Codebook Design for Noncoherent MIMO Communications Via Reflection Matrices

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***Abstract*— This paper studies the codebook design problem for noncoherent multiple input multiple output (MIMO) communications. Each codeword in the codebook is considered to be a point in a Grassmann manifold. In this paper, the codebook design is formulated as an inverse eigenvalue problem. A new algorithm using reflection matrices is proposed to obtain the optimal codebook for noncoherent block fading channels where the channel state information (CSI) is unknown at both the receiver and transmitter. The key contribution of this paper is that our algorithm is able to construct an optimal codebook via a sequence of reflection matrices.**

I. Introduction

Multiple antennas at the transmitter and receiver have been shown to yield higher data rates in wireless communications systems. In these multiple-input multiple-output (MIMO) approaches, reliable knowledge of channel state information (CSI) plays an vital role in achieving the higher data rates. CSI can be used in precoding at the transmitter or (space-time) equalization in the receiver.

For slowly varying channels, CSI can be estimated in the receiver using training data, or in the transmitter through reciprocity or feedback from the receiver. If the channel varies rapidly or the number of antennas is very large, then the receiver may not have enough time to reliably estimate the CSI before the CSI has significantly changed. In this case, one solution is to jointly estimate and predict the CSI. Another solution is to design the transmitter and receiver without assuming knowledge of the CSI. The latter is the approach taken in this paper.

When the CSI is not known at either the transmitter or receiver, the communication system is noncoherent. The choice of codebook used in the transmitter and receiver is critical for signal detection at the receiver. With a certain detection criterion, the receiver chooses one codeword from the codebook as the decoded transmitted signal. This paper proposes a codebook design via a finite sequence of reflection matrices for noncoherent communication with multiple transmit and receive antennas. We compare our proposed method with previous methods in terms of the computational complexity to create the codebook and the computational complexity to decode received signals.

A noncoherent block fading model in [1] is used in this paper. This model assumes that the propagation coefficients between transmit and receive antennas are statistically independent and unknown. The Grassmann manifold and its application in noncoherent communications was exploited in [2]. A design criterion for space-time unitary codebooks have been proposed in [3]. This design criterion requires that rows of each codeword are orthogonal if each codeword is represented by a matrix. The codebook design for noncoherent MIMO communications employs this criterion. The previous methods as well as our proposed algorithm all comply with this criterion.

Algorithm	Searching complexity	*Notes*
DFT [4]	O(2^{RTM_t})	
Coherent codes [5]	O($2^{RT}(T-M_t)$)	
PSK [6]	O($2^{RT}M_t$)	$T=2M_t$
Orthogonal matrices [6]	O($2^{RT}\log_2 M_t$)	$T=2M_t, M_t \in \{1,2,4,8\}$
Training [7]	O($2^{RT}T$)	

TABLE I

NONCOHERENT CODEBOOK DESIGN FOR MIMO COMMUNICATIONS. R: TRANSMIT DATA RATE IN UNITS OF BITS/SYMBOL PERIOD. T: COHERENCE TIME OF THE CHANNEL IN UNITS OF SYMBOL PERIODS. M_t: NUMBER OF TRANSMIT ANTENNAS.

A simple systematic code construction by the discrete Fourier transform (DFT) method was proposed in [4]. A method to use existing coherent codes to design noncoherent codewords was proposed in [5]. Phase-shift keying (PSK) constellations were used to construct the noncoherent codebook in [6]. Another method in [6] to design the noncoherent codebook was based on orthogonal matrices. There is a training method to design the noncoherent codebook in [7].

We consider the search complexity to obtain an entire codebook, i.e. how many steps we must take to find all codewords. We compare these methods in Table I. R is the transmit data rate in units of bits/symbol period, T is the coherence time of the channel in units of symbol periods, M_t is the number of transmit antennas and M_r is the number of receive antennas. Here, $T \geq 2M_t$ as explained in Section II. As seen in Table I, "DFT" method needs an exhaustive search. The "PSK" and "Orthogonal matrices" methods have the second lowest and lowest search complexity, respectively but limit values of T and M_t. This motivates us to propose a design method that has low search complexity but supports arbitrary numbers of transmit antennas and channel coherence time.

Two numerical algorithms using the plane rotation matrices on the inverse eigenvalue problem in matrix analysis were presented in [8]. They presented pure math algorithms on matrix analysis and did not explicity solve the noncoherent MIMO codebook design problem. We notice that two rotation operations with rotating angle θ are equivalent to one reflection operation with reflecting angle 2θ. Thus, this motivates us to make use of reflection operations to solve the inverse eigenvalue problem.

In this paper, we formulate the codebook design for noncoherent MIMO communications as an inverse eigenvalue problem. We propose an algorithm using the reflection matrices to solve the corresponding inverse eigenvalue problem. The algorithm reduces the search complexity compared to other algorithms with arbitrary numbers of transmitter antennas and channel coherence time. We analyze and compare the searching and decoding computational complexity among these different algorithms, including our algorithm, in Section IV.

The outline of the paper is given as follows. In Section II, the necessary background and definitions are provided. Section III introduces the necessary theorem and presents the algorithm. In Section IV, the simulation results are provided. We conclude and suggest the future work in Section V.

II. Background

Throughout this paper, we denote $(.)^H$ as the conjugate transpose of a matrix; $\|.\|_F$ as the Frobenius norm of the matrix; $\mathbb{C}^T$ as a *T*-dimensional complex vector; and $\mathbb{C}^{M\times T}$ as an $M \times T$ complex matrix.

A. System Model

We assume the system has M_t transmit and M_r receive antennas. We also assume the $M_r \times M_t$ MIMO channel matrix remains constant for T consecutive symbol periods, and changes to another independent realization in the next T symbol periods, where T is denoted as the coherence time of the channel. Within the coherence time, the system model is

$$\mathbf{Y} = \mathbf{HX} + \mathbf{W} \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{M_t\times T}$ and $\mathbf{Y} \in \mathbb{C}^{M_r\times T}$. $\mathbf{H} \in \mathbb{C}^{M_r\times M_t}$ is the channel matrix and the entries of $\mathbf{W} \in \mathbb{C}^{M_r\times T}$ are i.i.d. Additive White Gaussian Noise(AWGN) is distributed according to $\mathcal{CN}(0, N_0)$.

B. Grassmann Manifold

In [2], the geometric of the Stiefel and Grassmann manifolds are applied for noncoherent communications. The Stiefel manifold *S(T,M)* is the set of all *M*-dimensional subspaces in a *T*-dimensional space. We present one element, an *M*-dimensional subspace in *S(T,M)*, as an $M \times T$ matrix. Two elements in *S(T,M)* are equivalent if their rows span the same subspace. We classify all elements of *S(T,M)* into a same one class if they are equivalent.

The Grassmann manifold *G(T,M)* is defined as the set of all *M*-dimensional classes in *S(T,M)*. Thus, any one point in *G(T,M)* represents a class of $M\times T$ unitary matrices. An $M_t \times T$ transmitted block $\mathbf{X}$ is represented as the product of $\mathbf{X}_g$, a point in $G(T, M_t)$ spanned by the rows of *X*, with a $M_t \times M_t$ matrix $\mathbf{C}_x$ [2], i.e. $\mathbf{X} = \mathbf{C}_x\mathbf{X}_g$. The transformation $\mathbf{X} \to (\mathbf{C}_x, \mathbf{X}_g)$ is a change of coordinates $\mathbb{C}^{M_t\times T} \to \mathbb{C}^{M_t\times M_t} \times G(T, M_t)$. When the signal to noise ratio (SNR) is high, it has been shown that the channel matrix $\mathbf{H}$ only changes $\mathbf{C}_x$ and keeps $\mathbf{X}_g$ fixed in noncoherent communications [2]. Thus, codewords can be regarded as points in $G(T, M_t)$ [2]. Since $G(T, M_t)$ and $G(T, T - M_t)$ are complementary, we can assume that $M_t \leq \frac{T}{2}$.

If the distance measure between matrices $\mathbf{P}$ and $\mathbf{Q}$ is differentiable everywhere, then it gives structure to an optimization problem or adaptive formulation that uses it. A relevant distance measure on a Grassmann manifold is the chordal distance between $\mathbf{P}$ and $\mathbf{Q}$ in [9] is given by

$$d_c(\mathbf{P}, \mathbf{Q}) = \sqrt{\sum_{i=0}^{M_t} \sin^2\theta_i}. \tag{2}$$

θ_i, for $i = 1, .., M$ where $0 \leq \theta_i \leq \frac{\pi}{2}$, is the principal angles between two points $\mathbf{P}$ and $\mathbf{Q}$ in $G(T, M_t)$.

The square of chordal distance is differentiable everywhere. In [9], it has been also shown that the chordal distance can be expressed as

$$d_c(\mathbf{P}, \mathbf{Q}) = \sqrt{M_t - \left\|\mathbf{PQ}^H\right\|_F^2}. \tag{3}$$

The computational complexity in (3) is dominated by the matrix product $\mathbf{PQ}^H$, which is relatively simple to compute.

C. Codebook Model

A codebook with *N* codewords on Grassmann manifold is defined as

$$\mathbf{S} = \{\mathbf{X}_i, i = 1, 2, ..., N\} \tag{4}$$

The codeword $\mathbf{X}_i \in \mathbb{C}^{M_t\times T}$ is a point in $G(T, M_t)$. To achieve low pairwise error probability in [10], any two distinct $\mathbf{X}_i$ and $\mathbf{X}_j$ should be as far apart as possible. This indicates that the minimum distance should be as large as possible. Thus, an optimal codebook is defined as

$$\begin{aligned} \mathbf{S} &= \arg\max_{\mathbf{S}\subseteq G(T,M_t)} \{\min d_c(\mathbf{X}_i, \mathbf{X}_j)\} & (5)\\ &= \arg\min_{\mathbf{S}\subseteq G(T,M_t)} \left\{\max \left\|\mathbf{X}_i\mathbf{X}_j^H\right\|_F^2\right\} & (6)\end{aligned}$$

where $i \neq j$ and $1 \leq i, j \leq N$.

D. Gram Matrix

A Gram matrix denoted by $\mathbf{G}$ for a codebook $\mathbf{S}$ in (4) is defined as

$$\begin{aligned} \mathbf{X}_s &= \begin{bmatrix} \mathbf{X}_1^H & \mathbf{X}_2^H & \cdots & \mathbf{X}_N^H \end{bmatrix}^H & (7)\\ \mathbf{G} &= \mathbf{X}_s\mathbf{X}_s^H & (8)\end{aligned}$$

where $\mathbf{X}_s$ is the $M_t N \times T$ matrix formed by stacking all elements in the codebook. With (8), $\mathbf{G}$ is an $M_t N \times M_t N$ matrix containing N^2 block matrices. Each block matrix in $\mathbf{G}$ is denoted as $\mathbf{G}_{i,j} = \mathbf{X}_i \mathbf{X}_j^H$. Notice that the rank of $\mathbf{G}$ is T. Since any $\mathbf{X}_i$, $1 \leq i \leq N$ is a unitary matrix, $\mathbf{G}_{i,i}$ is an $M_t \times M_t$ matrix with identical diagonal elements, and the trace of $\mathbf{G}$ equals $M_t N$. Further, $\mathbf{G}$ is a Hermitian matrix.

E. Total Squared Correlation

To find a codebook with the maximum minimum distance between two different codewords, we define the Total Squared Correlation (TSC) of a matrix $\mathbf{X}_s$ in (7) which is defined as

$$\mathrm{TSC}(\mathbf{X}_s) = \left\|\mathbf{X}_s \mathbf{X}_s^H\right\|_F^2 = \|\mathbf{G}\|_F^2 = M_t N + 2 \sum_{i \neq j} \left\|\mathbf{X}_i \mathbf{X}_j^H\right\|_F^2 \tag{9}$$

Thus, one way to minimize the maximum value of $\left\|\mathbf{X}_i \mathbf{X}_j^H\right\|_F^2$,where $i \neq j$ and $1 \leq i, j \leq N$ is to minimize $\mathrm{TSC}(\mathbf{X}_s)$. The codebook design now is reduced to the design of a Gram Matrix $\mathbf{G}$ such that $\mathrm{TSC}(\mathbf{X}_s)$ is minimized. Notice that the rank of $\mathbf{X}_s$ is T. The T non-zero eigenvalues of $\mathbf{G}$ are denoted as $\lambda_1, \lambda_2, .., \lambda_T$. Assuming that the total transmission power P for the entire codebook is fixed, we have

$$P = \sum_{i=1}^{T} \lambda_i. \tag{10}$$

On the other hand, we assume that each codeword is equally likely and each single dimension of each codeword is allocated with the same power p. The power of each codeword is equal to pM_t since each codeword has M_t dimensions. The total transmission power of the entire codebook

$$P = \mathrm{Trace}\{\mathbf{G}\} = pM_t N. \tag{11}$$

Using (9), (10) and (11), we write

$$\begin{aligned}
\mathrm{TSC}(\mathbf{X}_s) &= \left\|\mathbf{X}_s \mathbf{X}_s^H\right\|_F^2 && (12)\\
&= \sum_{i=1}^{T} \lambda_i^2 && (13)\\
&= \sum_{i=1}^{T} (\lambda_i - \frac{P}{T})^2 + \frac{2P^2}{T} - \frac{P^2}{T^2} && (14)\\
&\geq \frac{2P^2}{T} - \frac{P^2}{T^2} && (15)
\end{aligned}$$

The equality holds in the last inequality when $\lambda_i = \frac{P}{T}$ for $i = 1, ..., T$. Therefore, the Gram matrix $\mathbf{G}$ for the optimal codebook should have T identical non-zero eigenvalues, which are all equal to $\frac{P}{T}$. Let $\boldsymbol{\lambda}$ denote the vector containing all the eigenvalues of $\mathbf{G}$ in ascending order, i.e.

$$\boldsymbol{\lambda} = \left[\underbrace{0 \quad 0 \quad \cdots \quad 0}_{M_t N - T} \underbrace{\frac{P}{T} \quad \frac{P}{T} \quad \cdots \quad \frac{P}{T}}_{T} \right]. \tag{16}$$

Meanwhile, notice that $\mathbf{G}$ contains N diagonal block matrices and the diagonal entries of any diagonal block matrix $\mathbf{G}_{i,i}$, $1 \leq i \leq N$ are powers for single dimensions, and equal to $p = \frac{P}{M_t N}$ in (11) by the assumption of equally likely codewords. We denote the diagonal vector of $\mathbf{G}$ as

$$\boldsymbol{\omega} = \underbrace{\left[\frac{P}{M_t N} \quad \frac{P}{M_t N} \quad \cdots \quad \frac{P}{M_t N} \right]}_{M_t N}. \tag{17}$$

Hence, the codebook design can be considered as an inverse eigenvalue problem for a given P, i.e. how to construct a Hermitian matrix $\mathbf{G}$ given its eigenvalues in (16) and diagonal elements in (17). An algorithm to solve this problem will be discussed in the next section.

III. Constructing the Codebook

A. Majorization

Before we discuss the algorithm, we first refer to a relationship between two vectors called *Majorization* in [11].

Definition "Let α and β be two given M-dimensional vectors. The vector β is said to majorize the vector α if $\min\left\{\sum_{j=1}^{k} \beta_{i_j} : 1 \leq i_1 < \cdots < i_k \leq M\right\} \geq \min\left\{\sum_{j=1}^{k} \alpha_{i_j} : 1 \leq i_1 < \cdots < i_k \leq M\right\}$ for all $k = 1, 2, ..., M$ with equality for $k = M$."

It turns out that majorization precisely defines the relationship between the diagonal vector of a Hermitian matrix and the vector composed of its eigenvalues in [11]. We can see that $\boldsymbol{\omega}$ in (17) majorizes $\boldsymbol{\lambda}$ in (16).

(Schur-Horn's Theorem) "The diagonal entries of a Hermitian matrix majorizes its eigenvalues. Conversely, if $\boldsymbol{\omega}$ majorizes $\boldsymbol{\lambda}$, there exists a Hermitian matrix with diagonal elements listed by $\boldsymbol{\omega}$ and eigenvalues listed by $\boldsymbol{\lambda}$" [11].

It can be verified that the Gram matrix G for the optimal codebook satisfies the Schur-Horn's theorem. Further, $\boldsymbol{\omega}$ majorizes any other vector with sum of all elements being P.

B. Reflection Matrix

A reflection matrix can be used to modify the specified diagonal entries of a Hermitian matrix while preserving its eigenvalues. A 2×2 reflection matrix is defined by

$$\mathbf{F} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{18}$$

where $\theta \in \left(0, \frac{\pi}{2}\right)$ is a reflection angle. Obviously, $\mathbf{F}$ is a unitary symmetric matrix. For any given four non-negative numbers $x_1 \leq y_1 \leq y_2 \leq x_2$ and a matrix $\mathbf{A}$, $\begin{bmatrix} x_1 & x_{12}^* \\ x_{12} & x_2 \end{bmatrix}$, a reflection matrix $\mathbf{F}$ can be explicitly constructed so that the first diagonal element of $\mathbf{FAF}$ is equal to y_1. The matrix equation is shown by

$$\mathbf{FAF} = \begin{bmatrix} y_1 & y_{12}^* \\ y_{12} & \hat{y}_2 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x_1 & x_{12}^* \\ x_{12} & x_2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{20}$$

Thus, we have that following equations:

$$y_1 = x_1 \cos^2\theta + 2Re\left\{x_{12}\right\}\sin\theta\cos\theta + x_2\sin^2\theta \tag{21}$$

$$= \frac{x_1 - x_2}{2}\cos 2\theta + Re\left\{x_{12}\right\}\sin 2\theta + \frac{x_1 + x_2}{2} \tag{22}$$

and

$$\hat{y}_2 = x_1 + x_2 - y_1 \tag{23}$$

The equation (22) is a second-order equation with one variable $\cos 2\theta$. Solving (22), we can obtain θ where $0 \le \cos 2\theta \le 1$ and $\theta \in (0, \frac{\pi}{2})$. Then the reflection matrix $\mathbf{F}$ can be constructed with (18) using θ.

In the next subsection, we propose an algorithm to construct a Hermitian matrix given its diagonal entries and eigenvalues for codebook design in noncoherent communications.

C. Algorithm

Assuming that $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$ are two arbitrary vectors with their elements in ascending order, and $\boldsymbol{\omega}$ majorizes $\boldsymbol{\lambda}$. To construct $\mathbf{G}$, we use a diagonal matrix with diagonal vector $\boldsymbol{\lambda}$ as the initial matrix. With a majorization relationship, we can find the indices $i < j$ so that $\lambda_i < \omega_i \le \omega_j < \lambda_j$. From (20), we can apply a reflection matrix to transform λ_i to ω_i and λ_j to $\lambda_i + \lambda_j - \omega_i$ with the rest of the diagonal entries unchanged. A permutation matrix can be used to rearrange the rest of the diagonal entries in ascending order. The permutation matrix is also a unitary matrix that preserves the eigenvalues of a Hermitian matrix. It can be verified that $\boldsymbol{\omega}$ still majorizes the updated $\boldsymbol{\lambda}$ and the updated $\boldsymbol{\lambda}$ majorizes the old one because $\lambda_i < \{\omega_i, \lambda_i + \lambda_j - \omega_i\} < \lambda_j$; i.e., the difference between two elements becomes smaller. Thus, based on this criterion, we can update $\boldsymbol{\lambda}$ by finite steps until $\boldsymbol{\lambda} = \boldsymbol{\omega}$ finally. We use one reflection matrix and a set of permutation matrices in each step. The algorithm is described as follows:

Algorithm Suppose that the codebook $\mathbf{S}$ is defined in (4) and each codeword has identical power $p = \frac{P}{M_t N}$, where P is the total transmission power for the codebook. Let $\mathbf{G}$ be an initial $M_t N \times M_t N$ diagonal Hermitian matrix. The algorithm in Table II obtains an optimal codebook from the initial matrix $\mathbf{G}$.

Step	*Operations*
1	Set $i = 1$ and $\mathbf{G}$=diag($\boldsymbol{\lambda}$) where $\boldsymbol{\lambda}$ is given by (16)
2	Find the smallest $j > i$ such that$\lambda_{j-1} \le p \le \lambda_j$.
3	Swap λ_{i+1} and λ_j in $\mathbf{G}$ by a permutation matrix.
4	Reflect $\begin{bmatrix} \lambda_i & \lambda^*_{i,i+1} \\ \lambda_{i+1,i} & \lambda j \end{bmatrix}$ to $\begin{bmatrix} p & \hat{\lambda}^*_{i,i+1} \\ \hat{\lambda}_{i+1,i} & \lambda_i + \lambda j - 1 \end{bmatrix}$ by a Reflection Matrix $\mathbf{F}$ defined in (18).
5	Arrange the rest of the diagonal entries of $\mathbf{G}$, $\begin{bmatrix} \lambda_i + \lambda_j - 1 & \lambda_{i+2} & \cdots & \lambda_{j-1} & \lambda_{i+1} & \lambda_{j+1} & \cdots & \lambda_{M_t N} \end{bmatrix}$ in ascending order by multiplying symmetric permutation matrices.
6	$i := i + 1$. If $i \le M_t N - 1$, go to the Step 2.
7	Decompose $\mathbf{G} = \Sigma\Xi\Sigma^H$ by the eigenvalue decomposition since $\mathbf{G}$ is a square Hermitian matrix. Take $\mathbf{X}_s = \Sigma\Xi^{\frac{1}{2}}$.
8	Split $\mathbf{X}_s$ by rows into N codewords $\mathbf{X}_i \in \mathbb{C}^{M_t \times T}$, for $i = 1, ..., N$ which provides the codebook for noncoherent MIMO communications we desire.

TABLE II

CODEBOOK DESIGN FOR NONCOHERENT MIMO COMMUNICATIONS VIA REFLECTION MATRICES

Algorithm	Searching complexity	*Notes*
DFT [4]	O(2^{RTM_t})	
Coherent codes [5]	O($2^{RT}(T - M_t)$)	
PSK [6]	O($2^{RT}M_t$)	$T = 2M_t$
Orthogonal matrices [6]	O($2^{RT}\log_2 M_t$)	$T = 2M_t, M_t \in \{1, 2, 4, 8\}$
Training [7]	O($2^{RT}T$)	
Reflection matrices	O($2^{RT}M_t$)	

TABLE III

THE COMPARISON FOR THE SEARCH COMPLEXITY. R: TRANSMIT DATA RATE IN UNITS OF BITS/SYMBOL PERIOD. T: COHERENCE TIME OF THE CHANNEL IN UNITS OF SYMBOL PERIODS. M_t: NUMBER OF TRANSMIT ANTENNAS. NOTE THAT $T \ge 2M_t$.

IV. SIMULATION

First, we compare the codebook generated by our algorithm with the codebook in [12]. For the case $M_t = 1, N = 4$, $P = 4$ and $T = 3$, $\mathbf{X}_s$ from our codebook design algorithm is generated as:

$$\mathbf{X}_s = \begin{bmatrix} -1.0000 & 0.0000 & 0.0000 \\ 0.3333 & 0.5369 & -0.7750 \\ 0.3333 & -0.9396 & -0.0775 \\ 0.3333 & 0.4027 & 0.8525 \end{bmatrix}$$

The codebook in [12] is

$$\hat{\mathbf{X}}_s = \begin{bmatrix} -0.7416 & 0.1716 & -0.6485 \\ -0.1266 & 0.5723 & 0.8102 \\ 0.8759 & 0.2195 & -0.4296 \\ -0.0077 & -0.9634 & 0.2680 \end{bmatrix}.$$

Here, $\mathbf{X}_s\mathbf{Q} = \hat{\mathbf{X}}_s$, where $\mathbf{Q}$ is a unitary matrix. Thus, $\mathbf{X}_s$ and $\hat{\mathbf{X}}_s$ are equivalent in that $\mathbf{X}_s$ and $\hat{\mathbf{X}}_s$ span the same space.

For the case, $M_t = 2, N = 4$, $P = 8$ and $T = 8$, our algorithm finds $\mathbf{X}_s$ for our codebook to be equal to an $\mathbf{I}_8$, an 8×8 identity matrix. This simple result tells us that we only need to group the standard basis for 8-dimensional Euclidean space into four groups; i.e., each group has two bases. The optimal codebook for this case consists of these four groups.

Second, we compare the search and computational complexity among different design algorithms. R is the transmit data rate in units of bits per symbol period. N is the size of codebook and equals $\lceil 2^{RT} \rceil$. We first discuss the search complexity, i.e. how many steps to construct a codebook. Since our algorithm only needs to compute at most $M_t N$ reflection matrices, the search complexity for the entire codebook is O($2^{RT}M_t$) which is the second lowest in these methods shown in Table III. The lowest one is “Orthogonal matrices” method but it limits $M_t \in \{1, 2, 4, 8\}$. Our method does not have this limit.

Finally, the computational complexity of decoding is discussed. The generalized likelihood ratio test (GLRT) detection for noncoherent communications at the receiver was derived in [13]. In our method, we use the GLRT detection and obtain

Algorithm	Decoding method	Computational complexity
DFT [4]	GLRT	$O(2^{RT})$
Coherent codes [5]	GLRT	$O(2^{RT})$
PSK [6]	ML	$O(M_t M_r)$
Orthogonal matrices [6]	ML	$O(M_t^2 M_r)$
Training [7]	MMSE	$O(M_t^3 M_r^3)$
Reflection matrices	GLRT	$O(2^{RT})$

TABLE IV

THE COMPARISON FOR THE COMPUTATIONAL COMPLEXITY IN DECODING. R: TRANSMIT DATA RATE IN UNITS OF BITS/SYMBOL PERIOD. T: COHERENCE TIME OF THE CHANNEL IN UNITS OF SYMBOL PERIODS. M_t: NUMBER OF TRANSMIT ANTENNAS. M_r: NUMBER OF RECEIVE ANTENNAS. NOTE THAT $T \geq 2M_t$.

the decoder decision in [13] as follows

$$\hat{\mathbf{X}} = \arg \max_{\mathbf{X} \in \textbf{codebook S}} \mathrm{Trace}\left(\mathbf{Y}^H \mathbf{Y} \mathbf{X}^H \mathbf{X}\right) \qquad (24)$$

Compared with other methods in Table IV, our method does not have low computational complexity in decoding. We hope to improve it in future. The "PSK" method has the lowest computational complexity but limits $T = 2M_t$ as shown in Table III.

V. CONCLUSION

In this paper, we analyze the codebook design for noncoherent MIMO communications using a Grassmann manifold. We show that designing a codebook for noncoherent MIMO communications is equivalent to finding an optimal Gram matrix with given diagonal entries and eigenvalues. Hence, the problem can be considered as an inverse eigenvalue problem. A sequence of reflection matrices can be used to construct the Gram matrix. An algorithm using reflection matrices is proposed to generate the optimal codebook. We compare search complexity for constructing a codebook and computational complexity for decoding at the receiver for noncoherent MIMO communications of different codebook design methods.

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