Solution Set for Homework #8

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PROBLEM 1: CONTINUOUS-TIME SYSTEM PROPERTIES. ***20 points.***

Signal Processing First, problem P-9.2, page 279.

In each of the following cases, state whether or not the continuous time system is (i) linear, (ii) time-invariant, (iii) stable, and (iv) causal. In each case, x(t) he represents the input and y(t) represents the corresponding output of the system. Provide a brief justification, either in the form of mathematical equations or statements in the form of complete, crack, push sentences. Remember, in order to show the system does not have the property, all you have to do is give an example and put up with is not satisfy the condition of the property.

(a) An exponential system: $y(t) = e^{x(t+2)}$. **Used in speech denoising and machine learning**.

(b) A phase modulator: $y(t) = \cos(\omega_c t + x(t))$. **Phase modulation is used for low-power transmission in IoT systems. The digital version, Phase Shift Keying, is used in RFID and Bluetooth and higher transmit power systems, such as Wi-Fi and cellular communications.**

(c) An amplitude modulator: $y(t) = (A + x(t))\cos(\omega_c t)$. **Used in AM radio. Amplitude modulation (without the offset of *A*) is used in Wi-Fi, cellular and cable modems.**

(d) Take the even part of the input signal: $y(t) = \frac{x(t)+x(-t)}{2}$. **Primarily for theoretical analysis.**

***Solution:* Linearity**

When checking each system for linearity, we can use the quick test of input signal of 0 for all time, which is a by-product of the homogeneity property when the input signal is scaled by a = 0. If the output signal is not zero for all time, then the system is not linear. If the output is zero for all time, then we'll have to apply the mathematical definitions for homogeneity and additivity.

(a) $\boldsymbol{y(t) = e^{x(t+2)} = e^0 = 1}$. **Fails all-zero input test. Not linear.**

(b) $\boldsymbol{y(t) = \cos(\omega_c t + x(t)) = \cos(\omega_c t)}$. **Fails all-zero input test. Not linear.**

(c) $\boldsymbol{y(t) = (A + x(t))\cos(\omega_c t) = A\cos(\omega_c t)}$. **Fails all-zero input test. Not linear.**

(d) $\boldsymbol{y(t) = \frac{x(t)+x(-t)}{2} = 0}$. **Passes all-zero input test. Check for homogeneity and additivity.**

- ***Homogeneity*. Input** $\boldsymbol{a\,x(t)}$. **Output** $\boldsymbol{y_{scaled}(t) = \frac{(a\,x(t))+(a\,x(-t))}{2} = a\,\frac{x(t)+x(-t)}{2} = a\,y(t)}$
- ***Additivity*. Input** $\boldsymbol{x_1(t) + x_2(t)}$. **Output**

$$\boldsymbol{y_{additive}(t) = \frac{(x_1(t)+x_2(t))+(x_1(-t)+x_2(-t))}{2} = \frac{x_1(t)+x_1(-t)}{2} + \frac{x_2(t)+x_2(-t)}{2} = y_1(t) + y_2(t)}$$

Yes, system (d) is linear.

***Solution:* Time-Invariance**

For time-invariant system, shift of the input signal by any real-valued t causes the same shift in output signal, i.e. *x*(*t* - t) means *y*(*t* - t) for all t.

(a) $y(t) = e^{x(t+2)}$**. Input** $x(t-\tau)$**. Output** $y_{shifted}(t) = e^{x((t-\tau)+2)} = e^{x(t-\tau+2)} = y(t-\tau)$**.** ***Time-invariant.***

(b) $y(t) = \cos(\omega_c t + x(t))$**. The signal** $\cos(\omega_c t)$ **is part of the system and does not shift in time when the input shifts in time.** ***Time-varying.***

(c) $y(t) = (A + x(t))\cos(\omega_c t)$**. The signal** $\cos(\omega_c t)$ **is part of the system and does not shift in time when the input shifts in time.** ***Time-varying.***

(d) $y(t) = \frac{x(t)+x(-t)}{2}$**. The copy of the input signal shifts in the same way that the input signal shifts. The copy that is reversed in time gives the negated shift.** ***Time-varying.***

<u>Solution:</u> **<u>Stability</u>**

A stable system will always produce a bounded amplitude output signal when given a bounded amplitude input signal. Let $|x(t)| < B < \infty$

(a) $|y(t)| = \left|e^{x(t+2)}\right| \le |e^B|$**. Bounded output.** ***Stable.***

(b) $y(t) = \cos(\omega_c t + x(t))$**. Output will always be in range [-1, 1] regardless of the value of** $x(t)$**. Bounded output.** ***Stable.***

(c) $|y(t)| = |(A + x(t))\cos(\omega_c t)| \le |A + x(t)|\,|\cos(\omega_c t)| \le |A + x(t)| \le |A| + |x(t)| \le |A| + B$**. Bounded output.** ***Stable.***

(d) $|y(t)| = \left|\frac{x(t)+x(-t)}{2}\right| \le \frac{|x(t)|}{2} + \frac{|x(-t)|}{2} \le B$**. Bounded output.** ***Stable.***

<u>Solution:</u> **<u>Causality</u>**

A causal system depends only on current and previous input values and/or previous output value to compute an output value.

(a) $y(t) = e^{x(t+2)}$**. Depends on input 2 seconds in the future.** ***Not Causal.***

(b) $y(t) = \cos(\omega_c t + x(t))$**. Only depends on the current input value** $x(t)$**.** ***Causal.***

(c) $y(t) = (A + x(t))\cos(\omega_c t)$**. Only depends on the current input value** $x(t)$**.** ***Causal.***

(d) $y(t) = \frac{x(t)+x(-t)}{2}$ **. Check specific values of time** ***t*****.**

- **When** ***t*** **= 0,** ***y*****(0) = (** ***x*****(0) +** ***x*****(0)) / 2 =** ***x*****(0). Causal.**
- **When** ***t*** **= 2,** ***y*****(2) = (** ***x*****(2) +** ***x*****(-2)) / 2. Causal.**
- **When** ***t*** **= -2,** ***y*****(-2) = (** ***x*****(-2) +** ***x*****(2)) / 2. Depends on future input** ***x*****(2).** ***Not Causal.***

<u>PROBLEM 2: CONTINUOUS-TIME AVERAGING FILTERS.</u> ***32 points.***

For a **continuous**-time LTI system with input signal $x(t)$ and impulse response $h(t)$, the output signal $y(t)$ is the convolution of $h(t)$ and $x(t)$:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$$

(a) Compute the output $y(t)$ when the input $x(t)$ is a rectangular pulse of amplitude 1 for $t \in [0, T_x]$ and amplitude 0 otherwise and $x[n]$ is filtered by an LTI unnormalized averaging filter whose impulse response $h(t)$ is a rectangular pulse of amplitude 1 for $t \in [0, T_h]$ and amplitude 0 otherwise. Assume $T_x \neq T_h$.

i. Write an equation relating output $y(t)$ and input $x(t)$. *4 points*

Solution: **With $h(t) = 1$ for $t \in [0, T_h]$,**

$$\boldsymbol{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{0}^{T_h} x(t-\lambda)\, d\lambda}$$

We can apply the change of variables $u = t - \lambda$. $du = -d\lambda$. As $\lambda \to 0$, $u \to t$. As $\lambda \to T_h$, $u \to t - T_h$. This gives

$$\boldsymbol{y(t) = \int_{0}^{T_h} x(t-\lambda)\, d\lambda = \int_{t-T_h}^{t} x(u)\, du}$$

The averaging filter integrates the input signal over the previous T_h seconds.
Although not asked, this filter is stable unlike the integrator over all time in part (b).

ii. What is(are) the initial condition(s) and what value should it(they) be set to? *3 points*

Solution: **The initial conditions are the memory of the previous T_h seconds of the input signal. This memory (signal buffer) would have to be initially zeroed out.**

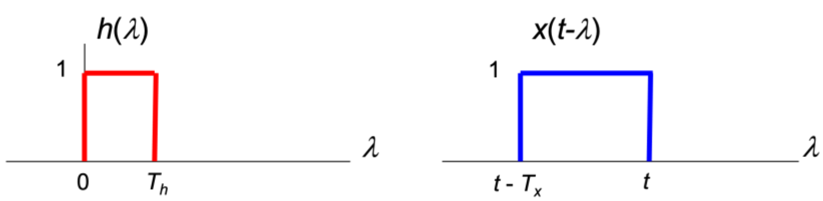
iii. Develop a formula for $y(t) = h(t) * x(t)$ using the convolution definition in terms of T_x and T_h. Show the intermediate steps in computing the convolution. *6 points*

Solution: **Trapezoid of duration $T_y = T_h + T_x$. Let $T_{min} = \min(T_h, T_x)$ and $T_{max} = \max(T_h, T_x)$. As we flip and slide one rectangular pulse against the other, partial overlap occurs from 0 to T_{min} seconds, complete overlap from T_{min} to T_{max} seconds, and partial overlap from T_{max} to T_y seconds.**

$$\boldsymbol{y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \leq t < T_{min} \\ T_{min} & \text{for } T_{min} \leq t < T_{max} \\ T_y - t & \text{for } T_{max} \leq t < T_y \\ 0 & \text{for } t > T_y \end{cases}}$$

The details of the flip-and-slide are analogous to problem 1(a)iii for the discrete-time convolution of two rectangular pulses, and explained next.
We will hold $h(t)$ in place and flip and slide $x(t)$about $h(t)$:

There are five cases to consider:



1. **No overlap. $t < 0$. Amplitude is 0.**
2. **Partial overlap. $0 \leq t < T_{min}$. Amplitude is t.**
   Initial overlap at the origin, and integration of a point is zero. Each shift by a time unit adds that much to the area.

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{0}^{t} d\lambda = t$$

3. **Complete overlap. $T_{min} \le t < T_{max}$. Amplitude is T_{min}.**
   Here, T_{min} seconds overlap, and each amplitude has a value of one.
4. **Partial overlap. $T_{max} \le n < T_y$. Amplitude is $T_y - t$.**
   Amplitude reduces by the same amount that t is shifted.

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda = \int_{t-T_x}^{T_h} d\lambda = T_h - (t - T_x) = T_y - t$$

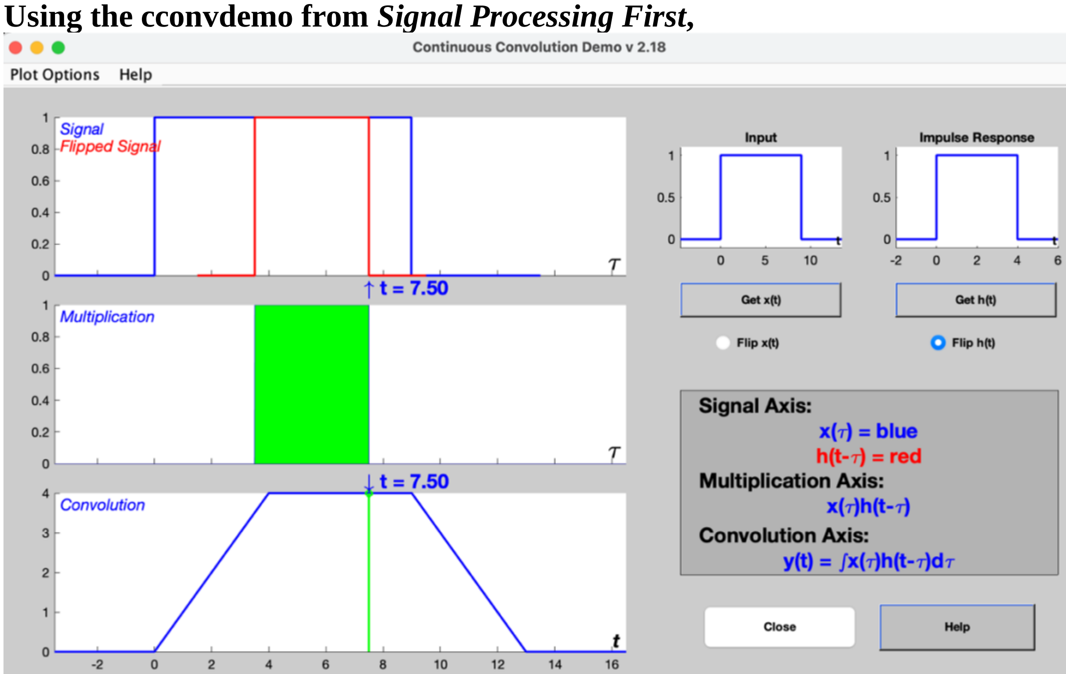
5. **No overlap. $t \ge T_y$. Amplitude is 0.**

iv. Validate the formula for $y(t)$ to compute the convolution for $T_x = 9$ seconds and $T_h = 4$ seconds. *3 points.*

***Solution*: Trapezoid of duration $T_y = 4 + 9 = 13$ seconds. Let $T_{min} = \min(4, 9) = 4$ seconds and $T_{max} = \max(4, 9) = 9$ seconds. As we flip and slide one rectangular pulse against the other, partial overlap occurs from 0 to T_{min} seconds, complete overlap from T_{min} to T_{max} seconds, and partial overlap from T_{max} to T_y seconds.**

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } 0 \le t < 4 \\ 4 & \text{for } 4 \le t < 9 \\ 13 - t & \text{for } 9 \le t < 13 \\ 0 & \text{for } t > 13 \end{cases}$$

Using the cconvdemo from *Signal Processing First*,



(b) When an input signal has an average value of zero, i.e. the DC component is zero, an LTI integrator can be used as an averaging filter. The differential equation governing the input-output relationship is

$$y(t) = \int_{0^-}^{t} x(\tau)\, d\tau \quad \text{for } t \geq 0$$

The integrator was operating before *t* = 0 seconds, but we weren't able to observe that. We are observing the system starting at *t* = 0 seconds.

Since we are starting the integration at *t* = 0 seconds, there is ambiguity as to whether Dirac delta signal would be included at time 0. We can use 0^- as the lower limit to indicate that integration starts at time 0 before the impulse occurs.

i. What is(are) the initial condition(s) and what value should it(they) be set to? *3 points*
***Solution*: The initial condition is the initial integration value *y*(0). It should be set to zero as a necessary condition for LTI properties to hold.**

ii. What is the impulse response? *3 points*
***Solution:* For input $\boldsymbol{x(t) = \delta(t)}$, the output is the impulse response**

$$\boldsymbol{h(t) = \int_{0^-}^{t} \delta(\tau)\, d\tau = u(t)}$$

iii. Develop a formula for $y(t) = h(t) * x(t)$ using the convolution definition when the input signal is $x(t) = u(t)$. Note that $x(t)$ has bounded amplitude. *9 points*
***Solution:* Using the convolution definition,**

$$\boldsymbol{y(t) = h(t) * x(t) = u(t) * u(t) = \int_{-\infty}^{\infty} u(\lambda)\, u(t-\lambda)\, d\lambda}$$

$\boldsymbol{u(\lambda)}$ is 1 for $\boldsymbol{\lambda \geq 0}$ and 0 otherwise, whereas $\boldsymbol{u(t-\lambda)}$ is 1 for $\boldsymbol{t - \lambda \geq 0}$ or $\boldsymbol{\lambda \leq t}$. Also, note that $\boldsymbol{t \geq 0}$ because $\boldsymbol{\lambda \geq 0}$:

$$\boldsymbol{y(t) = \int_{0}^{t} d\lambda = t\, u(t)}$$

iv. Is the LTI integrator bounded-input bounded-output (BIBO) stable? Your work in part iii might be helpful. *3 points*
***Solution:* For input signal $\boldsymbol{x(t) = u(t)}$, whose amplitude is bounded in [0, 1], the output $\boldsymbol{y(t) = t\, u(t)}$ grows without bound as $\boldsymbol{t \to \infty}$. LTI integrator is not stable.**