Rectangles are Better than Chains for Encoding Partially Ordered Sets

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Abstract.
Partial orders have applications in various areas in computer science such as distributed systems, object oriented languages, knowledge representation systems and databases. In this paper we present a new technique to encode partially ordered sets or posets. Specifically, we use the family of two-dimensional posets, which we refer to as *rectangles*, to encode a given poset; an element x is less than another element y in the partial order if and only if x is less than y in at least one of the rectangular orders and y is not less than x in any of the rectangular orders. The least number of rectangular orders needed to represent a poset is referred to as its *rectangular dimension*. We establish upper bounds on rectangular dimension of posets in general and special families of posets in particular. We also provide examples of families of posets that can be encoded *optimally* using our technique but require much larger number of bits per element using other techniques such as adjacency matrix, adjacency list and dimension theory.

Keywords: partial order, poset, dimension, encoding, graph, implicit representation

1. Introduction

The importance of managing the transitive closure of relationships has been acknowledged in several areas in computer science. Examples include the subsumption relation in knowledge representation systems (Agrawal et al., 1989; Ellis, 1995) and object oriented languages (OOL) (Caseau, 1993), the causality relation in distributed systems (Fidge, 1991; Mattern, 1989), and the transitive closure query in deductive database systems (Agrawal et al., 1989). These hierarchies of objects are expected to grow larger in the future. Therefore it will become important to have techniques that enable such hierarchies to be stored

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in a compact manner while, at the same time, admitting fast response to queries such as *"is the object x related to the object y?"*. In distributed systems, for example, the order between any set of events is partial, and in many applications such as electronic commerce systems, the set of events and their order may be recorded for later queries such as "is bid x concurrent with bid y?"

A popular method to store a partially ordered set or *poset* is to view it as a directed acyclic graph and represent the graph using adjacency matrix or adjacency list. The adjacency matrix representation uses a matrix of size $n \times n$, where n is the number of elements in the set. Elements are assumed to be numbered from 1 to n. The $(i, j)^{th}$ entry of the matrix is 1 if and only if the i^{th} element is less than the j^{th} element in the partial order; otherwise it is 0. To compare two distinct elements, at most two entries of the matrix need to be examined. Thus the adjacency matrix representation uses $O(n)$ bits per element but allows $O(1)$ response to the *less-than* query ("is x less than y?"). In the adjacency list representation, a list is associated with each element; the list for element x contains all elements greater than x. The adjacency list representation is only useful for those posets which have low average degree.

Another method to represent a partial order on a set of elements is based on dimension theory introduced by Dushnik and Miller (Dushnik and Miller, 1941). In this theory, a poset is represented as intersection of a collection of total orders on the set (of elements) (or *chains*), each of which extends the partial order. The collection of total orders whose intersection is the given partial order constitutes a *realizer* of the partial order. An element x is less than another element y in the partial order if and only if x is less than y in all the total orders in the realizer. A total order can be represented optimally using $\log n$ bits per element, where n is the number of elements, thereby furnishing a technique to represent a poset. If the realizer contains k total orders, then each element can be encoded using $O(k \log n)$ bits per element. Every element is simply represented by its rank in each total order. The *dimension* of a poset is the least number of total orders required to realize the corresponding partial order. Clearly, the dimension of a poset is one if and only if it is a total order. Yannakakis (Yannakakis, 1982) established that it is in general NP-complete to test whether the dimension of a poset is at most k for any fixed $k \geqslant 3$. Poset dimension has been studied extensively; the interested reader can consult the book (Trotter, 1992). When the dimension is small, the dimension-theoretic representation is more concise than adjacency matrix representation.

In our approach, we use posets with dimension at most two as "building blocks" for realizing a poset. For convenience, we refer to

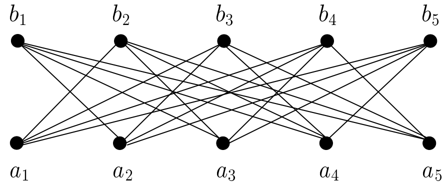

Figure 1. The standard example $\mathbf{S}_5$.

two-dimensional posets as *rectangles*. An element x is less than another element y in the partial order if and only if x is less than y in at least one of the rectangular orders and y is not less than x is any of the rectangular orders. The set of rectangular orders that realizes a partial order constitutes its *rectangular realizer*. Also, the *rectangular dimension* of a poset is the least number of rectangular orders needed to realize the corresponding partial order. Clearly, by definition, the rectangular dimension of a poset is one if and only if its dimension is at most two. Trivially, the rectangular dimension of a poset is upper bounded by its dimension.

It turns out that there are posets with arbitrarily high dimension but only constant rectangular dimension. As an illustration, consider the family of bipartite posets called *standard examples*. The standard example $\mathbf{S}_n$ for $n \geqslant 3$ is the poset induced by the 1-element and $(n-1)$-element subsets of n distinct elements when ordered by set containment. The graph representation of $\mathbf{S}_5$, for example, is shown in Figure 1. In the figure, all edges are directed upwards. It can be proved that a standard example has "large" dimensions (Trotter, 1992, Chapter 1). Specifically, the dimension of $\mathbf{S}_n$ is given by n for each $n \geqslant 3$. This implies that $O(n \log n)$ bits per element are required to encode $\mathbf{S}_n$ using chains. On the other hand, we prove in this paper that the rectangular dimension of $\mathbf{S}_n$ is at most two for each $n \geqslant 3$. Each rectangle can be encoded using $O(\log n)$ bits. Therefore using rectangles leads to a much more efficient representation of $\mathbf{S}_n$. We further prove that the rectangular dimension of the generalized crown $\mathbf{S}_n^k$ for $n \geqslant 3$ and $k \geqslant 0$ (Trotter, 1992, Chapter 2), which is a generalization of the standard example, is also two. Its dimension, however, is given by $\lceil 2(n+k)/(k+2) \rceil$. Note that encoding $\mathbf{S}_n$ and $\mathbf{S}_n^k$ requires a large number of bits per element (specifically, $O(n \log n)$ and $O(n \log(n+k))$, respectively) using adjacency list representation as well.

We describe two methodologies to compute a rectangular realizer of a poset. The first methodology involves partitioning the ground set into one or more subsets; we refer to it as *point decomposition* (also known as *removal theorem* in the literature). The second methodology involves decomposing the partial order into one or more suborders; we refer to it as *order decomposition*. Using point decomposition, we establish upper bounds of $n/3$ and $\lceil n/4 \rceil$ on the rectangular dimension of arbitrary posets and bipartite posets, respectively, where n is the number of elements with $n \geqslant 3$. Using order decomposition, we derive efficient representations for special classes of posets such as bounded-degree posets and generalized crowns. For encoding each rectangular order in a rectangular realizer, we use the *string representation* introduced in (Garg and Skawratananond, 2001). If a rectangle is represented using two chains, then each element requires $O(\log n)$ bits per element per rectangle. However, by exploiting string representation, the number of bits required can be as small as $O(1)$ per element per rectangle for some posets.

The paper is organized as follows. We formally define the notions of rectangle, rectangular realizer and rectangular dimension in Section 2. The two methodologies for computing rectangular realizers of posets, namely point decomposition and order decomposition, are illustrated in Section 3 and Section 4, respectively. Some applications are also provided. The related work is discussed in Section 5. In Section 6, we present our conclusions and outline some directions for future research.

2. Rectangles

We use the notation and terminology given in (Trotter, 1992). We assume that the reader is familiar with the notions of realizer and dimension of a poset. Sometimes, we use the terms "chain realizer" and "chain dimension" instead of the terms "realizer" and "dimension". In this section, we formally define the notions of rectangle, rectangular realizer and rectangular dimension of a poset.

Definition 1. (rectangle) A *rectangle* is a two-dimensional poset.

When a poset $\mathbf{P} = (X, P)$ is a rectangle, we refer to P as *rectangular order*. For a rectangular order R, we use $R.1$ and $R.2$ to refer to the two total orders that realize R. In case the dimension of R is one, both $R.1$ and $R.2$ refer to the same total order. Clearly, $R = R.1 \cap R.2$. Also, trivially, a chain as well as an antichain is a rectangle.

Definition 2. (rectangular realizer) Let $\mathbf{P} = (X, P)$ be a poset. A family $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ of rectangular orders on X is called a *realizer* of P on X (also, $\mathcal{R}$ *realizes* $\mathbf{P}$) if for every $x, y \in X$, $x < y$ in P if and only if $y \not< x$ in R_i for each $i \in [1, t]$ and $x < y$ in R_j for some $j \in [1, t]$.

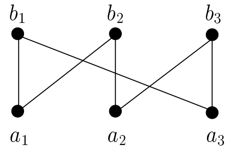

Figure 2. An example of a poset.

If $x \parallel y$ in P, then in a rectangular realizer $\mathcal{R}$ of P two cases are possible. Either $x \parallel y$ in all rectangular orders in $\mathcal{R}$, or $x < y$ in some rectangular order in $\mathcal{R}$ and $y < x$ in some other rectangular order in $\mathcal{R}$. On the other hand, if $x < y$ in P, then $x < y$ in some rectangular order in $\mathcal{R}$ and $x < y$ or $x \parallel y$ in all other rectangular orders in $\mathcal{R}$. The notion of rectangular dimension can now be defined as follows:

Definition 3. (rectangular dimension) The *rectangular dimension* of a poset $\mathbf{P} = (X, P)$, denoted by $\mathrm{rdim}(X, P)$ (or $\mathrm{rdim}(\mathbf{P})$), is the least positive integer t for which there exists a family $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ of t rectangular orders on X so that $\mathcal{R}$ realizes $\mathbf{P}$.

As an example, the rectangular dimension of the poset depicted in Figure 2 is two. The two rectangular orders realizing it are given by $(a_1 < a_2 < a_3 < b_1 < b_2 < b_3\ ;\ a_3 < a_2 < a_1 < b_3 < b_2 < b_1)$ and $(b_1 < a_2 < b_2 < a_3 < b_3 < a_1\ ;\ b_3 < a_1 < b_2 < a_3 < b_1 < a_2)$.

The rectangular dimension of a poset and its dual are identical because the dual of a two-dimensional poset is again a two-dimensional poset. The notions of rectangular realizer and rectangular dimension defined for a poset can be generalized to any (acyclic) relation on a ground set. For a collection of rectangular orders $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$, let $\mathrm{rel}(\mathcal{R})$ denote the relation realized by $\mathcal{R}$. For example, consider the rectangular orders $(a < b < c\ ;\ c < a < b)$ and $(a < b < c\ ;\ b < c < a)$. The relation realized by the two orders collectively is given by $\{(a, b), (b, c)\}$. Note that the relation is not transitive because it does not contain the ordered pair (a, c). Next, we define two concepts we use when deriving rectangular realizers for posets using the point and order decomposition methods. Let $X = Y \cup Z$ be a partition of X, and let Q and R be rectangular orders on Y and Z, respectively. Then $P = Q \cup R$ is also a rectangular order on X. Evidently, two chains realizing P are given by $P.1 = Q.1 < R.1$ and $P.2 = R.2 < Q.2$. Moreover, P satisfies the following properties:

1. $P(Y) = Q$ and $P(Z) = R$, and

2. $y \parallel z$ in P for all $y \in Y$ and $z \in Z$.

Definition 4. (disjoint composition) Let $X = Y \cup Z$ be a partition of X, and let Q and R be rectangular orders on Y and Z, respectively. Then we say that the rectangular order $P = Q \cup R$ on X is obtained by *disjoint composition* of the rectangular orders Q on Y and R on Z.

Definition 5. (non-interference) A rectangular order R is said to be *non-interfering* with a partial order P if $R \subseteq P$.

Also, a rectangular realizer is *non-interfering* with a partial order if every rectangular order in the realizer is non-interfering with the partial order. In other words, if two elements are incomparable in the partial order, they are also incomparable in all rectangular orders in the realizer. In that case, the partial order is given by the union of all rectangular orders in its realizer. Evidently, every partial order (even a relation) has a non-interfering rectangular realizer.

3. Point Decomposition Method and its Applications

3.1. The Main Idea

Given a poset $\mathbf{P} = (X, P)$, we partition the ground set X into two: Y and $X \setminus Y$. We then compute rectangular realizers, say $\mathcal{Q}$ and $\mathcal{R}$, for the two induced subposets $(Y, P(Y))$ and $(X \setminus Y, P(X \setminus Y))$, respectively. A rectangular realizer $\mathcal{S}$ of $P(Y) \cup P(X \setminus Y)$ on X can be easily obtained by disjoint composition of $\mathcal{Q}$ and $\mathcal{R}$; each rectangular order of $\mathcal{S}$ is obtained by disjointly composing the corresponding rectangular orders of $\mathcal{Q}$ and $\mathcal{R}$ (padding if necessary). Finally, we compute a rectangular realizer, say $\mathcal{T}$, of the relation $P \setminus (P(Y) \cup P(X \setminus Y))$ on X. To ensure that $\mathcal{S} \cup \mathcal{T}$ indeed constitutes a rectangular realizer of P on X, it suffices to ensure that $\mathcal{T}$ is non-interfering with P. Note that it is not necessary that the relation realized by $\mathcal{T}$, given by $\mathrm{rel}(\mathcal{T})$, be exactly $P \setminus (P(Y) \cup P(X \setminus Y))$. It is sufficient that $P \setminus (P(Y) \cup P(X \setminus Y) \subseteq \mathrm{rel}(\mathcal{T}) \subseteq P$. We often choose Y such that U is a rectangular order.

Before we present our removal theorems, we describe the notion of *indistinguishable elements* that we use extensively in our proofs. Let $\mathbf{P} = (X, P)$ be a poset and consider an element $x \in X$ and a subset $Y \subseteq X$. We denote the subset of elements in Y that are less than x in P, that is, $\{y \in Y \mid y < x \text{ in } P\}$ by $D(x, Y)$ (called the *down set* of x in Y). Similarly, the subset of elements in Y that are greater than x in P, that is, $\{y \in Y \mid x < y \text{ in } P\}$ is denoted by $U(x, Y)$ (called the *up set* of x in Y). For convenience, we abbreviate $D(x, X)$ by $D(x)$ and $U(x, X)$ by $U(x)$.

Definition 6. (indistinguishable elements) Two elements $x, y \in X$ are said to be *indistinguishable* with respect to Y in P if $D(x,Y) = D(y,Y)$ and $U(x,Y) = U(y,Y)$.

The elements of $X \setminus Y$ can be partitioned into equivalence classes such that elements in the same class are mutually indistinguishable with respect to Y in P. These equivalence classes are referred to as *Y–indistinguishable classes* of $X \setminus Y$ in P. The class for which $D(x,Y) = U(x,Y) = \emptyset$ for each element x in the class is called *Y–disconnected class.* (No element in the Y–disconnected-class is comparable to any element in Y.) When deriving the rectangular realizer $\mathcal{T}$ to represent $P \setminus (P(Y) \cup P(X \setminus Y))$ described earlier, it suffices to consider at most one representative element from every Y–indistinguishable class.

For example, for the poset shown in Figure 2, $D(b_1, \{a_3, b_3\}) = \{a_3\}$. Furthermore, there are three $\{a_3, b_3\}$–indistinguishable classes of $\{a_1, a_2, b_1, b_2\}$, namely $\{b_1\}$, $\{a_1, b_2\}$ and $\{a_2\}$. The class $\{a_1, b_2\}$ is the $\{a_3, b_3\}$–disconnected class.

3.2. Applications

We now present our main removal theorem that forms the basis for many of our other removal theorems. The theorem is based on the notion of *critical subposet.* A pair (x, y) from X with $x \parallel y$ in P forms a *critical pair* in the subposet $(Y, P(Y))$ if $D(x,Y) \subseteq D(y,Y)$ and $U(x,Y) \supseteq U(y,Y)$. A subposet $(Y, P(Y))$ is called a *critical subposet* of the poset (X, P) if for every incomparable pair (x, y) from Y either (x, y) or (y, x) forms a critical pair in the subposet $(X \setminus Y, P(X \setminus Y))$.

Theorem 1. (critical subposet removal theorem) Let $(Y, P(Y))$ be a critical subposet of the poset $\mathbf{P} = (X, P)$ where $X \setminus Y \neq \emptyset$. Then,

$$\mathrm{rdim}(X,P) \leqslant 1 + \max\{\mathrm{rdim}(X \setminus Y, P(X \setminus Y)),\ \dim(Y, P(Y))\}$$

Proof. First, we compute a rectangular order R that contains all ordered pairs in the relation $P \setminus (P(Y) \cup P(X \setminus Y))$. However, R may interfere with P. But, the ordered pairs in R that do not belong to P only involve the elements of Y, that is, $R \setminus P \subseteq Y \times Y$. Such ordered pairs are reversed later. Now, to compute R, we claim that the elements of Y can be "viewed" as a chain with respect to the elements of $X \setminus Y$. More precisely, it is possible to linearize the partial order $P(Y)$ to obtain a total order T on Y that satisfies the following property: for all elements $x, y \in Y$ if $x < y$ in T then $D(x, X \setminus Y) \subseteq D(y, X \setminus Y)$ and $U(x, X \setminus Y) \supseteq U(y, X \setminus Y)$. Let Q_D and Q_U be the relations as defined:

$$Q_D = \{(x,y) \mid x, y \in Y,\ x \parallel y \text{ in } P(Y) \text{ and } D(x, X \setminus Y) \subsetneq D(y, X \setminus Y)\}$$

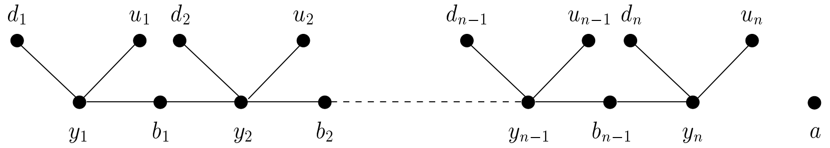

Figure 3. The Y–indistinguishable classes of $X \setminus Y$ in P where $(Y, P(Y))$ is a critical subposet of the poset (X, P) (note that in the Hasse diagram, the line segments between elements are directed from left to right instead of the usual bottom to top).

$$Q_U = \{(x, y) \mid x, y \in Y,\ x \parallel y \text{ in } P(Y) \text{ and } U(x, X \setminus Y) \supsetneq U(y, X \setminus Y)\}$$

We establish that the relation $P(Y) \cup Q_D$ is acyclic. The main idea is that any cycle in $P(Y) \cup Q_D$, if it exists, must involve a pair from Q_D because $P(Y)$ is acyclic. Note that for every pair $(x, y) \in P(Y)$, $D(x, X \setminus Y) \subseteq D(y, X \setminus Y)$, and for every pair $(x, y) \in Q_D$, $D(x, X \setminus Y) \subsetneq D(y, X \setminus Y)$. Hence if an element p is involved in a cycle then $D(p, X \setminus Y) \subsetneq D(p, X \setminus Y)$—a contradiction. Similarly, it can be proved that the relation $P(Y) \cup Q_D \cup Q_U$ is acyclic as well. All incomparable pairs that remain after taking the transitive closure of $P(Y) \cup Q_D \cup Q_U$ are $(X \setminus Y)$–indistinguishable in P and therefore can be ordered either way in T. The required total order T on Y is given by any linearization of the relation $P(Y) \cup Q_D \cup Q_U$.

We use y_i to refer to the i^{th} element in the total order T for $i = 1, 2, \ldots, n$ where $n = |Y|$. We now compute the Y–indistinguishable classes of $X \setminus Y$ in P. Figure 3 shows various Y–indistinguishable classes of $X \setminus Y$ in P and how they relate to Y. In the figure we depict each equivalence class by a single representative element, namely a for A, and d_i, u_i and b_i for D_i, U_i and B_i, respectively, for each i. The classes other than the Y–disconnected class, say A, can be partitioned into three categories. The first family of classes, denoted by $\mathcal{D}$, consists of classes D_i for $i = 1, 2, \ldots, n$ so that an element $x \in X \setminus Y$ belongs to D_i if $x < y_i$ in P but $y_{i-1} \parallel x$ in P. Note that when $i = 1$ only the first condition is applicable, that is, $x \in D_1$ if $x < y_1$ in P. The second family of classes, denoted by $\mathcal{U}$, consists of classes U_i for $i = 1, 2, \ldots, n$ so that an element $x \in X \setminus Y$ belongs to U_i if $y_i < x$ in P but $x \parallel y_{i+1}$ in P. Again, note that when $i = n$ only the first condition applies, that is, $x \in U_n$ if $y_n < x$ in P. The third family of classes, denoted by $\mathcal{B}$, contains classes B_i for $i = 1, 2, \ldots, n-1$ such that an element $x \in X \setminus Y$ is contained in B_i if $y_i < x < y_{i+1}$. Clearly, all Y–indistinguishable classes of $X \setminus Y$ in P are covered by $A \cup \mathcal{D} \cup \mathcal{U} \cup \mathcal{B}$. The required rectangular order R is given by:

$$\begin{aligned} R.1 \ = \ & a < d_n < d_{n-1} < \cdots < d_1 < y_1 < b_1 < y_2 < b_2 < \cdots < y_n < \\ & u_n < u_{n-1} < \cdots < u_1 \end{aligned}$$

$$R.2 = d_1 < y_1 < u_1 < b_1 < d_2 < y_2 < u_2 < b_2 < \cdots < b_{n-1} < d_n < y_n < u_n < a$$

Now, we independently compute representations for the subposets $(Y, P(Y))$ and $(X \setminus Y, P(X \setminus Y))$. To represent the former, we use a chain realizer, and, to represent the latter, we use a rectangular realizer. Set $t = \max\{\dim(Y, P(Y)),\ \mathrm{rdim}(X \setminus Y, P(X \setminus Y))\}$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ be a chain realizer of $P(Y)$ on Y, and let $\mathcal{S} = \{S_1, S_2, \ldots, S_t\}$ be a rectangular realizer of $P(X \setminus Y)$ on $X \setminus Y$. We construct a family of t rectangular orders $\mathcal{T} = \{T_1, T_2, \ldots, T_t\}$ where the order T_i on X is given by disjoint composition of the order C_i on Y and the order S_i on $X \setminus Y$ for $i = 1, 2, \ldots, t$. We choose a chain realizer and not a rectangular realizer for representing $(Y, P(Y))$ because when we linearize $P(Y)$ we may introduce ordered pairs from $Y \times Y$ that are not present in P. These pairs are reversed by the chain realizer. Finally, $\{R\} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X. □

A chain trivially constitutes a critical subposet of any poset because it does not contain any incomparable pair. Also, the dimension of a chain is one. Thus, from critical subposet removal theorem, it follows that:

Theorem 2. (chain removal theorem) Let C be a chain in the poset $\mathbf{P} = (X, P)$. Then,

$$\mathrm{rdim}(X, P) \leqslant 1 + \mathrm{rdim}(X \setminus C, P(X \setminus C))$$

Does a similar theorem exist for an antichain? The answer is in general no. But in case the antichain consists of only two elements, a removal theorem can indeed be provided.

Theorem 3. (incomparable pair removal theorem) Let $\mathbf{P} = (X, P)$ be a poset and (x, y) be an incomparable pair P. Then,

$$\mathrm{rdim}(X, P) \leqslant 1 + \mathrm{rdim}(X \setminus \{x, y\}, P)$$

Proof. For convenience, set $Y = \{x, y\}$. We give a rectangular order R containing the relation $P \setminus P(X \setminus Y)$ (in this case, $P(Y) = \emptyset$) that is non-interfering with P. Figure 4 depicts various Y-indistinguishable classes and how they relate to Y. As shown, there are seven such classes represented by a, b, c, d, e, f and g. The required rectangular order R is given by:

$$R.1 = a < b < c < x < e < d < y < f < g$$
$$R.2 = d < c < y < g < b < x < f < e < a$$

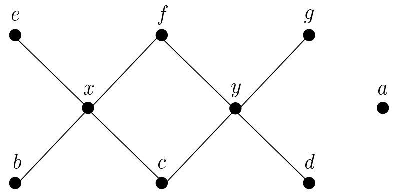

Figure 4. The Y-indistinguishable classes of $X \setminus Y$ in P where $Y = \{x, y\} \subseteq X$ with $x \parallel y$ in P.

It can be verified that R contains $P \setminus P(X \setminus Y)$ and does not interfere with P. Let $\mathcal{S}$ be a rectangular realizer of $P(X \setminus Y)$ on $X \setminus Y$. We can obtain a rectangular realizer $\mathcal{T}$ of $P(X \setminus Y)$ on X by disjointly composing each rectangular order of $\mathcal{S}$ with the empty order on Y. Then $\{R\} \cup \mathcal{T}$ constitutes a rectangular realizer of P on X. □

3.2.1. *Establishing Upper Bounds on Rectangular Dimension*

Since $\mathrm{rdim}(X, P) \leqslant \dim(X, P)$ and $\dim(X, P) \leqslant |X|/2$ when $|X| \geqslant 4$, trivially, $\mathrm{rdim}(X, P) \leqslant |X|/2$ when $|X| \geqslant 4$. The upper bound on chain dimension is tight even for bipartite posets. The bound is not tight for rectangular dimension as shown in this section.

Theorem 4. Let $\mathbf{P} = (X, P) = (L, U, P)$ be a bipartite poset with nonempty L and U. Then,

$$\mathrm{rdim}((X, P)) \leqslant \min\{\lceil |L|/2 \rceil, \lceil |U|/2 \rceil\}$$

Proof. Without loss of generality, assume that $\lceil |L|/2 \rceil \leqslant \lceil |U|/2 \rceil$ and further that $|L|$ is even. In case $|L|$ is odd, we add an element to L that is not connected to any other element. Set $t = \lceil |L|/2 \rceil$. Using incomparable pair removal theorem repeatedly, we successively remove two elements from L which, by definition of L, are incomparable in P until we have exhausted all elements in L. Clearly, t pairs of elements are removed. Thus,

$$\mathrm{rdim}(X, P) \leqslant t + \mathrm{rdim}(U, P(U)) = t + \mathrm{rdim}(U, \emptyset) = t + 1$$

Note, however, that we do not need the $(t+1)^{st}$ rectangular order to represent the poset $(U, \emptyset)$. This is because the rectangular order constructed in the proof of the incomparable pair removal theorem is non-interfering with P. This implies that for all $x, y \in U$, $x \parallel y$ in each one of the t rectangular orders. In other words, the fact that the elements of U form an antichain has been captured in (all) the t rectangular orders already. Thus $\mathrm{rdim}(X, P) \leqslant t = \lceil |L|/2 \rceil$. □

Clearly, either $|L| \leqslant |X|/2$ or $|U| \leqslant |X|/2$. Thus, from Theorem 4, it follows that:

Corollary 5. Let $\mathbf{P} = (X, P)$ be a bipartite poset. Then,

$$\operatorname{rdim}(X, P) \leqslant \lceil |X|/4 \rceil$$

For a general poset, a slightly weaker upper bound can be given.

Theorem 6. Let $\mathbf{P} = (X, P)$ be a poset with $|X| \geqslant 3$. Then,

$$\operatorname{rdim}(X, P) \leqslant |X|/3$$

Proof. The proof is by induction on the number of elements in X.

Base Case ($|\mathbf{X}| \leqslant \mathbf{5}$): It can be verified by doing case analysis that whenever $|X| \leqslant 5$, $\dim(X, P) \leqslant 2$ implying that $\operatorname{rdim}(X, P) \leqslant 1$ (Trotter, 1992, Page 23).

Induction Step: Now suppose that $\operatorname{rdim}(X, P) \leqslant |X|/3$ whenever $|X| \leqslant k$ where $k \geqslant 5$. Then consider a poset (X, P) with $k+1$ elements. In case the poset (X, P) contains a chain C involving three elements, from Theorem 2, $\operatorname{rdim}(X, P) \leqslant 1+\operatorname{rdim}(X\backslash C, P(X\backslash C)) \leqslant 1+|X\backslash C|/3$ $= 1 + (|X| - 3)/3 = |X|/3$. Thus assume that the poset (X, P) does not contain any chain involving more than two elements, or, in other words, it is a bipartite poset. When $6 \leqslant |X| \leqslant 8$, from Corollary 5, $\operatorname{rdim}(X, P) \leqslant \lceil |X|/4 \rceil \leqslant 2 \leqslant |X|/3$. Similarly, when $9 \leqslant |X| \leqslant 11$, $\operatorname{rdim}(X, P) \leqslant \lceil |X|/4 \rceil \leqslant 3 \leqslant |X|/3$. Finally, when $|X| \geqslant 12$, from Corollary 5, $\operatorname{rdim}(X, P) \leqslant \lceil |X|/4 \rceil \leqslant |X|/4 + 1 \leqslant |X|/3$. □

3.2.2. *Bounding Rectangular Dimension of Interval Orders*

Recall that a poset $\mathbf{P} = (X, P)$ is an *interval order* if there is an *interval representation* function F assigning to each element $x \in X$ a non-degenerate closed interval $F(x) = [a_x, b_x]$ of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$ (Trotter, 1992, Page 190). We define the *range* of an interval order (X, P) with respect to an interval representation function F, denoted by $\operatorname{range}(X, P, F)$, as the set $\{b_x - a_x \mid x \in X \text{ with } F(x) = [a_x, b_x]\}$. In case $|\operatorname{range}(X, P, F)| = 1$, the interval order (X, P) is called a *semi-order* (Trotter, 1992, Page 192). The following theorem generalizes the result from dimension theory that if $|\operatorname{range}(X, P, F)| = 1$ then $\dim(X, P) \leqslant 3$ (Trotter, 1992, Page 196).

Theorem 7. Let $\mathbf{P} = (X, P)$ be an interval order with interval representation F. Then,

$$\operatorname{rdim}(X, P) \leqslant |\operatorname{range}(X, P, F)| + 2$$

Proof. Set $t = |\text{range}(X, P, F)|$. Clearly, the ground set X can be partitioned into t subsets X_i for $i = 1, 2, \ldots, t$ such that each induced subposet $(X_i, P(X_i))$ is a semi-order. We claim that each subposet $(X_i, P(X_i))$ for $i \in [1, t]$ is also a critical subposet of the poset (X, P).

Consider a subposet $(X_i, P(X_i))$ and elements $x, y \in X_i$ with $x \parallel y$ in $P(X_i)$. Let $b_x - a_x = b_y - a_y = d$ (say) where $F(x) = [a_x, b_x]$ and $F(y) = [a_y, b_y]$. Either $a_x \leqslant a_y$ or $a_y \leqslant a_x$. Without loss of generality, assume that $a_x \leqslant a_y$. We establish that $D(x) \subseteq D(y)$ and $U(x) \supseteq U(y)$ which in turn implies that (x, y) is a critical pair in (X, P) and hence a critical pair in $(X \setminus Y, P(X \setminus Y))$. Consider an element $z \in X$ with $F(z) = [a_z, b_z]$. If $z \in D(x)$ then $b_z < a_x \leqslant a_y$ implying that $z \in D(y)$. Similarly, if $z \in U(y)$ then $a_z > b_y = a_y + d \geqslant a_x + d = b_x$ or, equivalently, $a_z > b_x$ implying that $z \in U(x)$.

The dimension of a semi-order is at most three. Thus, by repeatedly apply the critical subposet removal theorem, we obtain:

$$\text{rdim}(X, P) \leqslant |\text{range}(X, P, F)| - 1 + \max\{\text{rdim}(X_i, P(X_i)), 3\}$$

Since the rectangular dimension of a semi-order is also at most three, $\text{rdim}(X, P) \leqslant |\text{range}(X, P, F)| - 1 + 3 = |\text{range}(X, P, F)| + 2$. □

4. Order Decomposition Method and its Applications

4.1. The Main Idea

Given a poset $\mathbf{P} = (X, P)$, we decompose the partial order P into n suborders P_i for $i = 1, 2, \ldots, n$; the suborders need not be disjoint. Each subposet (X, P_i) is represented using rectangles; let $\mathcal{R}_i$ be a rectangular realizer of P_i on X for $i = 1, 2, \ldots, n$. Then the collection of rectangular orders $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$ constitutes a rectangular realizer of P on X provided that each rectangular realizer $\mathcal{R}_i$ is non-interfering with P_i.

Note that when we specify a decomposition of a partial order into suborders we do not enumerate the reflexive pairs which can always be added later. For a relation R on $X \times X$, let $\text{tr}(R)$ denote the reflexive transitive closure of R. Also, let $\text{rf}(R)$ denote its reflexive closure.

4.2. Applications

Let $\mathbf{P} = (X, P)$ be a poset. Suppose there is a relation Q with $P = \text{tr}(Q)$ such that for every element $x \in X$ there is at most one element $y \in X$, $x \neq y$, so that $(x, y) \in Q$. In that case, it is easy to see that Q is a forest of disjoint trees. Specifically, it can be verified that for every element $x \in X$, $U(x)$ forms a chain in P. A forest poset is also a *series-parallel* poset (Kaerkes and Möhring, 1978; Lawler, 1976; Valdes et al., 1982). The chain dimension of a series-parallel poset is at most two and hence its rectangular dimension is at most one (Kaerkes and Möhring, 1978; Lawler, 1976; Valdes et al., 1982). In the rest of the section, we establish upper bounds on the rectangular dimension of special classes of posets.

For a poset $\mathbf{P} = (X, P)$ and an element $x \in X$, define the *indegree* of x in P, denoted $\deg_{\mathrm{D}}(x)$, as the number of elements less than x in P (Trotter, 1992). Let $\Delta_{\mathrm{D}}(X, P)$ denote $\max\{\deg_{\mathrm{D}}(x) \mid x \in X\}$. If $\Delta_{\mathrm{D}}(X, P) = k$ then $\dim(X, P) < 2(k+1)\log(|X|)$(Trotter, 1992). We improve upon this result in the next theorem.

Theorem 8. Let $\mathbf{P} = (X, P)$ be a poset with $\Delta_{\mathrm{D}}(X, P) \leqslant k$ where $k \geqslant 1$. Then,

$$\mathrm{rdim}(X, P) \leqslant k$$

Proof. The central idea is to decompose the partial order P into at most k suborders such that the subposet induced by each suborder resembles a forest (that is, its Hasse diagram is a set of disjoint trees) which in turn can be encoded using at most one rectangle. For ease of exposition, assume that for every element $x \in X$, the elements in $D(x)$ are numbered from 1 to $|D(x)|$ (order is not important). The i^{th} suborder P_i is given by the reflexive transitive closure of the set $\{(x, y) | x \in X$ and y is the i^{th} element in $D(x)$, if it exists$\}$. Clearly, by definition, (X, P_i) is a lower singular poset, and $P = P_1 \cup P_2 \cup \cdots \cup P_k$. □

It turns out that we can further generalize the above theorem as follows. A poset $\mathbf{P} = (X, P)$ is called *k-lower bounded* if for each element $x \in X$ either $|D(x)| \leqslant k$ or $|D(x)| \geqslant |X| - k$.

Theorem 9. Let $\mathbf{P} = (X, P)$ be a k-lower bounded poset. Then,

$$\mathrm{rdim}(X, P) \leqslant \lceil 3k/2 \rceil + 1$$

Proof. It suffices to prove that $\mathrm{rdim}(X, P) \leqslant \lceil 3k/2 \rceil + 1$ when $k \leqslant \lfloor |X|/2 \rfloor$. Our approach is to partition the ground set X into two disjoint subsets L and U such that for every element $x \in L$, $|D(x)| \leqslant k$, and for every element $x \in U$, $|D(x)| \geqslant |X| - k$. Clearly, there is no element in U that is less than some element in L; otherwise $D(x) \supset D(y)$ with $x \in L$, $y \in U$, and $y < x$ in P implying that $|D(x)| > |D(y)| \geqslant |X| - k \geqslant \lceil |X|/2 \rceil$—a contradiction. Mathematically, $P \cap (U \times L) = \emptyset$. Consider a minimal element x of U, that is, there is no element $y \in U$ with $y < x$

in P. By definition, $D(x) \subseteq L$. Therefore $|L| \geqslant |D(x)| \geqslant |X| - k$ which in turn implies that $|U| \leqslant k$.

Set $t = \lceil 3k/2 \rceil + 1$. Let T be some total order on $L \cup U$. In the first rectangular order R on X, we make every element of L less than every element of U as follows:

$$\begin{aligned} R.1 &= T(L) < T(U) \\ R.2 &= T^d(L) < T^d(U) \end{aligned}$$

It can be verified that $R(L) = \emptyset$ and $R(U) = \emptyset$. Next, we derive a relation Q from the partial order P as follows:

$$Q = P(L) \cup P(U) \cup \{ (y,x) \mid x \in L,\ y \in U \text{ and } x \parallel y \text{ in } P \}$$

Let $Q_1 = P(L) \cup P(U)$ and $Q_2 = Q \setminus Q_1 = \{ (y,x) \mid x \in L,\ y \in U \text{ and } x \parallel y \text{ in } P \}$. Informally, Q_2 reverses those ordered pairs in R that are not present in P, that is, $Q_2 = (R \setminus P)^d$. We establish that Q is actually a partial order on $L \cup U$. Clearly, it is reflexive. It is easy to see that Q_1 is a partial order. Consequently, any cycle in Q, if it exists, must involve at least one pair from Q_2. Since $Q_2 \subseteq U \times L$, for a cycle to exist, Q must contain a pair from $L \times U$. However, $Q \cap (L \times U) = \emptyset$, thereby implying that Q is cycle free. It remains to be shown that Q is transitive. Consider elements $x, y, z \in X$ with $x < y$ in Q and $y < z$ in Q. There are two cases to consider:

Case 1 ($x \in L$): Since $Q \cap (L \times U) = \emptyset$, $\{y, z\} \subseteq L$. Further, $Q(L) = P(L)$ and $P(L)$ is a partial order on L which implies that $x < z$ in Q.

Case 2 ($x \in U$): There are three subcases to consider:

Case 2.1 ($y \in U$ and $z \in U$): In this case, since $Q(U) = P(U)$ and $P(U)$ is a partial order on U, $x < z$ in Q.

Case 2.2 ($y \in U$ and $z \in L$): In this case, since $y < z$ in Q and $Q \cap (U \times L) = Q_2$, by definition of Q_2, $y \parallel z$ in P. We claim that $x \parallel z$ in P which in turn, by definition of Q, implies that $x < z$ in Q. If not, since $P \cap (U \times L) = \emptyset$, $x \not< z$ in P or $z < x$ in P. Further, as $x < y$ in P, by transitivity, $z < y$ in P—a contradiction.

Case 2.3 ($y \in L$ and $z \in L$): In this case, since $x < y$ in Q and $Q \cap (U \times L) = Q_2$, by definition of Q_2, $x \parallel y$ in P. We claim that $x \parallel z$ in P which in turn, by definition of Q, implies that $x < z$ in Q. If not, since $P \cap (U \times L) = \emptyset$, $x \not< z$ in P or $z < x$ in P. Further, as $y < z$ in P, by transitivity, $y < x$ in P—a contradiction.

The fourth subcase when $\{x, z\} \subseteq U$ and $y \in L$ is not possible because $y < z$ in Q and $Q \cap (L \times U) = \emptyset$. To ensure that the rectangular order R along with a rectangular realizer of Q constitutes a rectangular realizer of P, it suffices to ensure that the rectangular realizer of Q is non-interfering with Q. To accomplish this, we give non-interfering rectangular realizers for the subposets (X, Q_1) and (X, Q_2). It may be observed that $\Delta_{\mathrm{D}}(X, Q_1) \leqslant k$. This is because $Q_1 \cap (X \times L) = P(L)$ and, by definition of L, for every element $x \in L$, $|D(x)| \leqslant k$. Moreover, $Q_1 \cap (X \times U) \subseteq U \times U$ and $|U| \leqslant k$. From Theorem 8, the poset (X, Q_1) can be realized using at most k rectangular orders. Further, it can be verified that each rectangular order in the realizer that we construct in the proof of Theorem 8 is non-interfering with Q_1. Finally, observe that the poset (X, Q_2) is a bipartite poset. Then $(X, Q_2) = (L', U', Q_2)$ where $L' = U$ and $U' = L$ with $|L'| = |U| \leqslant k$. From Theorem 4, the poset (X, Q_2) can be realized using at most $\lceil k/2 \rceil$ rectangular orders. Further, it can be verified that each rectangular order in the realizer that we construct in the proof of Theorem 4 is non-interfering with Q_2. □

The upper bound can be further improved for a bipartite poset. First, we modify the definition of k-lower bounded poset slightly for a bipartite poset. A bipartite poset $\mathbf{P} = (L, U, P)$ is called *k-lower bounded* if for each element $x \in U$ either $|D(x)| \leqslant k$ or $|D(x)| \geqslant |L| - k$.

Theorem 10. Let $\mathbf{P} = (X, P) = (L, U, P)$ be a k-lower bounded bipartite poset. Then,

$$\mathrm{rdim}(X, P) \leqslant k + 2$$

Proof. It suffices to prove that $\mathrm{rdim}(X, P) \leqslant k+2$ when $k \leqslant \lfloor |L|/2 \rfloor$. First, we partition the set U into two disjoint subsets U_1 and U_2 such that for every element $x \in U_1$, $|D(x)| \leqslant k$, and for every element $x \in U_2$, $|D(x)| \geqslant |L| - k$. In the first rectangular order R, we make every element of L less than every element of U_2. Let T be some total order on $L \cup U$. The rectangular order R is given by:

$$\begin{aligned} R.1 &= T(L) < T(U_1) < T(U_2) \\ R.2 &= T^d(U_1) < T^d(L) < T^d(U_2) \end{aligned}$$

The rectangular order R may contain ordered pairs that do not belong to the partial order P. Such ordered pairs can be divided into two categories. The first category contains ordered pairs from $U_1 \times U_2$. In fact, R contains *all* ordered pairs in $U_1 \times U_2$. The second category consists of ordered pairs from $L \times U_2$. To obtain a realizer of P on X, the ordered pairs from both categories need to be reversed. In the second rectangular order S, we reverse the ordered pairs in the first category.

$$\begin{aligned} S.1 &= T(L) < T(U_2) < T(U_1) \\ S.2 &= T^d(U_2) < T^d(U_1) < T^d(L) \end{aligned}$$

Next, we define a relation Q on X such that (1) it contains all ordered pairs in $P \cap (L \times U_1)$, and (2) it reverses all ordered pairs in the second category, that is, it includes $((R \setminus P) \cap (L \times U_2))^d$. Formally,

$$Q = (P \cap (L \times U_1)) \bigcup \{(y, x) \mid x \in L,\ y \in U_2 \text{ and } x \parallel y \text{ in } P\}$$

Note that, to realize P, it suffices to construct a rectangular realizer $\mathcal{T}$ such that (1) $Q \subseteq \mathrm{rel}(\mathcal{T}) \subseteq \mathrm{tr}(Q)$, and (2) $\mathcal{T}$ is non-interfering with $\mathrm{tr}(Q)$. Due to transitivity, $\mathrm{rel}(\mathcal{T})$ may contain ordered pairs from $U_2 \times U_1$. Such ordered pairs, however, have already been reversed in R.

For clarity, for every element $x \in X$, we distinguish between the subset of elements that are less than x in P, denoted by $D_P(x)$, and the subset of elements that are less than x in $\mathrm{tr}(Q)$, denoted by $D_Q(x)$. Similarly, we define $U_P(x)$ and $U_Q(x)$. For ease of exposition, assume that for every element $x \in U_1$, the elements in $D_Q(x) \cap L = D_P(x)$—at most k in number—are numbered from 1 to $|D_P(x)|$. Similarly, for every element $x \in U_2$, the elements in $U_Q(x) \cap L = L \setminus D_P(x)$—at most k in number—are numbered from 1 to $(|L| - |D_P(x)|)$. We next define k suborders Q_i for $i = 1, 2, \ldots, k$ as follows:

$$Q_i \;=\; \mathrm{tr}\left(\begin{array}{l} \{(y, x) \mid x \in U_1 \text{ and } y \text{ is the } i^{th} \text{ element in } D_Q(x) \cap L, \\ \qquad\qquad \text{if it exists}\} \bigcup \\ \{(x, y) \mid x \in U_2 \text{ and } y \text{ is the } i^{th} \text{ element in } U_Q(x) \cap L, \\ \qquad\qquad \text{if it exists}\} \end{array}\right)$$

It is easy to see that every ordered pair in Q is present in some Q_i where $i \in [1, k]$. Formally,

$$Q \;\subseteq\; \bigcup_{i=1}^{k} Q_i$$

Finally, consider the subposet (X, Q_i). Note that its Hasse diagram will comprise of disjoint subsets of X where the Hasse diagram of each disjoint subset resembles a "star" with an element of L at the center. For an illustration refer to Figure 5. Clearly, a "star" can be realized using at most one rectangle. The rectangular order T_i realizing Q_i on X can then be obtained by disjoint composition of the rectangular orders corresponding to each "star". Finally, the required rectangular realizer $\mathcal{T}$ is given by $\{T_1, T_2, \ldots, T_k\}$. □

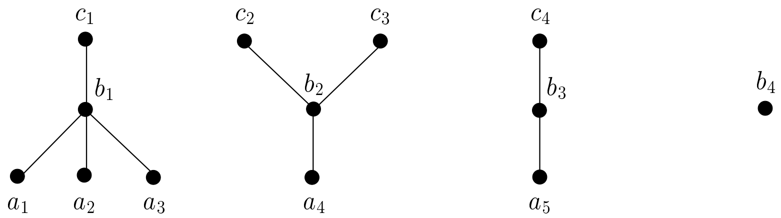

Figure 5. A disjoint set of "stars" where $L = \{b_1, b_2, b_3, b_4\}$, $U_1 = \{c_1, c_2, c_3, c_4\}$ and $U_2 = \{a_1, a_2, a_3, a_4, a_5\}$.

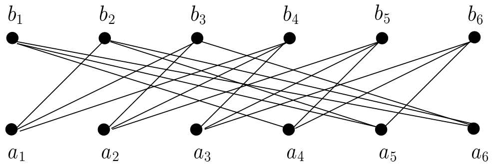

Figure 6. The crown $\mathbf{S}_4^2$.

In case for every element $x \in U$, $|D(x)| \geqslant |L| - k$, it can be proved that $\mathrm{rdim}(X, P) \leqslant k+1$. The standard example $\mathbf{S}_n$ for $n \geqslant 3$, shown in Figure 1, is a bipartite poset formed by singletons and $(n-1)$-element subsets of n distinct elements ordered by set containment (Trotter, 1992). Note that for the standard example, $k = 1$, and for every element $x \in U$, $|D(x)| = |L| - 1 = n - 1$. Thus $\mathrm{rdim}(\mathbf{S}_n) \leqslant 1 + 1 = 2$ for each $n \geqslant 3$ which is a substantial improvement over the chain dimension of the standard example given by $\dim(\mathbf{S}_n) = n$.

We next establish that even a generalized crown can be encoded optimally using only a small number of rectangles. A *generalized crown* $\mathbf{S}_n^k$ (Trotter, 1992), for integers $n \geqslant 3$ and $k \geqslant 0$, is a bipartite poset with $L = \{a_1, a_2, \ldots, a_{n+k}\}$ and $U = \{b_1, b_2, \ldots, b_{n+k}\}$. For each $i = 1, 2, \ldots, n+k$, $b_i \parallel \{a_i, a_{i+1}, \ldots, a_{i+k}\}$ in $\mathbf{S}_n^k$, and $b_i > a_j$ in $\mathbf{S}_n^k$ for each $j = i+k+1, i+k+2, \ldots, i+k+n-1$. In the definition, the subscripts are to be interpreted cyclically. Figure 6 depicts $\mathbf{S}_4^2$. The dimension of the crown $\mathbf{S}_n^k$, for each $n \geqslant 3$ and $k \geqslant 0$, is given by $\dim(\mathbf{S}_n^k) = \lceil 2(n+k)/(k+2) \rceil$ (Trotter, 1992). We demonstrate that the rectangular dimension of a generalized crown is at most two. To accomplish this, we rely on two properties defined for a bipartite poset in (Spinrad et al., 1987) as follows.

*Definition 7. (adjacency property (*Spinrad et al., 1987*))* A bipartite poset $\mathbf{P} = (X, P) = (L, U, P)$ satisfies the *adjacency property* if there is an ordering of elements of U such that for each element $x \in L$, elements of $U(x) \subseteq U$ occur consecutively in the ordering.

Bipartite posets satisfying the adjacency property can be encoded optimally as proved next.

Theorem 11. Let $\mathbf{P} = (X, P) = (L, U, P)$ be a bipartite poset that satisfies the adjacency property. Then,

$$\mathrm{rdim}(X, P) \leqslant 2$$

Proof. Our approach is to first construct a rectangular order R that contains P. However, R may contain ordered not present in P but such pairs only involve the elements of L which are reversed by the second rectangular order S, if required.

Let T be some total order on $L \cup U$ that is consistent with the total order on U as dictated by the adjacency property. Further, let $u_i \in U$ denote the i^{th} element in the total order $T(U)$ for $i = 1, 2, \ldots, n$ where $n = |U|$.

The total orders $R.1$ and $R.2$ on X that realize the rectangular order R are both constructed recursively. To construct the total order $R.1$, we start with the empty order G_0. At the i^{th} step for $i = 1, 2, \ldots, n$, we "append" to G_{i-1} the element u_i and those elements of $D(u_i)$ that are not already in G_{i-1} as follows:

$$G_i = G_{i-1} < T(D(u_i) \setminus G_{i-1}) < u_i$$

Finally, to obtain $R.1$, we "append" to G_n the remaining elements of L (that are not already in there) as follows: $R.1 = G_n < T(L \setminus G_n)$. The total order $R.2$ is constructed in a similar fashion except that the elements of U are traversed in the reverse order. More precisely, starting from the empty order H_0, H_i for $i = 1, 2, \ldots, n$ is constructed recursively as given:

$$H_i = H_{i-1} < T(D(u_{n-i+1}) \setminus H_{i-1}) < u_{n-i+1}$$

Finally, we "prepend" to H_n the remaining elements of L (that are not already in there) to obtain $R.2$ as follows: $R.2 = T(L \setminus H_n) < H_n$. We demonstrate that the rectangular order R satisfies two important properties. First, it contains all ordered pairs in P. Second, all other ordered pairs in R, given by $R \setminus P$, only involve the elements of L, that is, $R \setminus P \subseteq L \times L$.

Since $R.1(U) = T(U)$ and $R.2(U) = T^d(U)$, $R(U) = \emptyset$ which means that the elements of U form an antichain in R. Note that $L \setminus G_n = L \setminus H_n = A$ (say). Informally, an element of A is a "loose" point not connected to any other element in X. Clearly, by construction, $A \parallel U$ in R. It now remains to be shown that $R \cap (U \times (L \setminus A)) = \emptyset$ and $R \cap ((L \setminus A) \times U) = P$.

Consider elements $a \in L \setminus A$ and $b \in U$. In case $b < a$ in $R.1$, there exists an element $c \in U$ so that $b < c$ in T and $a \in D(c)$. However, by construction, $c < b$ in $R.2$ implying that $a < b$ in $R.2$. As a result, $a \parallel b$ in R. Thus no element of U is less than an element of $L \setminus A$ in R or $R \cap (U \times (L \setminus A)) = \emptyset$. Now, consider the other case when $a < b$ in $R.1$. Suppose $a \in D(b)$. By construction, $a < b$ in $R.2$ which implies that $a < b$ in R. Thus every dependency in P is a dependency in R or $P \subseteq R$. Finally, suppose $a \notin D(b)$. Then there is an element $c \in U$ with $c < b$ in T such that $a \in D(c)$. We claim that $a \not< b$ in $R.2$. Assume the contrary, that is, $a < b$ in $R.2$. Since $a \notin D(b)$, there is an element $d \in U$ such that $d < b$ in T^d (or, $b < d$ in T) with $a \in D(d)$. Therefore $a \in D(c) \cap D(d)$, $c < b < d$ in T and $a \notin D(b)$. This implies that elements in $U(a)$ do not occur consecutively in T, thereby violating the adjacency property—a contradiction. Thus if an element of $L \setminus A$ is less than an element of U in R then so is the case in P or $R \cap ((L \setminus A) \times U) \subseteq P$.

Finally, the rectangular order S on X that reverses the ordered pairs in $R \setminus P$ is obtained by disjoint composition of the rectangular order $R^d(L)$ on L with the empty order on U. □

We now define the second property needed to prove the result about generalized crowns.

*Definition 8. (enclosure property (*Spinrad et al., 1987*))* A bipartite poset $\mathbf{P} = (X, P) = (L, U, P)$ that satisfies the adjacency property also satisfies the *enclosure property* if for all elements $x, y \in L$ with $U(x) \subseteq U(y)$, elements in $U(y) \setminus U(x) \subseteq U$ occur consecutively in the ordering dictated by the adjacency property.

It can be proved that if a bipartite poset satisfies both the adjacency and enclosure properties then its rectangular dimension is at most one.

*Theorem 12. ((*Spinrad et al., 1987*))* Let $\mathbf{P} = (X, P)$ be a bipartite poset that satisfies the adjacency and enclosure properties. Then,

$$\mathrm{rdim}(X, P) \leqslant 1$$

Proof. Spinrad et al. (Spinrad et al., 1987) prove that a bipartite poset that satisfies the adjacency and enclosure properties has the chain dimension of at most two, thereby implying that its rectangular dimension is at most one. □

For a bipartite poset $\mathbf{P} = (X, P) = (L, U, P)$, let $\mathrm{adjnum}(X, P)$ denote the least number of suborders into which P can be decomposed such that the bipartite poset induced by each suborder (on X) satisfies the adjacency property. Observe that the ordering on U imposed by the adjacency property may be different for various suborders. Clearly, $\mathrm{adjnum}(X, P) \leqslant \lceil |L|/2 \rceil$. The heuristics described in (Gambosi et al., 1990) can be used to compute a decomposition of the partial order into suborders satisfying the required property; the decomposition may not be optimal. The next theorem generalizes the previous theorem.

Theorem 13. Let $\mathbf{P} = (X, P) = (L, U, P)$ be a bipartite poset. Then,

$$\mathrm{rdim}(X, P) \leqslant \mathrm{adjnum}(X, P) + 2$$

Further, if the bipartite poset induced by each suborder also satisfies the enclosure property then,

$$\mathrm{rdim}(X, P) \leqslant \mathrm{adjnum}(X, P)$$

Proof. Set $t = \mathrm{adjnum}(X, P)$. Let $P = P_1 \cup P_2 \cup \cdots \cup P_t$ be a decomposition of P into t suborders such that each bipartite subposet (X, P_i) satisfies the adjacency property. For each subposet (X, P_i), we construct a rectangular order R_i using the construction given in the proof of Theorem 11 for the first rectangle. The rectangle R_i satisfies two important properties. First, it completely contains P_i. Second, the ordered pairs in R_i that do not belong to P_i involve only the elements of L, that is, $R_i \setminus P_i \subseteq L \times L$. Thus the family of rectangular orders $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ "almost" realizes P on X but for some ordered pairs involving the elements of L that need to be reversed. To that end, we use two additional rectangles. Let T denote some total order on L. The first rectangular order, denoted R_{t+1}, is obtained by disjoint composition of the total order T on L with the empty order on U. The second rectangular order, denoted R_{t+2}, is obtained by disjoint composition of the dual of the total order T on L, given by T^d, with the empty set on U. It is easy to verify that $\mathcal{R} \cup \{R_{t+1}, R_{t+2}\}$ realizes P on X.

In case each suborder also satisfies the enclosure property, from Theorem 12, there is a rectangular order R_i that realizes P_i on X. Moreover, trivially, each rectangular order is non-interfering with P. Thus $\mathcal{R} = \{R_1, R_2, \ldots, R_t\}$ is a rectangular realizer of P on X. □

We establish that $\mathrm{adjnum}(\mathbf{S}_n^k) = 2$ for each $n \geqslant 3$ and $k \geqslant 0$. Moreover, the bipartite poset induced by each suborder satisfies the adjacency and enclosure properties. Thus we obtain:

Theorem 14. The rectangular dimension of a generalized crown is at most two.

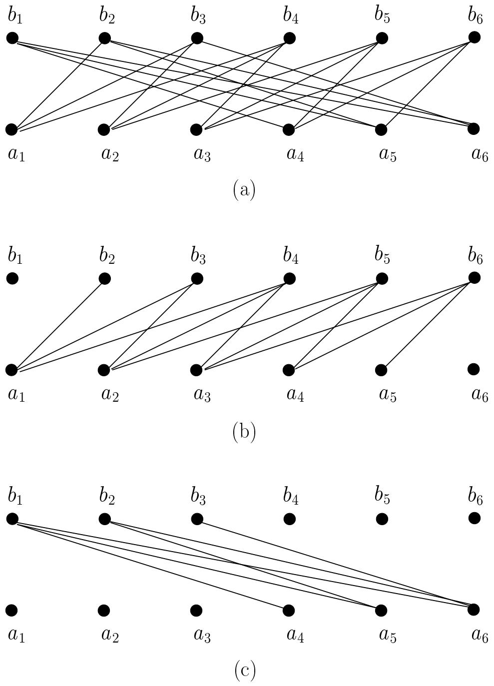

Figure 7. (a) The crown $\mathbf{S}_4^2$, (b) and (c) its two subposets that satisfy the adjacency and enclosure properties.

Proof. We establish that for the generalized crown $\mathbf{S}_n^k$, $\mathrm{adjnum}(\mathbf{S}_n^k) \leqslant 2$. Moreover, the subposets induced by the two suborders also satisfy the enclosure property. The theorem then follows immediately from Theorem 13. To accomplish this, it helps to redefine the generalized crown $\mathbf{S}_n^k$, this time from the point of view of the elements in L. For each $i = 1, 2, \ldots, n+k$, $a_i \parallel \{b_{i-k}, b_{i-k+1}, \ldots, b_i\}$ in $\mathbf{S}_n^k$, and $a_i < b_j$ in $\mathbf{S}_n^k$ for each $j = i+1, i+2, \ldots, i+n-1$. Of course, the subscripts are to be interpreted cyclically.

For each element $a_i \in L$, we partition the set $U(a_i) = \{b_{i+1}, b_{i+2}, \ldots, b_{i+n-1}\}$ into two subsets denoted by $U_1(a_i)$ and $U_2(a_i)$. The first subset $U_1(a_i)$ consists of elements b_j for $j = i+1, i+2, \ldots, \min\{i+n-1, n+k\}$. In case $i+n-1 > n+k$, the second subset $U_2(a_i)$ consists of elements b_j for $j = n+k+1, n+k+2, \ldots, i+n-1$ (interpreted cyclically); otherwise it is empty. Equivalently, if $i-1 > k$ then $U_2(a_i) = \{b_1, b_2, \ldots, b_{i-k-1}\}$ otherwise $U_2(a_i) = \emptyset$.

Finally, the first suborder P_1 is given by $\{(a,b) | a \in L \text{ and } b \in U_1(a)\}$, and the second suborder P_2 by $\{(a,b) \mid a \in L \text{ and } b \in U_2(a)\}$. Clearly, both subposets $(L \cup U, P_1)$ and $(L \cup U, P_2)$ satisfy the adjacency and enclosure properties. For an illustration, refer to Figure 7. $\square$

5. Related Work

The *boolean dimension* of a poset $\mathbf{P} = (X, P)$, defined by Gambosi, Nešetřil and Talamo (Gambosi et al., 1990), is the least positive integer t so that there exist a boolean formula $F(b_1, b_2, \ldots, b_t)$ on boolean variables $b_1, b_2, \ldots, b_t$ and total orders $\{C_1, C_2, \ldots C_t\}$ satisfying the following: for every $x, y \in X$, $x \leqslant y$ in P if and only if $F(b_1, b_2, \ldots, b_t)$ evaluates to true where b_i is true if and only if $x \leqslant y$ in C_i. The notions of dimension and rectangular dimension can be derived from the notion of boolean dimension by appropriately defining the boolean formula. For example, to derive the notion of dimension, the boolean formula can be set to $b_1 \wedge b_2 \wedge \cdots \wedge b_t$. On the other hand, to derive the notion of rectangular dimension, the boolean formula can be set to $\Big((b_1 \wedge b_2) \vee (b_3 \wedge b_4) \vee \cdots \vee (b_{t-1} \wedge b_t)\Big) \bigwedge \Big((b_1 \vee b_2) \wedge (b_3 \vee b_4) \wedge \cdots \wedge (b_{t-1} \vee b_t)\Big)$. Clearly, the boolean dimension of a poset is less than or equal to its rectangular dimension. However, using boolean dimension entails the added complexity of representing the boolean formula F.

The *encoding dimension* of a poset $\mathbf{P} = (X, P)$ is defined as the least positive integer t with $t = \sum_{i=1}^{m} \lceil \log_2(k_i) \rceil$ such that P can be embedded into $K_1 \times K_2 \times \cdots \times K_m$ where K_i denotes a chain of length k_i (Habib et al., 1995). The encoding dimension minimizes the number of bits as opposed to the chain dimension which minimizes the number of chains in a chain realizer. Clearly, if the objective is to minimize the number of bits, the code assigned to each element when rectangles are used is at most two times the encoding dimension of a poset. However, posets with large encoding dimension that can be encoded optimally using rectangles include standard examples and generalized crowns.

Garg and Skawratananond introduce the notion of *string* in (Garg and Skawratananond, 2001). A poset is said to form a string if there is a function f assigning integer values to each element in the set so that an element x is less than another element y in the partial order if $f(x) < f(y)$. In case $f(x) \leqslant f(y)$, we say that x is less than or equal to y in the string. Trivially, a chain is a string and the dimension of a string is at most two. A set of strings realizes a partial order if an element x is less than another element y in the partial order if and only if x is less than or equal to y in all the strings and x is less than y in at least one of the strings. The *string dimension* of a poset is the least number of strings required to realize the corresponding partial order. Garg and Skawratananond (Garg and Skawratananond, 2001) prove that the string dimension of a poset, which is itself not a string, is equal to its dimension. However, encoding posets using strings could potentially use much fewer bits than chains. For example, encoding an antichain using strings requires only 1 bit per element;

every element is assigned bit 0. It can be proved that every poset can be encoded using strings with at most n bits per element, where n is the number of elements. Algorithms by Fidge (Fidge, 1991) and Mattern (Mattern, 1989) for encoding the causality relation between events in a distributed computation using vector clocks can be viewed as special cases of encoding using strings.

It is possible to expand the class of extensions available for use in realizers in dimension theory. In particular, interval orders can be used instead of chains. In that case, the *interval dimension* of a poset is defined as the least number of interval orders whose intersection is same as the given partial order (Trotter, 1992). Posets with arbitrarily large interval dimension but constant rectangular dimension include standard examples and generalized crowns.

Capelle (Capelle, 1994) propose a technique in which the given partial order is decomposed into a set of interval orders for which optimal encoding is already known. The scheme is similar to our order decomposition method but for the "building block" which is an interval order instead of a two-dimensional poset. While the focus in (Capelle, 1994) is on developing a heuristic for decomposition and evaluating its performance experimentally, we primarily concern ourselves with establishing theoretical upper bounds on the rectangular dimension of a poset in general and special families of posets in particular.

In *PQ-encoding*, proposed by Zibin and Gil (Zibin and Gil, 1997), the set of elements is partitioned into *slices*; each element belongs to exactly one slice. Each slice satisfies the following property: there exists an ordering of elements such that for every element x in the slice, the elements in $x \cup D(x)$ appear consecutively in the ordering. An element is assigned a code which consists of (number of slices + 3) integers. Using PQ-encoding, a *less-than* query can be answered in $O(1)$ time. However, PQ-encoding may not produce an optimal encoding for a poset. For a poset $\mathbf{P} = (X, P)$, let numslices(X, P) denote the least number of such slices into which X can be partitioned. For the standard example $\mathbf{S}_n$, no slice can contain more than two maximal elements implying that numslices$(\mathbf{S}_n) = \lceil n/2 \rceil$. In this case, therefore, PQ-encoding assigns a code consisting of $O(n \log n)$ bits to each element which requires much more space than the encoding obtained using rectangles. Similarly, for the generalized crown $\mathbf{S}_n^k$, no slice can contain more than $k+1$ maximal elements. Therefore numslices$(\mathbf{S}_n^k) = \lceil (n+k)/(k+1) \rceil$, which implies that the PQ-encoding for the generalized crown is suboptimal. In fact, for a bipartite poset (X, P), adjnum$(X, P) \leqslant$ numslices(X, P).

Madej and West (Madej and West, 1991) give the notion of *interval inclusion number* of a poset. An *interval inclusion representation* of a poset is a set-valued function f that assigns to every element in the set a union of intervals from the real line $\mathbf{R}$ so that an element x is less than another element y in the partial order if $f(x) \subsetneq f(y)$. The interval inclusion number of a poset is the least positive integer t for which there is an interval inclusion representation of the poset that maps to every element a union of at most t intervals from the real line $\mathbf{R}$. The algorithm proposed by Agrawal, Borgida and Jagdish (Agrawal et al., 1989) for managing transitive relationships in a directed acyclic graph can be viewed as computing an interval inclusion representation of the poset. Many results on the rectangular dimension of a poset in this paper also hold for its interval inclusion number. However, for the boolean lattice B_n, which contains 2^n elements and whose interval inclusion number is equal to $\lceil n/2 \rceil$, $O(n \log n)$ bits per element are required if interval inclusion representation is used. Our method, in contrast, assigns optimal $O(n)$ bits to each element (by utilizing string representation).

A graph with n vertices has an *implicit representation* if it is possible to assign a label of size $O(\log n)$ to every vertex so that the adjacency relation between every pair of vertices can be ascertained by simply examining the labels of the two vertices only (Kannan et al., 1988). Due to the limit on the size of the label, not every graph (or poset) has an implicit representation. Kannan, Naor and Rudich study in (Kannan et al., 1988) which graphs permit implicit representation; the coding scheme may vary with graphs. We, on the other hand, investigate the behaviour of a specific coding scheme, based on rectangles, for different classes of posets.

6. Conclusion and Future Work

In this paper, we present a novel technique for encoding posets using rectangles. We establish upper bounds on rectangular dimension of posets in general and special families of posets in particular. We also provide examples of posets that can be encoded optimally using our technique but require may more bits per element element using other techniques such as adjacency matrix, adjacency list, dimension, interval dimension and PQ-encoding (Trotter, 1992; Habib et al., 1995; Zibin and Gil, 1997; Garg and Skawratananond, 2001).

Many interesting questions still remain. For instance, given n distinct elements, it can be proved that as many as $2^{\theta(n^2)}$ different posets can be constructed. Using information theory, therefore, a lower bound of $\theta(n/\log n)$ can be obtained on the rectangular dimension of a poset (even a bipartite poset) in the worst case. In this paper, we have only established upper bounds of $n/3$ and $\lceil n/4 \rceil$ on the rectangular dimension of general posets and bipartite posets, respectively, when $n \geqslant 3$. Can we tighten the upper bound to match the lower bound? What classes of posets have large rectangular dimension? Is it possible to compute the rectangular dimension of a poset efficiently or is the problem NP-hard?

We also plan to investigate in detail the relationship between the rectangular dimension of a poset and other measures of dimension such as interval dimension and interval inclusion number.

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