Solution Set for Homework #9

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PROBLEM 9.1: CONTINUOUS TIME-FREQUENCY RESPONSE. *48 points.*

Signal Processing First, problem P-10.9, page 305. In addition, for each of the seven filters given, describe the frequency selectivity in the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

Consider an LTI system whose frequency response $H(j\omega)$ is unknown. The system has a periodic input whose spectrum is shown in Fig. P-10.9.

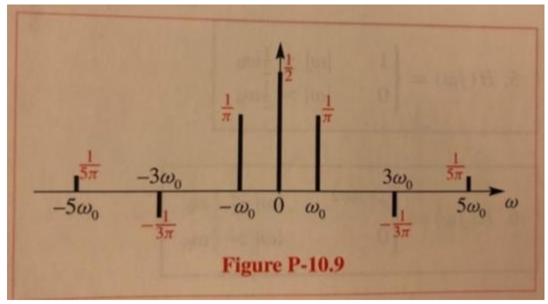


Figure P-10.9

For each part of this problem, the output of the system is given and the frequency response must be determined by selecting from the list numbered 1-7 below. Chose the frequency response $H(j\omega)$ of the system that could have produced the specific output when the input is the signal with the spectrum in Fig. P-10.9.

Solution: Fig. P-10.9 plots the following Fourier series coefficients

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{\pi}, & k = \pm 1 \\ 0, & k = \pm 2 \\ -\frac{1}{3\pi}, & k = \pm 3 \\ 0, & k = \pm 4 \\ \frac{1}{5\pi}, & k = \pm 5 \end{cases}$$

which can be used in the Fourier series formula

$$x(t) = \sum_{k=-5}^{5} a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

Due to the system having linear and time-invariant properties, all the frequency components in the output signal had to be present in the input signal. That is, a linear time-invariant (LTI) system cannot create new frequencies.

Using LTI system properties, the output signal is simply the sum of the system's response to each frequency component of the input signal:

$$y(t) = \frac{1}{2}H(j0) + \frac{2}{\pi}\cos(\omega_0 t)H(j\omega_0) - \frac{2}{3\pi}\cos(3\omega_0 t)H(j3\omega_0) + \frac{2}{5\pi}\cos(5\omega_0 t)H(j5\omega_0)$$

We can write the frequency response into polar form as $H(j\omega) = |H(j\omega)|e^{j\,\angle H(j\omega)}$:

$$\begin{aligned} y(t) = \frac{1}{2}H(j0) + \frac{2}{\pi}|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) - \\ \frac{2}{3\pi}|H(j3\omega_0)|\cos(3\omega_0 t + \angle H(j3\omega_0)) + \\ \frac{2}{5\pi}|H(j5\omega_0)|\cos(5\omega_0 t + \angle H(j5\omega_0)) \end{aligned}$$

Please see lecture slide 14-6 and *Signal Processing First* Section 10-2.

a) $y(t) = \frac{1}{2}$

The output of system can be obtained by the following formula

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

Hence, we can find the value of $H(j\omega)$ for frequencies that are present in the input.

$$H(jk\omega_0) = \begin{cases} 1, & k = 0 \\ 0, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

Filter 5 has similar response, so the input has passed through this lowpass filter.

$$H(j\omega) = \begin{bmatrix} 1 & |\omega| \le \frac{1}{2}\omega_0 \\ 0 & |\omega| > \frac{1}{2}\omega_0 \end{bmatrix}$$

b) $y(t) = \frac{1}{2} + \frac{2}{\pi}\cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t} = \frac{1}{2} + \frac{2}{\pi}\cos\left[\omega_0\left(t - \frac{1}{2}\right)\right]$$

$$H\left(jk\omega_0\right) = \begin{cases} 1, & k = 0 \\ e^{-j\omega_0/2}, & k = \pm 1 \\ 0, & k = \pm 3, \pm 5 \end{cases}$$

Filter 6 has this property and will give similar output. This lowpass filter removes frequencies above $\frac{3\omega_0}{2}$ and delays the input by ½ sample. We can obtain the delay by computing the group delay for the filter as follows: Group Delay$(\omega) = -\frac{d}{d\omega}\angle H(j\omega) = -\frac{d}{d\omega}\left(-\frac{\omega}{2}\right) = \frac{1}{2}$.

$$H(j\omega) = \begin{bmatrix} e^{-j\frac{\omega}{2}} & |\omega| \le \frac{3}{2}\omega_0 \\ 0 & |\omega| > \frac{3}{2}\omega_0 \end{bmatrix}$$

c)

$$y(t) = \frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t} = \frac{2}{\pi}\cos\left(\omega_0 t\right)$$

$$H\left(jk\omega_0\right) = \begin{cases} 1, & k = \pm 1 \\ 0, & k = 0, \pm 3, \pm 5 \end{cases}$$

The original signal has passed through a bandpass filter which removes all frequencies present in the input signal except ω_0. Filter 7 shows a bandpass filter with this property.

$$H(j\omega) = \begin{cases} 1, & \frac{1}{2}\omega_0 < |\omega| < \frac{3}{2}\omega_0 \\ 0, & |\omega| < \frac{1}{2}\omega_0 \text{ or } |\omega| > \frac{3}{2}\omega_0 \end{cases}$$

d)

$$y(t) = x(t) - \frac{1}{2} = \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t} = \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\cos(5\omega_0 t)$$

$$H\left(jk\omega_0\right) = \begin{cases} 0, & k = 0 \\ 1, & k = \pm 1, \pm 3, \pm 5 \end{cases}$$

This filter passes all frequencies except $\omega = 0$, therefore it acts as a highpass filter or a bandpass filter. Filter 1 is a highpass filter that can produce this output.

$$H(j\omega) = \begin{cases} 0 & |\omega| \le \frac{1}{2}\omega_0 \\ 1 & |\omega| > \frac{1}{2}\omega_0 \end{cases}$$

e)

$$y(t) = x(t - \frac{1}{2})$$

$$y(t) = \sum_{k=-5}^{k=5} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

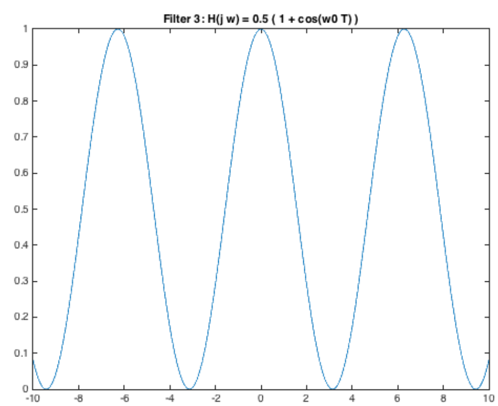
$$H\left(jk\omega_0\right) = e^{-jk\omega_0/2}$$

Filter 2 gives this response, which is allpass.

$$H(j\omega) = e^{-j\omega/2}$$

Filters 3 and 4 cannot produce any of the output signals.

<u>Filter 3:</u>



$$H(j\omega) = \frac{1}{2}\left[1 + \cos(\omega T_0)\right]$$

$$for\ \omega = k\omega_0 \rightarrow H(jk\omega_0) = \frac{1}{2}\left[1 + \cos(k\omega_0 T_0)\right] = \frac{1}{2}\left[1 + \cos(2\pi k)\right] = 1$$

For $\omega = k\omega_0$, this filter passes all the harmonic frequencies; however, it rejects sub-harmonic frequencies at $\omega = \frac{k\omega_0}{2}$. This filter has a periodic magnitude response as shown above.

Filter 4: This filter removes frequencies above $\frac{3\omega_0}{2}$ and is a lowpass filter.

Epilogue: The LTI ideal delay is a building block in continuous-time systems.

An LTI system with a constant non-zero magnitude response such as $|H(j\omega)| = 1$ passes all frequencies through equally well. This is called an *allpass filter*.

From the phase response, we can determine the group delay in seconds through the LTI system for a particular frequency by taking the derivative of the phase response and negating it. For a phase response of $\angle H(j\omega) = -\frac{\omega}{2}$, the group delay would be $\frac{1}{2}$ seconds, which is the delay in the ideal delay system. See also problem 8.2(b) below.

If we could only observe the ideal delay for time $t \geq 0$, then we would have to set the initial conditions to zero as a necessary condition for the ideal delay to be LTI. Please see Handout U Property of Time-Invariance (Shift-Invariance) for a System Under Observation for an example of an ideal delay under observation for time $t \geq 0$.

PROBLEM 9.2: FORWARD CONTINUOUS-TIME FOURIER TRANSFORMS.

Compute the continuous-time Fourier transform $X(j\omega)$ for continuous-time signal $x(t)$ using the definition in Signal Processing First in equation (11.1)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

for the following time-domain signals $x(t)$. In addition, for each part, describe the frequency selectivity of the magnitude response as lowpass, highpass, bandpass, bandstop, allpass, or notch.

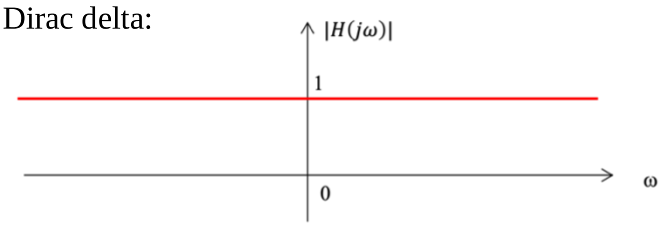
Signal Processing First Section 11.4 covers examples (a)-(d). You can check your answers using continuous-time Fourier transform pairs in Table 11-2 of on page 338 in *Signal Processing First.*

Solution:

a) $x(t) = \delta(t)$. We can use the sifting property for the Dirac delta:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big]_{t=0} = 1$$

Allpass filter. All frequencies pass through equally well. See the epilogue in the solution to Problem 9.1 above. Magnitude response plotted on the right.



b) Rectangular pulse of unit amplitude that lasts from $-\frac{T}{2}$ to $\frac{T}{2}$ seconds.

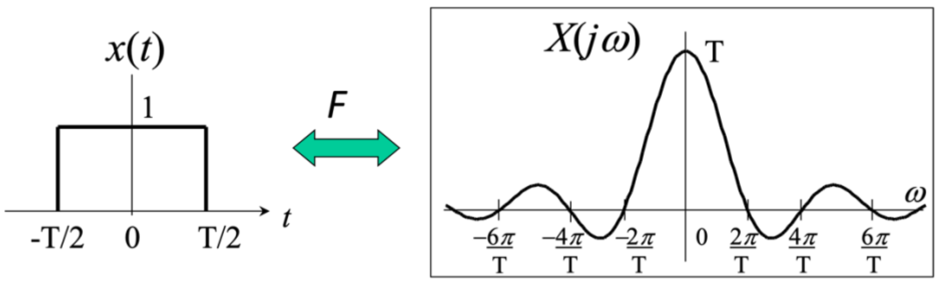
$$x(t) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \left[u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)\right] e^{-j\omega t} dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j\omega t}\, dt$$

$$X(j\omega) = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-\frac{T}{2}}^{\frac{T}{2}} = -\frac{1}{j\omega}\left(e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}}\right) = \frac{2j \sin\left(\frac{T}{2}\omega\right)}{j\omega} = T\frac{\sin\left(\frac{T}{2}\omega\right)}{\frac{T}{2}\omega}$$

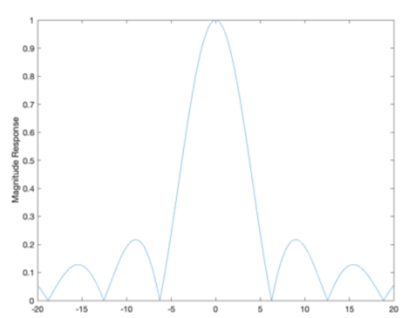
$$X(j\omega) = T\,\text{sinc}\left(\frac{\omega T}{2\pi}\right) \text{ where } \text{sinc(x)} = \frac{\sin\,(\pi\, x)}{\pi\, x}$$

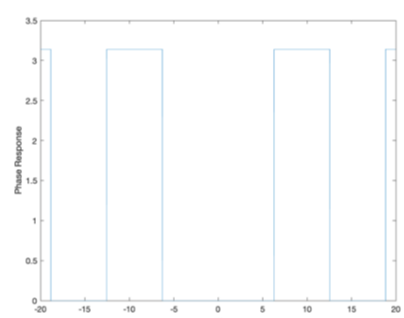
See also *Signal Processing First*, pp. 314-315. ***Lowpass filter.*** Lecture slide 15-6:



Magnitude and phase plots in MATLAB for $T = 1$ for $-20 < \omega < 20$ rad/s. In the magnitude response, the first zero to the right of zero frequency occurs at $\omega = \frac{2\pi}{T} = 2\pi$. The phase response is π whenever the sinc pulse goes negative.

```
T = 1;
w = -20 : 0.01 : 20;
X = T*sinc(w*T/(2*pi));
figure;
plot(w, abs(X));
xlabel('w');
ylabel('Magnitude Response');
figure;
plot(w, angle(X));
xlabel('w');
ylabel('Phase Response');
```



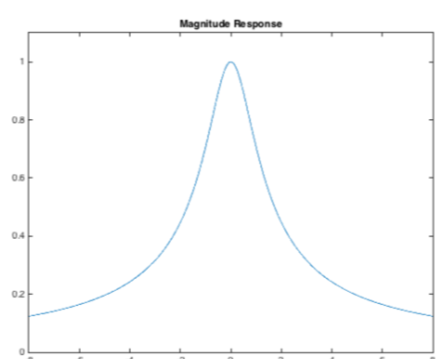


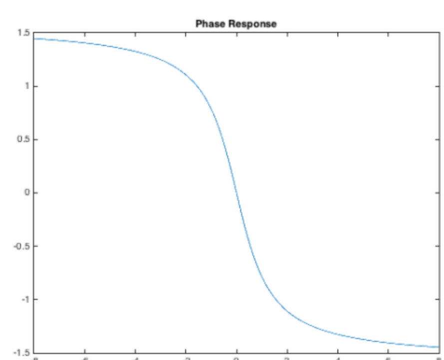
c) $x(t) = e^{-at}u(t)$ for positive and real-valued a

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt = -\frac{1}{a+j\omega}\left[e^{-(a+j\omega)t}\right]_0^{\infty} = \frac{1}{a+j\omega}$$

See *Signal Processing First*, pp. 309 & 313. ***Lowpass filter.*** From Lecture Slide 15-5 for $a = 1$:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 + j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
```





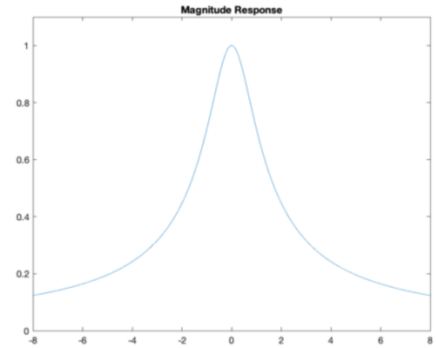
```
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```

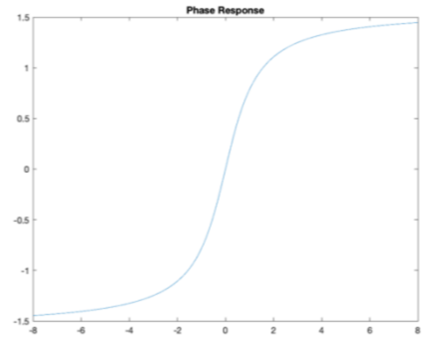
d) $x(t) = e^{bt}u(-t)$ for positive and real-valued b

$$X(j\omega) = \int_{-\infty}^{\infty} e^{bt}u(-t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(b-j\omega)t}dt = \frac{1}{b-j\omega}\left[e^{(b-j\omega)t}\right]_{-\infty}^{0} = \frac{1}{b-j\omega}$$

See *Signal Processing First*, pp. 314. ***Lowpass filter.*** Matlab plots for the magnitude and phase for b = 1. Magnitude response is the same as part (c) and phase response is flipped in frequency:

```
w = -8 : 0.01 : 8;
H = 1 ./ (1 - j*w);
Hmag = abs(H);
Hphase = phase(H);
figure;
plot(w, Hmag);
title('Magnitude Response');
ylim( [-0.0 1.1] );
figure;
plot(w, Hphase);
title('Phase Response');
```





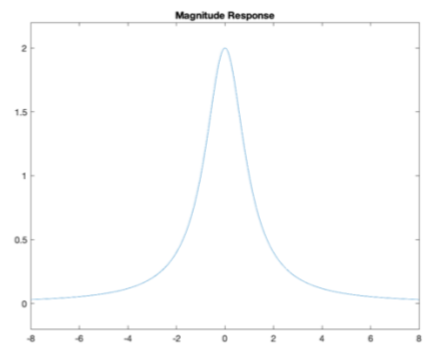
e) $x(t) = e^{-a|t|}$ for $-\infty < t < \infty$ for positive and real-valued a. When $t < 0$, $|t| = -t$. When $t > 0$, $|t| = t$. We can reuse the results from parts (c) and (d).

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{-a(-t)}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$X(j\omega) = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

$$X(j\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{a-j\omega+a+j\omega}{(a+j\omega)(a-j\omega)} = \frac{2a}{a^2+\omega^2}$$

Lowpass filter. See magnitude response plot on the right for a = 1:



PROBLEM 9.3: CONTINUOUS-TIME FOURIER TRANSFORMS USING TRANSFORM PROPERTIES AND PAIRS.

Signal Processing First, problem P-11.8, page 343. In the following, the Fourier transform $X(j\,\omega)$ is given. Using the tables of Fourier transforms and Fourier transform properties to determine the inverse Fourier transform for each case. You may give your answer either as an equation or a carefully labeled plot, whichever is most convenient.

(a) $X(j\omega) = \frac{e^{-j3\omega}}{2+j\omega}$

Solution for (a): We can rewrite the Fourier transform as $X(j\omega) = \frac{e^{-j3\omega}}{2+j\omega} = \left(\frac{1}{2+j\omega}\right)e^{-j3\omega}$

The Fourier transform pair related to the first term is

$$e^{-2\,t}\,u(t) \leftrightarrow \frac{1}{2+j\omega}$$

The second term $e^{-j3\omega}$ relates to the time delay property, where $t_d = 3$:

$$x(t) = \; e^{-2(t-3)}u(t-3)$$

(b) $X(j\omega) = \frac{j\omega}{2+j\omega}$

Solution for (b) #1: (from a student's solution) We rearrange this expression as follows:

$$X(j\omega) = \frac{j\omega + 2 - 2}{2+j\omega} = 1 - \frac{2}{2+j\omega}$$

and then use the Fourier transform pairs

$$e^{-2\,t}\,u(t) \leftrightarrow \frac{1}{2+j\omega} \quad \text{and} \quad \delta(t) \leftrightarrow 1$$

Using the linearity property of the Fourier transform

$$2\,e^{-2t}\,u(t) \leftrightarrow \frac{2}{2+j\omega}$$

we obtain

$$x(t) = \delta(t) - 2\,e^{-2\,t}\,u(t)$$

Solution for (b) #2: We rearrange the Fourier transform expression as follows:

$$X(j\omega) = \frac{j\omega}{2+j\omega} = \left(\frac{1}{2+j\omega}\right)(j\omega)$$

Using the differentiation-in-time property of the Fourier transform, we obtain

$$x(t) = \frac{d}{dt}\left(2\,e^{-2\,t}\,u(t)\right) = e^{-2\,t}\,\delta(t) - 2\,e^{-2\,t}\,u(t)$$

Solutions for (b) #1 and #2 are identical expressions under integration using the sifting property of the Dirac delta. That is, the value of $e^{-2\,t}$ is 1 when $t = 0$.

(c) $X(j\omega) = \frac{j\omega}{2+j\omega}e^{-j3\omega}$

Solution for (c): We build on the solution in part (b). We rearrange $X(j\omega)$ as

$$X(j\omega) = \frac{j\omega}{2+j\omega}e^{-j3\omega} = \left(\frac{1}{2+j\omega}\right)(j\omega)\left(e^{-j3\omega}\right)$$

We take the inverse Fourier transform of $\frac{1}{2+j\omega}$, differentiate the time-domain expression, and delay by 3s. That is, we take the solution in part (b) and delay it by 3s. From solution for (b) #1,

$$x(t) = \delta(t-3) - 2\,e^{-2\,(t-3)}\,u(t-3)$$

and from solution for (b) #2

$$x(t) = e^{-2\,(t-3)}\,\delta(t-3) - 2e^{-2(t-3)}u(t-3)$$

These are equivalent solutions (see part (b)).

(d) $X(j\omega) = \left(\frac{2\sin(\omega)}{\omega}\right)\left(\sum_{k=-\infty}^{\infty}\frac{\pi}{5}\delta\left(\omega - \frac{2\pi}{10}k\right)\right)$

Solution for (d): Using the convolution-in-time property:

$$x_1(t) * x_2(t) \leftrightarrow X_1(j\omega)\, X_2(j\omega)$$

we have

$$X_1(j\omega) = \frac{2\sin(\omega)}{\omega}$$

$$u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right) \leftrightarrow \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega}{2}}$$

$$u(t+1) - u(t-1) \leftrightarrow \frac{2\sin(\omega)}{\omega}$$

And

$$X_2(j\omega) = \sum_{k=-\infty}^{\infty}\frac{\pi}{5}\,\delta\left(\omega - \frac{2\pi}{10}k\right) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{10}\,\delta\left(\omega - \frac{2\pi}{10}k\right)$$

$$\sum_{n=-\infty}^{\infty}\delta(t - 10n) \leftrightarrow \frac{2\pi}{10}\sum_{k}\delta\left(\omega - \frac{2\pi}{10}k\right)$$

Therefore,

$$x(t) = x_1(t) * x_2(t) = \big(u(t+1) - u(t-1)\big) * \sum_{n=-\infty}^{\infty}\delta(t - 10n)$$

The expression $u(t+1) - u(t-1)$ is a rectangular pulse of duration 2s centered at the origin, i.e. rect(t/2). We interchange the summation and convolution due to linearity of convolution. When convolving a signal $g(t)$ with a delayed Dirac delta $\delta(t - t_0)$, we obtain $g(t - t_0)$:

$$x(t) = \text{rect}\left(\frac{t}{2}\right) * \sum_{n=-\infty}^{\infty}\delta(t - 10n) = \sum_{n=-\infty}^{\infty}\text{rect}\left(\frac{t}{2}\right) * \delta(t - 10n) = \sum_{n=-\infty}^{\infty}\text{rect}\left(\frac{t - 10n}{2}\right)$$

This signal is a square wave with period 10s. The rectangular pulse in each period lasts for 2s.

```
t = -20 : 0.01 : 20;
x = rectpuls((t+20)/2) + rectpuls((t+10)/2)
+ rectpuls(t/2) + rectpuls((t-10)/2) +
rectpuls((t-20)/2);
plot(t, x);
ylim( [-0.2 1.2] );
xlabel('t');
ylabel('x(t)');
```