Goal

- Design a new family of wavelets:
  Biorthogonal Quincunx Coifman Wavelets
  - 2-D nonseparable
  - compactly supported
  - biorthogonal
  - quincunx down- and up-sampling
- Useful properties:
  - zero-phase filterbanks
  - dyadic rational filter coefficients
  - cardinal scaling functions
  - converge to ideal filters asymptotically
  - closed-form formulae
- Promising applications:
  - image compression
  - sampling/interpolation

Motivation

- Why biorthogonal wavelets?
  - more flexible design
  - linear-phase filterbanks
  - cardinal scaling functions
- Why quincunx wavelets?
  - finer multiresolution analysis
  - allow *nonseparable* sampling
  - allow *nonseparable* filtering
  - more isotropic bases
  - better adaption to human visual system
- Why Coifman wavelets?
  - tradeoff between two types of vanishing moments
  - easy to achieve linear phase
  - useful in sampling/interpolation

1-D Biorthogonal Coiflets

- Synthesis filter of order l:

$$H_l(\omega) = \left(\frac{1+\cos\omega}{2}\right)^k \times \sum_{p=0}^{k-1} \binom{k-1+p}{p} \left(\frac{1-\cos\omega}{2}\right)^p$$

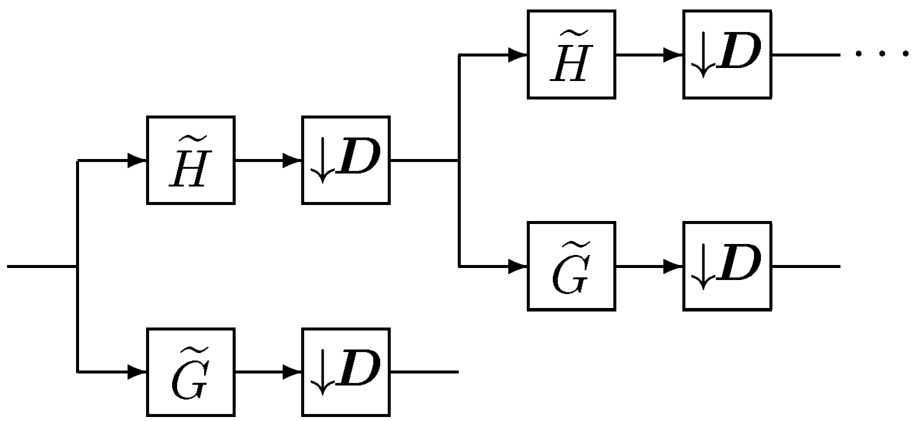
where $l = 2k$

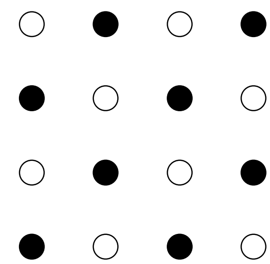
- Analysis filter of order (l, l'):

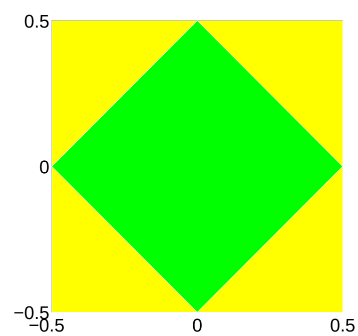
$$\widetilde{H}_{l,l'}(\omega) = 2H_{l'}(\omega) + H_l(\omega) - 2H_{l'}(\omega)H_l(\omega)$$

- Useful properties:
  - zero-phase filterbanks
  - dyadic rational filter coefficients
  - converges to ideal filters asymptotically
  - cardinal scaling functions
  - excellent for data compression

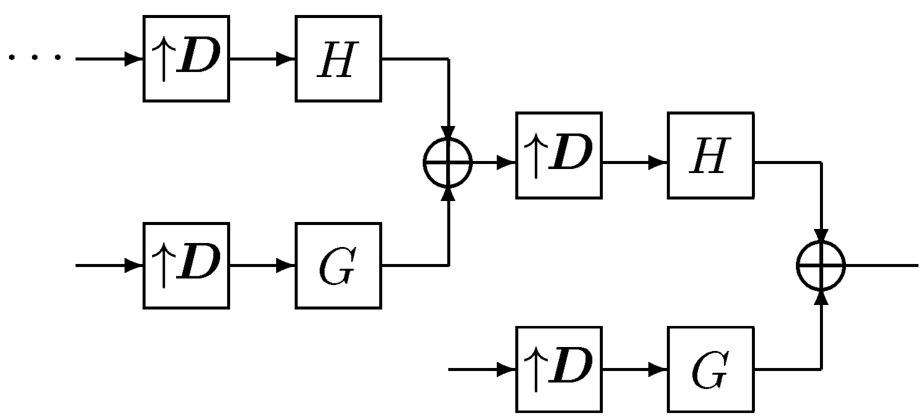
Quincunx Filterbank and Wavelet

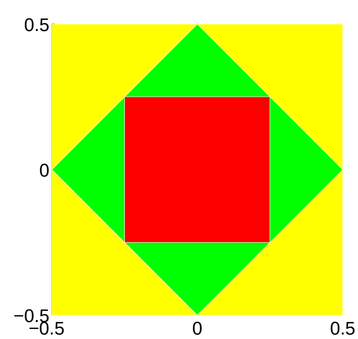




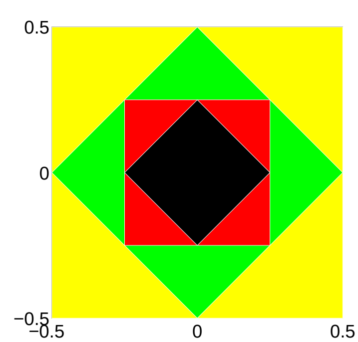


Iterated Analysis Filterbank





Iterated Synthesis Filterbank



$$\boldsymbol{D} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{or} \quad \boldsymbol{D} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Three Equivalent Definitions

- All moments up to order $(l-1)$ of the scaling function $\phi(\boldsymbol{t})$ and the wavelet $\psi(\boldsymbol{t})$ vanish:

$$\int_{\mathbf{R}^2} \boldsymbol{t}^{\boldsymbol{k}} \phi(\boldsymbol{t})\, d\boldsymbol{t} = \delta[\boldsymbol{k}], \qquad \int_{\mathbf{R}^2} \boldsymbol{t}^{\boldsymbol{k}} \psi(\boldsymbol{t})\, d\boldsymbol{t} = 0$$

for $\boldsymbol{k} = [k_1, k_2] \in \mathbf{Z}^2$, $0 \le k_1 \le l-1$, $0 \le k_2 \le l-1$, and $k_1 + k_2 \le l-1$, where $\delta[\boldsymbol{k}]$ denotes the Kronecker delta symbol and $\boldsymbol{t}^{\boldsymbol{k}}$ denotes $t_1^{k_1} t_2^{k_2}$.

- All moments up to order $(l-1)$ of the lowpass and highpass filters vanish:

$$\sum_{\boldsymbol{n} \in \mathbf{Z}^2} \boldsymbol{n}^{\boldsymbol{k}} h_l[\boldsymbol{n}] = \delta[\boldsymbol{k}], \qquad \sum_{\boldsymbol{n} \in \mathbf{Z}^2} \boldsymbol{n}^{\boldsymbol{k}} g_l[\boldsymbol{n}] = 0$$

for $\boldsymbol{k}$, k_1, and k_2 as above, where $\boldsymbol{n}^{\boldsymbol{k}}$ denotes $n_1^{k_1} n_2^{k_2}$.

- The frequency response of the lowpass filter has a zero of order l at the origin and the aliasing frequency $\boldsymbol{\pi} = [\pi, \pi]$:

$$\left. \frac{\partial^{k_1+k_2} H_l(\omega_1, \omega_2)}{\partial \omega_1^{k_1} \partial \omega_2^{k_2}} \right|_{\boldsymbol{\omega} = \mathbf{0}, \boldsymbol{\pi}} = 0$$

for k_1 and k_2 as above.

Why Vanishing Moments?

- Vanishing moments of wavelet
  - Combination of shifted scaling functions can approximate smooth functions accurately.
  - The wavelet coefficients of a smooth function decay rapidly.
  - Necessary condition for a wavelet to be smooth.
  - Correspond to zeros of $H_l(\boldsymbol{\omega})$ at $\boldsymbol{\pi}$.

- Vanishing moments of scaling function
  - The uniform samples of a smooth function can approximate its wavelet expansion coefficients accurately.
  - Imposing zero moments on a scaling function improves its symmetry and makes the filterbank close to linear-phase.
  - Correspond to zeros of $H_l(\boldsymbol{\omega})$ at $\mathbf{0}$.

Filterbank Design

- McClellan transformation-based design:
  - 1-D prototype filter:

$$H_l(\omega) = \frac{1}{2} + \sum_{k=1}^{l/2} 2h_l[2k-1]\, T_{2k-1}[\cos\omega]$$

    where $T_p[\cdot]$ denotes the pth-order Chebyshev polynomial

  - transformation function:

$$F(\boldsymbol{\omega}) = \frac{1}{2}(\cos\omega_1 + \cos\omega_2)$$

  - transformed 2-D filter:

$$H_l(\boldsymbol{\omega}) = \frac{1}{2} + \sum_{k=1}^{l/2} 2h_l[2k-1]\, T_{2k-1}[F(\boldsymbol{\omega})]$$

- Advantages:
  - simple
  - preserve the properties of 1-D filter

Sampling and Interpolation

- The synthesis scaling function in a BQCW system of any order is *cardinal*; i.e., for any $\boldsymbol{n} \in \mathbf{Z}^2$,

$$\phi(\boldsymbol{n}) = \delta[\boldsymbol{n}]$$

which is a useful property in sampling and interpolation.

- The approximation

$$\widetilde{f}(\boldsymbol{t}) = \sum_{\boldsymbol{k} \in \mathbf{Z}^2} f(\boldsymbol{D}^{-i}\boldsymbol{k})\,\phi(\boldsymbol{D}^i\boldsymbol{t} - \boldsymbol{k})$$

is exact at the quincunx sampling grid:

$$\widetilde{f}(\boldsymbol{D}^{-i}\boldsymbol{n}) = f(\boldsymbol{D}^{-i}\boldsymbol{n})$$

for any $\boldsymbol{n} \in \mathbf{Z}^2$ and any integer i.

- Wavelet expansion coefficients can be accurately approximated by function samples due to the vanishing moment conditions on the scaling function:

$$\langle f(\boldsymbol{t}), 2^{i/2}\phi(\boldsymbol{D}^i\boldsymbol{t} - \boldsymbol{k})\rangle \approx f(\boldsymbol{D}^{-i}\boldsymbol{k}).$$

Asymptotic Convergence

- The frequency responses of BQCW filters converge pointwise to the *ideal diamond-shaped halfband lowpass filters* as their orders tend to infinity:

$$\lim_{l\to\infty} H_l(\boldsymbol{\omega}) = \begin{cases} 1 & \text{if } |\omega_1| + |\omega_2| < \pi \\ 1/2 & \text{if } |\omega_1| + |\omega_2| = \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{l,l'\to\infty} \widetilde{H}_{l,l'}(\boldsymbol{\omega}) = \begin{cases} 1 & \text{if } |\omega_1| + |\omega_2| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

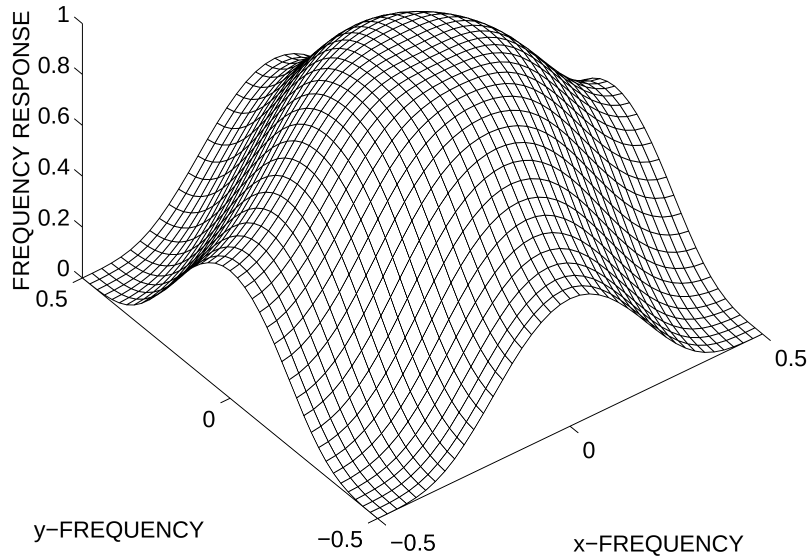
- The convergence of $H_l(\boldsymbol{\omega})$ is monotonic:

$$\begin{aligned} H_l(\boldsymbol{\omega}) \leq H_{l+1}(\boldsymbol{\omega}) \quad & \text{if } |\omega_1| + |\omega_2| \leq \pi \\ H_l(\boldsymbol{\omega}) \geq H_{l+1}(\boldsymbol{\omega}) \quad & \text{if } |\omega_1| + |\omega_2| > \pi \end{aligned}$$

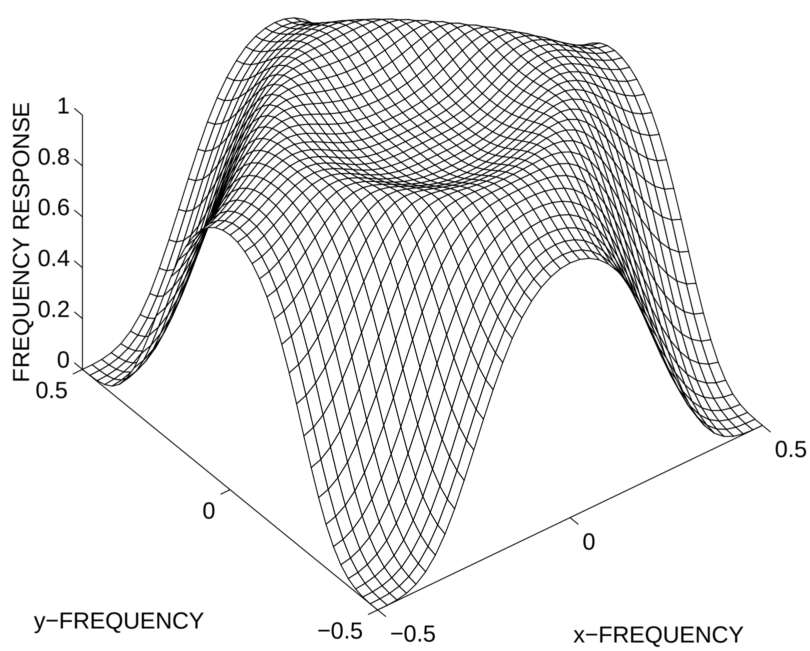
- The convergence of $H_l(\boldsymbol{\omega})$ does not exhibit any Gibbs-like phenomenon.

- The convergence of $\widetilde{H}_{l,l'}(\boldsymbol{\omega})$ exhibits a one-sided Gibbs-like phenomenon.

A Design Example

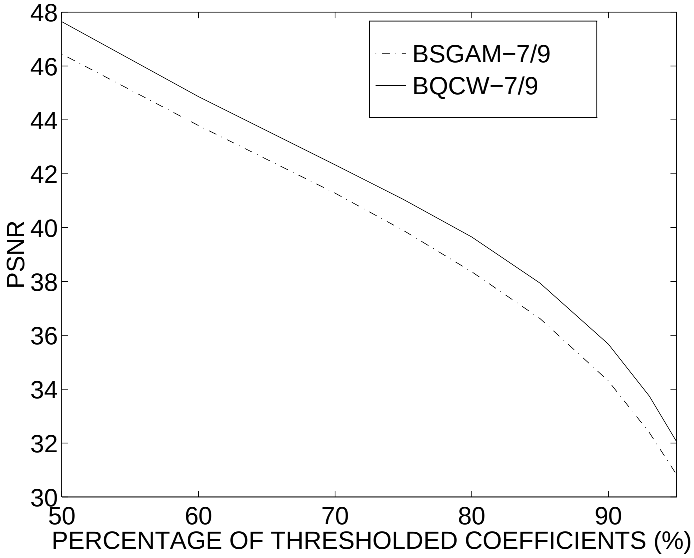


Frequency Response of Synthesis Filter

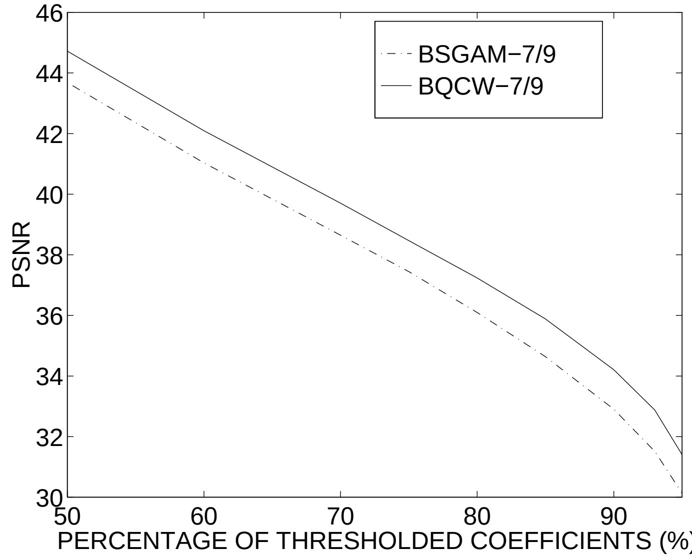


Frequency Response of Analysis Filter

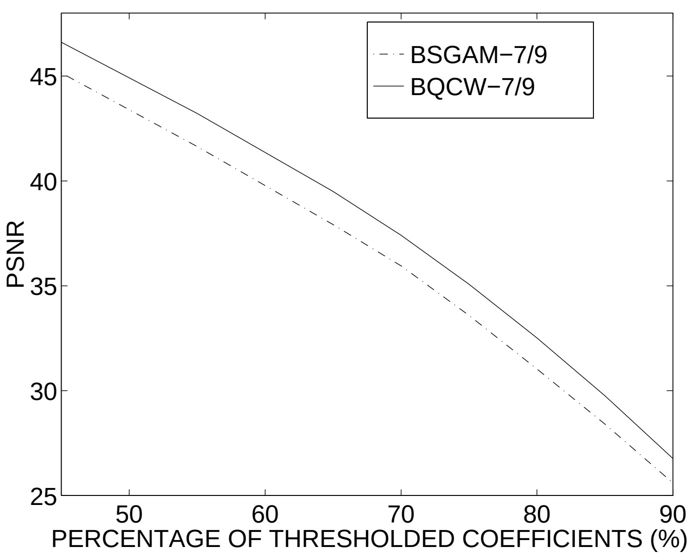
Energy Compaction



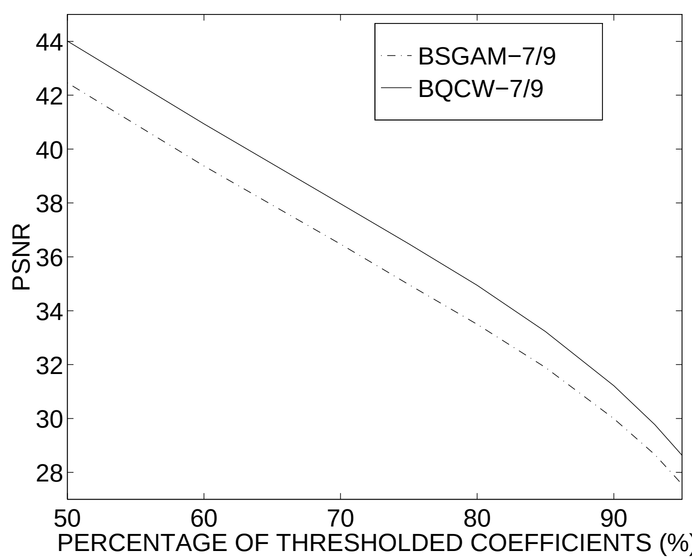
Lena



Peppers



Barbara



Goldhill

BSGAM: Barlaud-Sole-Gaidon-Antonini-Mathieu,
IEEE Trans. Image Process., Jul. 1994