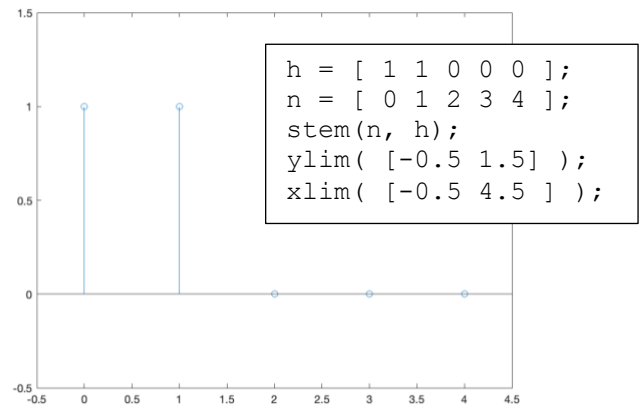
$$h[n] = \delta[n] + \delta[n-1]$$

We compute several impulse response values:

$h[0] = \delta[0] + \delta[-1] = 1 + 0 = 1$
$h[1] = \delta[1] + \delta[0] = 0 + 1 = 1$
$h[2] = \delta[2] + \delta[1] = 0 + 0 = 0$


```
h = [ 1 1 0 0 0 ];
n = [ 0 1 2 3 4 ];
stem(n, h);
ylim( [-0.5 1.5] );
xlim( [-0.5 4.5 ] );
```
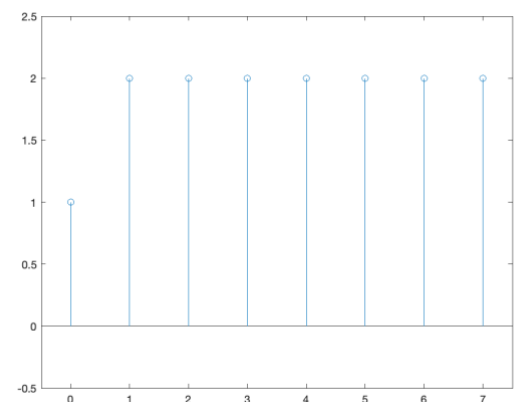
Solution for part (a) subpart 2: The step response $y_{step}[n]$ is the output (response) when the input is the discrete-time step function $u[n]$. The unit step is $u[n] = 1$ for $n \geq 0$ and 0 otherwise. When $x[n] = u[n]$, the output is

$$y_{step}[n] = u[n] + u[n-1]$$

Matlab code:

```
stepsignal = [ 1 1 1 1 1 1 1 1 ];
ystep = filter( [1 1], 1, stepsignal );
n = [ 0 1 2 3 4 5 6 7 ];
figure;
stem(n, ystep);
ylim( [-0.5 2.5] );
xlim( [-0.5 7.5 ] );
```

Solution for part (a) subpart 3: The frequency response of the LTI system is the discrete-time Fourier transform of the impulse response:

$$H_a\left(e^{j\hat{\omega}}\right) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\hat{\omega}n} = \sum_{n=0}^{1} h[n]\, e^{-j\hat{\omega}n} = h[0] + h[1]e^{-j\hat{\omega}} = 1 + e^{-j\hat{\omega}}$$

We can factor the frequency response into magnitude-phase form per lecture slide 9-9 by factoring out the phase shift corresponding to the delay equal to the index of the midpoint of the impulse response, which is at an index of ½, and hence the phase shift is $e^{-j\frac{\hat{\omega}}{2}}$:

$$H_a\left(e^{j\hat{\omega}}\right) = e^{-j\frac{\hat{\omega}}{2}}\left(e^{j\frac{\hat{\omega}}{2}} + e^{-j\frac{\hat{\omega}}{2}}\right) = e^{-j\frac{\hat{\omega}}{2}}\left(2\cos\left(\frac{\hat{\omega}}{2}\right)\right)$$

<u>Magnitude response:</u> $2\cos\left(\frac{\hat{\omega}}{2}\right)$ which is non-negative for $-\pi \le \omega < \pi$.

<u>Phase response:</u> $-\frac{\hat{\omega}}{2}$ which is a line of slope $-\frac{1}{2}$ that passes through the origin.
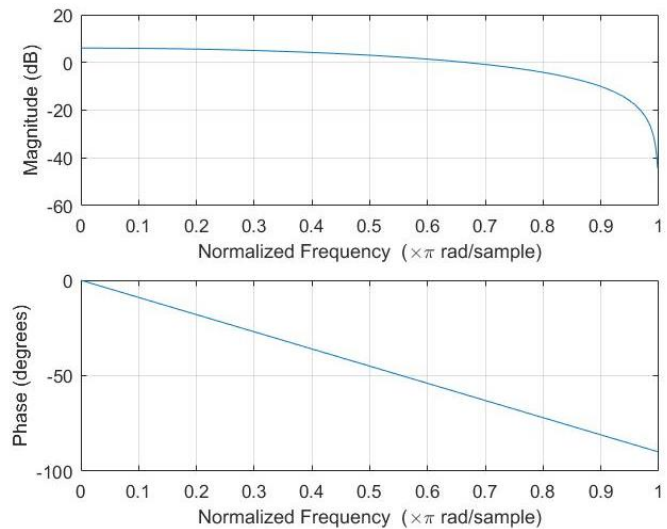
The MATLAB command:

```
freqz( [1 1] );
```

will plot the magnitude and phase response for an LTI system with impulse response

$$h[n] = \delta[n] + \delta[n-1]$$

The magnitude response, by default, will be plotted in decibels vs. discrete-time frequency, where $A_{dB} = 20\log_{10}|A|$ for $0 \le \omega < \pi$, as shown above. It is a lowpass filter.

Solution for part (b) subpart 1: The impulse response is the output (response) when the input is the discrete-time impulse $\delta[n]$. When $x[n] = \delta[n]$, the output is
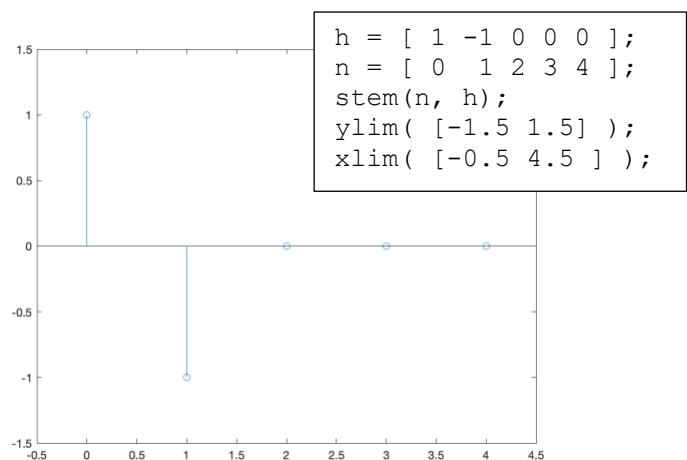
$$h[n] = \delta[n] - \delta[n-1]$$

We compute several impulse response values:

$h[0] = \delta[0] - \delta[-1] = 1 - 0 = 1$
$h[1] = \delta[1] - \delta[0] = 0 - 1 = -1$
$h[2] = \delta[2] - \delta[1] = 0 - 0 = 0$

…


```
h = [ 1 -1 0 0 0 ];
n = [ 0   1 2 3 4 ];
stem(n, h);
ylim( [-1.5 1.5] );
xlim( [-0.5 4.5 ] );
```
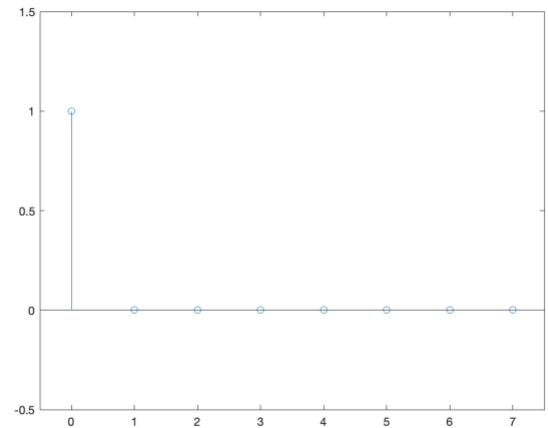
Solution for part (b) subpart 2: The step response $y_{step}[n]$ is the output (response) when the input is the discrete-time step function $u[n]$. When $x[n] = u[n]$, the output is

$$y_{step}[n] = u[n] - u[n-1]$$

Matlab code:

```
stepsignal = [ 1 1 1 1 1 1 1 1 ];
ystep = filter( [1 -1], 1, stepsignal );
n = [ 0 1 2 3 4 5 6 7 ];
figure;
stem(n, ystep);
ylim( [-0.5 1.5] );
xlim( [-0.5 7.5 ] );
```


Solution for part (b) subpart 3: The frequency response of the LTI system is the Fourier transform of the impulse response:

$$H_b\!\left(e^{j\widehat{\omega}}\right) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\widehat{\omega} n} = \sum_{n=0}^{1} h[n]\, e^{-j\widehat{\omega} n} = h[0] - h[1]\, e^{-j\widehat{\omega}} = 1 - e^{-j\widehat{\omega}}$$

We can factor the frequency response into polar form per lecture slide 9-9:

```
freqz( [1 -1] );
```

$$H_b\!\left(e^{j\widehat{\omega}}\right) = e^{-j\frac{\widehat{\omega}}{2}}\left(e^{j\frac{\widehat{\omega}}{2}} - e^{-j\frac{\widehat{\omega}}{2}}\right) = e^{-j\frac{\widehat{\omega}}{2}}\left(2j\,\sin\!\left(\frac{\widehat{\omega}}{2}\right)\right)$$

Using the fact that $j = e^{j\frac{\pi}{2}}$,

$$H_b\!\left(e^{j\widehat{\omega}}\right) = e^{-j\frac{\widehat{\omega}}{2}}\left(2\, e^{j\frac{\pi}{2}} \sin\!\left(\frac{\widehat{\omega}}{2}\right)\right) = 2\,\sin\!\left(\frac{\widehat{\omega}}{2}\right) e^{j\left(\frac{\pi}{2}-\frac{\widehat{\omega}}{2}\right)}$$

<u>Amplitude Response:</u> $2\,\sin\!\left(\frac{\widehat{\omega}}{2}\right)$ is non-positive for $-\pi \le \widehat{\omega} < 0$ (which is not a magnitude) and non-negative for $0 \le \widehat{\omega} < \pi$ (which is a magnitude).

<u>Phase Response:</u> $\frac{\pi}{2} - \frac{\widehat{\omega}}{2}$ which is a line of slope $-\frac{1}{2}$ that intercepts the y-axis at $\frac{\pi}{2}$. See next for more info.

We can put $H_b\!\left(e^{j\widehat{\omega}}\right)$ into magnitude-phase form for $-\pi \le \widehat{\omega} < 0$ by multiplying the amplitude term by -1 and the phase term by $-1 = e^{-j\pi}$:

$$H_b\!\left(e^{j\widehat{\omega}}\right) = \begin{cases} -2\,\sin\!\left(\dfrac{\widehat{\omega}}{2}\right) e^{j\left(-\frac{\pi}{2}-\frac{\widehat{\omega}}{2}\right)} & -\pi \le \omega < 0 \\[2mm] 2\,\sin\!\left(\dfrac{\widehat{\omega}}{2}\right) e^{j\left(\frac{\pi}{2}-\frac{\widehat{\omega}}{2}\right)} & 0 \le \omega < \pi \end{cases}$$
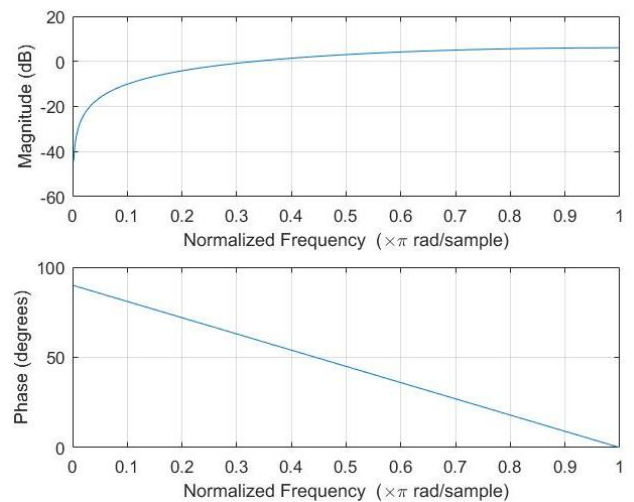
There is a discontinuity in the phase at $\widehat{\omega} = 0$, with a jump in phase of π rad/sample.

A student noticed the similarity between the impulse responses between parts (a) and (b):

$$h_a[n] = (-1)^n\, h_b[n] = \cos(\pi n)\, h_b[n]$$

This is amplitude modulation which shifts the frequency content by π and $-\pi$.

Solution for part (c) subpart 1: The impulse response is the output (response) when the input is the discrete-time impulse $\delta[n]$. When $x[n] = \delta[n]$, the output is

$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

We compute several impulse response values:

$h[0] = \delta[0] - 2\,\delta[-1] + \delta[-2] = 1 - 0 + 0 = 1$
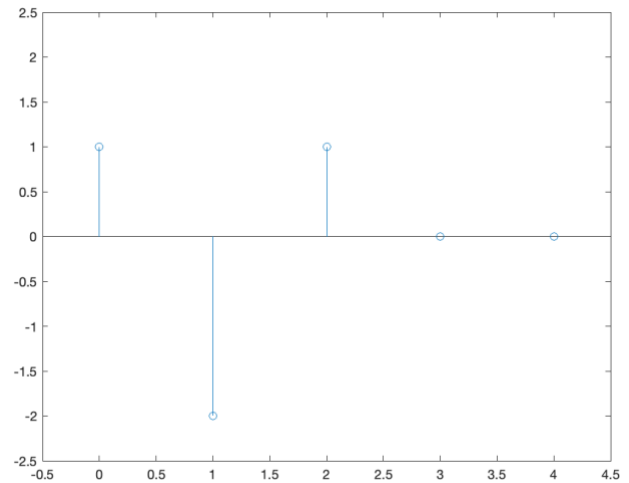$h[1] = \delta[1] - 2\,\delta[0] + \delta[-1] = 0 - 2 + 0 = -2$
$h[2] = \delta[2] - 2\,\delta[1] + \delta[0] = 0 - 0 + 1 = 1$
$h[3] = \delta[3] - 2\,\delta[2] + \delta[1] = 0 - 0 + 0 = 0$

…

MATLAB code for the plot of $h[n]$:

```
h = [ 1 -2 1 0 0 ];
n = [ 0  1 2 3 4 ];
stem(n, h);
ylim( [-2.5 2.5] );
xlim( [-0.5 4.5 ] );
```
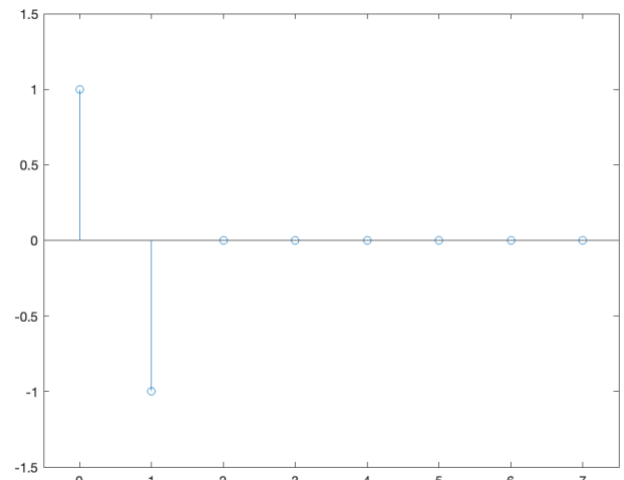
Solution for part (c) subpart 2: The step response $y_{\text{step}}[n]$ is the output (response) when the input is the discrete-time step function $u[n]$. When $x[n] = u[n]$, the output

$$y_{step}[n] = u[n] - 2\,u[n-1] + u[n-2]$$

Matlab code:

```
stepsignal = [ 1 1 1 1 1 1 1 1 ];
ystep = filter( [1 -2 1], 1,
stepsignal );
n = [ 0 1 2 3 4 5 6 7 ];
figure;
stem(n, ystep);
ylim( [-1.5 1.5] );
xlim( [-0.5 7.5 ] );
```
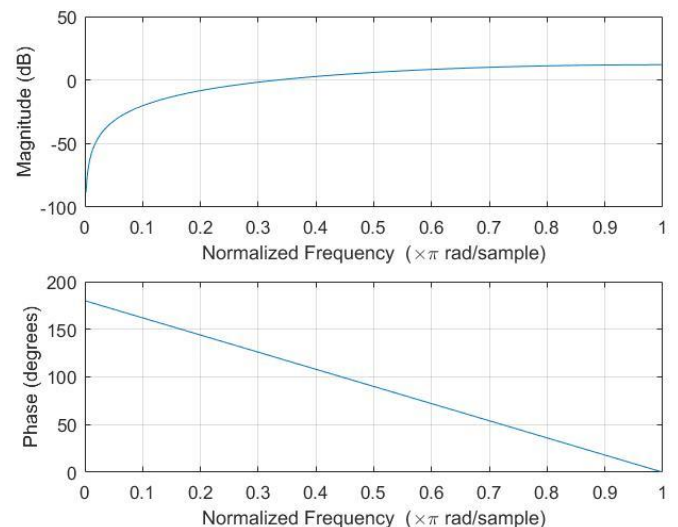
Solution for part (c) subpart 3: The frequency response of the LTI system is the Fourier transform of the impulse response:

$$H_c\left(e^{j\widehat{\omega}}\right) = \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\widehat{\omega}n} = \sum_{n=0}^{2} h[n]\, e^{-j\widehat{\omega}n}$$

$$= h[0] - 2h[1]e^{-j\widehat{\omega}} + h[2]e^{-2j\widehat{\omega}}$$

$$= 1 - 2e^{-j\widehat{\omega}} + e^{-2j\widehat{\omega}}$$

The MATLAB command:

```
freqz( [1 -2 1] );
```

PROBLEM 2: CASCADE OF THREE SYSTEMS

Suppose that three systems are connected in cascade; i.e., the output of S_1 is the input to S_2, and the output of S_2 is the input to S_3. The three systems are specified as follows

$$S_1 : y_1[n] = x_1[n] - x_1[n-1]$$
$$S_2 : y_2[n] = x_2[n] + x_2[n-2]$$
$$S_3 : y_3[n] = x_3[n-1] + x_3[n-2]$$

where the output of S_i is $y_i[n]$ and its input is $x_i[n]$.

(a) Determine the equivalent system that is a single operation from the input $x[n]$ (into S_1) to the output $y[n]$, which is the output of S_3. Thus, $x[n]$ is $x_1[n]$ and $y[n]$ is $y_3[n]$.

(b) Use the frequency response to write one difference equation that defines the overall system in terms of $x[n]$ and $y[n]$ only.

Solution to (a): Here's a block diagram of the cascade of the three systems.
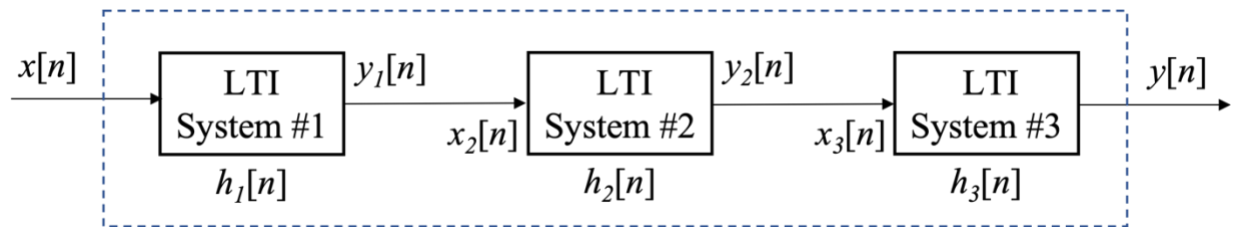
Approach #1 Using Impulse Responses. We find the impulse response of the overall system by letting $x[n] = \delta[n]$. The output of LTI System #1 will be its impulse response $h_1[n]$. The signal $h_1[n]$ is the input to LTI System #2 and the output is $h_1[n] * h_2[n]$ where * denotes convolution. (See Lecture Slides 8-8 and 8-11 and SP First Sections 5-6 and 5-7.) The response $h_1[n] * h_2[n]$ is the input to LTI System #3 and the output is $h_1[n] * h_2[n] * h_3[n]$ The impulse responses are

$$h_1[n] = \delta[n] - \delta[n-1] \quad h_2[n] = \delta[n] + \delta[n-2] \quad h_3[n] = \delta[n-1] + \delta[n-2]$$

We know that the length of the signal that results from the convolution of two finite-length signals is the sum of the lengths of the two finite-length signals. Here, $h_1[n]$ is of length 2 samples and $h_2[n]$ is of length 3 samples, and hence, the result will have 4 samples:

$$h_1[n] * h_2[n] = \sum_{k=0}^{3} h_1[k]\, h_2[n-k] = \sum_{k=0}^{3} (\delta[k] - \delta[k-1])\,(\delta[n-k] + \delta[n-k-2])$$

$$h_1[n] * h_2[n] = \sum_{k=0}^{3} \delta[k]\,\delta[n-k] + \delta[k]\,\delta[n-k-2] - \delta[k-1]\delta[n-k] - \delta[k-1]\,\delta[n-k-2]$$

The discrete-time impulse $\delta[n]$ is 1 for $n = 0$ and 0 otherwise. The product $\delta[k]\,\delta[n-k]$ is 1 when $k = 0$ and the product simplifies to $\delta[n]$:

$$h_1[n] * h_2[n] = \delta[n] + \delta[n-2] - \delta[n-1] - \delta[n-3]$$
$$h_1[n] * h_2[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$$

Convolution for finite-length signals is reminiscent of polynomial multiplication. In fact, the Matlab command conv can also be used to compute the product of two polynomials.

We could check our answers for the convolution calculations in MATLAB:

```
h1Convh2 = conv( [1 -1], [1 0 1] );     %% gives 1  -1   1   -1
h = conv(h1Convh2, [0 1 1] );           %% gives 0   1   0   0   0  -1
```

which gives [0   1   0   0   0   -1] or equivalently $h[n] = \delta[n-1] - \delta[n-5]$.

$$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k] = x[n-1] - x[n-5]$$

Approach #2 Rewriting Equations. The output of the second system is the input of the third system, $y_2[n] = x_3[n]$:

$$\begin{aligned}
y_3[n] &= x_3[n-1] + x_3[n-2] \\
&= y_2[n-1] + y_2[n-2] \\
&= x_2[n-1] + x_2[n-3] + x_2[n-2] + x_2[n-4] \\
&= x_2[n-1] + x_2[n-2] + x_2[n-3] + x_2[n-4]
\end{aligned}$$

The output of the first system is the input of the second system:

$$y_1[n] = x_2[n]$$

$$\begin{aligned}
y_3[n] &= x_2[n-1] + x_2[n-2] + x_2[n-3] + x_2[n-4] \\
&= y_1[n-1] + y_1[n-2] + y_1[n-3] + y_1[n-4] \\
&= x_1[n-1] - x_1[n-2] + x_1[n-2] - x_1[n-3] + x_1[n-3] - x_1[n-4] + x_1[n-4] - x_1[n-5] \\
&= x_1[n-1] - x_1[n-5]
\end{aligned}$$

The equivalent system that is a single operation from the input $x[n]$ (into $\mathcal{S}_1$) to the output $y[n]$, which is the output of $\mathcal{S}_3$, is the following:

$$y[n] = x[n-1] - x[n-5]$$

Solution for part (b): The frequency response form of the systems 1, 2, and 3 are shown below:

$$\mathcal{S}_1: H_1(e^{j\hat\omega}) = 1 - e^{-j\hat\omega}$$

$$\mathcal{S}_2: H_2(e^{j\hat\omega}) = 1 + e^{-2j\hat\omega}$$

$$\mathcal{S}_3: H_3(e^{j\hat\omega}) = e^{-j\hat\omega} + e^{-2j\hat\omega}$$

The three systems are connected in cascade. Therefore,

$$\begin{aligned}
H(e^{j\hat\omega}) = H_1(e^{j\hat\omega})H_2(e^{j\hat\omega})H_3(e^{j\hat\omega}) &= \left(1 - e^{-j\hat\omega}\right)\left(1 + e^{-2j\hat\omega}\right)\left(e^{-j\hat\omega} + e^{-2j\hat\omega}\right) \\
&= \left(1 + e^{-2j\hat\omega} - e^{-j\hat\omega} - e^{-3j\hat\omega}\right)\left(e^{-j\hat\omega} + e^{-2j\hat\omega}\right) \\
&= e^{-j\hat\omega} + e^{-2j\hat\omega} + e^{-3j\hat\omega} + e^{-4j\hat\omega} - e^{-2j\hat\omega} - e^{-3j\hat\omega} - e^{-4j\hat\omega} - e^{-5j\hat\omega} \\
&= e^{-j\hat\omega} - e^{-5j\hat\omega}
\end{aligned}$$

Converting back:

$$y[n] = x[n-1] - x[n-5]$$

Epilogue: In the time domain, the response (output) of an LTI system will be the convolution of the input signal and the system's impulse response. In the frequency domain, the output of the LTI system will be the product of the input frequency content and the system's frequency response. The discrete-time Fourier transform can compute the frequency content of a periodic or aperiodic signal. It is a generalization of the discrete-time Fourier series.

The convolution of two finite-length signals can be computed using polynomial multiplication. That is, each finite-length signal can be represented as a finite-length vector of amplitude values, and the process of convolving these vectors is the same as polynomial multiplication when interpreting the vectors as polynomial coefficients. See the Epilogue for Problem 7.1 which talks about the same connection in the z-domain, where it might be easier to see.