Lemma 4.2 that follows states that if there is a chain of events from s to t then $s.v \leq t.v$. Note that the proof of Lemma 4.2 does not use the initial conditions. Thus it holds independent of the initial values of the vectors.

Lemma 4.2 $s \rightarrow t \Rightarrow s.v \leq t.v$.

Proof: It is sufficient to show that for all $k > 0$: $s \stackrel{k}{\rightarrow} t \Rightarrow s.v \leq t.v$. We use induction on k.
Base $(k = 1)$:

$s \stackrel{1}{\rightarrow} t$
$\Rightarrow$ { definition of ml }
$s \prec_{im} t \ \vee \ s \rightsquigarrow t$
$\Rightarrow$ { expand $s \prec_{im} t$ and $s \rightsquigarrow t$ }
$(s, internal, t) \ \vee \ (s, send, t) \ \vee \ (\exists u : (s, rcv(u), t))$
$\vee \ (\exists u : (u, rcv(s), t))$
$\Rightarrow$ { send, rcv, and internal rules }
$(s.v = t.v) \ \vee \ (s.v < t.v) \ \vee \ (s.v \leq t.v)$
$\vee \ (s.v \leq t.v)$
$\Rightarrow$ { simplify }
$s.v \leq t.v$

Induction: $(k > 1)$

$s \stackrel{k}{\rightarrow} t \ \wedge \ (k > 1)$
$\Rightarrow$ { definition of ml }
$(\exists u : s \stackrel{k-1}{\rightarrow} u \ \wedge \ u \stackrel{1}{\rightarrow} t)$
$\Rightarrow$ { induction hypothesis }
$(\exists u : s.v \leq u.v \ \wedge \ u.v \leq t.v)$
$\Rightarrow$ { simplify }
$s.v \leq t.v$

■

Lemma 4.4 states that if two states s and t are on different processes, and s does not happen before t, then $t.v[s.p] < s.v[s.p]$. Our formal proof of this lemma is nontrivial. We first define the notion of rank of a state t.

Definition 4.3 (Rank of a state) *The rank of a state t is equal to the length of the longest chain from an initial state to t.*

$$rank(t) = ml(Init, t).$$

Lemma 4.4 $(\forall s, t : s.p \neq t.p : s \not\rightarrow t \Rightarrow t.v[s.p] < s.v[s.p])$.

Proof: The proof is by induction on $k = rank(t)$.
Base $(k = 0)$:

$$\begin{array}{ll} & s \not\rightarrow t \ \wedge \ s.p \neq t.p \\ \Rightarrow & \{\ rank(t) = 0\ \} \\ & initial(t) \ \wedge \ s.p \neq t.p \\ \Rightarrow & \{\ \text{let } u \text{ be initial state in } s.p\ \} \\ & initial(t) \ \wedge \ s.p \neq t.p \ \wedge \\ & (\exists u : initial(u) \ \wedge \ u.p = s.p : u = s \ \vee \ u \rightarrow s) \\ \Rightarrow & \{\ \text{lemma 4.2}\ \} \\ & initial(t) \ \wedge \ s.p \neq t.p \ \wedge \\ & (\exists u : initial(u) \ \wedge \ u.p = s.p : u.v = s.v \ \vee \ u.v \leq s.v) \\ \Rightarrow & \{\ \text{rule for initial states}\ \} \\ & t.v[s.p] = 0 \\ & \wedge \ (\exists u : u.v[s.p] = 1 : u.v = s.v \ \vee \ u.v \leq s.v) \\ \Rightarrow & \{\ \text{simplify}\ \} \\ & t.v[s.p] < s.v[s.p] \end{array}$$

Induction: $(k > 0)$

$$\begin{array}{ll} & s \not\rightarrow t \ \wedge \ s.p \neq t.p \ \wedge \ rank(t) > 0 \\ \Rightarrow & \{\ \text{let } u \text{ satisfy } u \prec_{im} t,\ u \text{ exists because } \neg initial(t)\ \} \\ & s \not\rightarrow t \ \wedge \ s.p \neq t.p \ \wedge \ u.p = t.p \ \wedge \ u \prec_{im} t \\ \Rightarrow & \{\ \text{definition of rank}\ \} \\ & s \not\rightarrow u \ \wedge \ rank(u) < k \ \wedge \ u.p \neq s.p \ \wedge \ u \prec_{im} t \\ \Rightarrow & \{\ \text{inductive hypothesis}\ \} \\ & u.v[s.p] < s.v[s.p] \ \wedge \ u \prec_{im} t \\ \Rightarrow & \{\ \text{expand } u \prec_{im} t\ \} \\ & u.v[s.p] < s.v[s.p] \\ & \wedge \ ((u, internal, t) \ \vee \ (u, send, t) \ \vee \ (u, rcv(w), t)) \end{array}$$

Consider each disjunct separately:

Case 1: $(u, internal, t)$

$$
\begin{array}{ll}
 & u.v[s.p] < s.v[s.p] \ \wedge \ (u, internal, t) \\
\Rightarrow & \{ \text{ internal event rule } \} \\
 & u.v[s.p] < s.v[s.p] \ \wedge \ t.v = u.v \\
\Rightarrow & \{ \text{ simplify } \} \\
 & t.v[s.p] < s.v[s.p]
\end{array}
$$

Case 2: $(u, send, t)$

$$
\begin{array}{ll}
 & u.v[s.p] < s.v[s.p] \ \wedge \ (u, send, t) \\
\Rightarrow & \{ \text{ Send rule, } s.p \neq t.p \ \} \\
 & u.v[s.p] < s.v[s.p] \ \wedge \ t.v[s.p] = u.v[s.p] \\
\Rightarrow & \{ \text{ simplify } \} \\
 & t.v[s.p] < s.v[s.p]
\end{array}
$$

Case 3: $(u, rcv(w), t)$

$$
\begin{array}{ll}
 & u.v[s.p] < s.v[s.p] \ \wedge \ (u, rcv(w), t) \\
\Rightarrow & \{ \ rcv \text{ rule } \} \\
 & u.v[s.p] < s.v[s.p] \ \wedge \ (u, rcv(w), t) \\
 & \wedge \ (t.v[s.p] = u.v[s.p] \ \vee \ t.v[s.p] = w.v[s.p]) \\
\Rightarrow & \{ \text{ simplify } \} \\
 & t.v[s.p] < s.v[s.p] \\
 & \vee \ ((u, rcv(w), t) \ \wedge \ t.v[s.p] = w.v[s.p])
\end{array}
$$

For case 3, it suffices to prove the following two cases.
Case 3A: $w.p = s.p$

$$
\begin{array}{ll}
 & t.v[s.p] = w.v[s.p] \ \wedge \ (u, rcv(w), t) \ \wedge \ w.p = s.p \\
\Rightarrow & \left\{ \begin{array}{l} \text{let } x \text{ satisfy } w \prec_{im} x, \ x \text{ exists because} \\ w \rightsquigarrow t \text{ implies } \neg final(w) \end{array} \right\} \\
 & t.v[s.p] = w.v[s.p] \ \wedge \ (w, send, x) \ \wedge \ w.p = s.p \\
\Rightarrow & \{ \text{ otherwise } s \rightarrow t \ \} \\
 & t.v[s.p] = w.v[s.p] \ \wedge \ (w, send, x) \ \wedge \ w.p = s.p \\
 & \wedge \ w \rightarrow s \\
\Rightarrow & \{ \text{ because } w \prec_{im} x \ \} \\
 & t.v[s.p] = w.v[s.p] \ \wedge \ (w, send, x) \ \wedge \ w.p = s.p \\
 & \wedge \ (x = s \ \vee \ x \rightarrow s) \\
\Rightarrow & \{ \text{ send rule } \} \\
 & t.v[s.p] = w.v[s.p] \ \wedge \ w.v[s.p] < x.v[s.p] \\
 & \wedge \ (x = s \ \vee \ x \rightarrow s) \\
\Rightarrow & \{ \text{ Lemma 4.2 } \} \\
 & t.v[s.p] = w.v[s.p] \ \wedge \ w.v[s.p] < x.v[s.p]
\end{array}
$$

$$
\begin{array}{ll}
 & \wedge\ x.v \leq s.v \\
\Rightarrow & \{\ \text{simplify}\ \} \\
 & t.v[s.p] < s.v[s.p]
\end{array}
$$

Case 3B: $w.p \neq s.p$

$$
\begin{array}{ll}
 & t.v[s.p] = w.v[s.p]\ \wedge\ (u, rcv(w), t)\ \wedge\ w.p \neq s.p \\
\Rightarrow & \{\ \text{definition of rank}\ \} \\
 & t.v[s.p] = w.v[s.p]\ \wedge\ w.p \neq s.p\ \wedge\ s \not\rightarrow w \\
 & \wedge\ rank(w) < k \\
\Rightarrow & \{\ \text{inductive hypothesis}\ \} \\
 & t.v[s.p] = w.v[s.p]\ \wedge\ w.v[s.p] < s.v[s.p] \\
\Rightarrow & \{\ \text{simplify}\ \} \\
 & t.v[s.p] < s.v[s.p]
\end{array}
$$

■

Lemma 4.5 is a refinement of Lemma 4.2 for the case when $s.p \neq t.p$, in which case $s.v < t.v$.

Lemma 4.5 $(\forall s, t : s.p \neq t.p : s \rightarrow t \Rightarrow s.v < t.v)$

Proof: From Lemma 4.2, we get that $s.v \leq t.v$. Furthermore, $s \rightarrow t$ implies that $t \not\rightarrow s$. From $t \not\rightarrow s$, $s.p \neq t.p$ and Lemma 4.4, we get that $s.v[t.p] < t.v[t.p]$. Combining this with $s.v \leq t.v$, we get the desired result.

■

Theorem 4.6 states the property that we set out to prove at the beginning of this section.

Theorem 4.6 $(\forall s, t : s.p \neq t.p : s \rightarrow t \Leftrightarrow s.v < t.v)$

Proof: Immediate from Lemmas 4.4 and 4.5.

■