THIS IS THE SEQUENTIAL BOOK VERSION

A Systematic Approach to Sequential Algorithms

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To my family

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Chapter 1

Introduction

The goal of the book is to present a unified treatment of a wide variety of algorithms. Linear programming, or integer programming, serves as a tool for providing insight into a large class of problems. The shortest path problem, the max-flow problem, the stable marriage problem, and the weighted bipartite matching problem can all be modeled and analyzed using linear programming techniques. Linear programming formulation provides additional insights such as notions of dual problems, certificates of optimality, and lower and upper bounds on the objective function. In this book, we present another general technique called *lattice-linear predicate detection* that can solve many problems. We use this method to solve the generalization of many fundamental problems in combinatorial optimization, including the stable marriage problem [GS62], the shortest path problem [Dij59], and the assignment problem [Mun57]. Due to the importance and applications of these problems, each one has been the subject of numerous books and thousands of papers. The classical algorithms to solve these problems are the Gale-Shapley algorithm [GS62] for the stable marriage problem, Dijkstra's algorithm [Dij59] for the shortest path problem, and Kuhn's Hungarian method [Mun57] to solve the assignment problem (or equivalently, Demange-Gale-Sotomayor auction-based algorithm [DGS86] for market clearing prices). Could there be a single efficient algorithm that solves all of these problems?

The book presents a technique that solves not only these problems, but also more *general* versions of each of the above problems. We seek the optimal solution for these problems that satisfies additional constraints modeled using a *lattice-linear* predicate [CG98]. When there are no additional constraints (i.e., when the set of constraints is empty), our approach reverts to addressing the classical versions of these problems.

Our technique requires the underlying search space to be viewed as a distributive lattice [Bir67, DP90, Gar15]. Common to all these seemingly disparate combinatorial optimization problems is the structure of the *feasible* solution space. The set of all stable marriages, the set of all feasible rooted trees for the shortest path problem, and the set of all market clearing prices are all closed under the meet operation of the lattice. If the order is appropriately defined, then finding the optimal solution (the man-optimal stable marriage, the shortest path cost vector, the minimum market clearing price vector) is equivalent to finding the infimum of all feasible solutions in the lattice.

We note here that it is well known that the set of stable marriages and the set of market-clearing price vectors form distributive lattices. The set of stable marriages forms a distributive lattice is given in [Knu97b] where the result is attributed to Conway. The set of market clearing price vectors forms a distributive lattice is given in [SS71]. However, algorithms to find the man-optimal stable marriage and

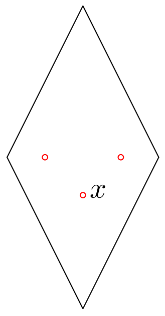


Figure 1.1: Search space with feasible solutions. The feasible solutions are shown in red. The solution x is the least feasible solution.

the minimum market clearing price vectors are not derived from the lattice property. In our method, once the lattice-linearity of the feasible solution space is established, the algorithm to find the optimal solution falls out as a consequence. Reference [Gar20] derives the Gale-Shapley algorithm, the Dijkstra algorithm, and the Demange-Gale-Sotomayor algorithm from a single algorithm by exploiting the lattice property.

The lattice-linear predicate detection method to solve the combinatorial optimization problem is as follows. The first step is to define a lattice of vectors, L, such that each vector is *assigned* a point in the search space. For the stable marriage problem, the vector corresponds to the assignment of men to women (or equivalently, the choice number for each man). For the shortest path problem, the vector assigns a cost to each node. For the market clearing price problem, the vector assigns a price to each item. The comparison operation ($\leq$) is defined on the set of vectors such that the least vector, if feasible, is the extremal solution of interest. For example, in the stable marriage problem, if each man orders women according to his preferences and every man is assigned the first woman in the list, then this solution is the man-optimal solution whenever the assignment is a matching and has no blocking pair. Similarly, in the shortest path problem and the minimum market clearing price problem, the zero vector would be optimal if it were feasible.

The second step in our method is to define a Boolean predicate B that models the feasibility of the vector. For the stable marriage problem, an assignment is feasible if and only if it is a matching and there is no blocking pair. For the shortest path problem, an assignment is feasible iff there exists a rooted spanning tree at the source vertex such that the cost of each vertex is greater than the cost of traversing the path in the rooted tree. For the minimum market clearing price problem, a price vector is feasible iff it is a market clearing price vector.

The third step is to show that the feasibility predicate is a lattice-linear predicate [CG98]. The lattice-linearity property allows one to search for a feasible solution efficiently. If any point in the search space is not feasible, it allows one to make progress towards the optimal feasible solution without any need for exploring multiple paths in the lattice. Moreover, multiple processes can make progress towards the feasible solution independently. In a finite distributive lattice, it is clear that the maximum number of such advancement steps before one finds the optimal solution or reaches the top element of the lattice is equal to the height of the lattice. Once this step is completed, we get the following outcomes.

First, by applying the lattice-linear predicate detection algorithm to unconstrained problems, we get the Gale-Shapley algorithm for the stable marriage problem, Dijkstra's algorithm for the shortest path problem, and Demange-Gale-Sotomayor's algorithm for the minimum market clearing price. In fact, the lattice-linear predicate detection method yields a parallel version of these algorithms, and by restricting these to their sequential counterparts, we get these classical sequential algorithms.

Second, we get solutions for the constrained version of each of these problems whenever the constraints are lattice-linear. We solve the *constrained stable marriage problem* where in addition to men's preferences and women's preferences, there may be a set of lattice-linear constraints. For example, we may require that Peter's regret [GI89] should be less than that of Paul, where the *regret* of a man in a matching is the choice number he is assigned. We note here that some special cases of constrained stable marriage problems have been studied. Dias et al. [DdFdFS03, CM16] study the problem of stable marriage with restricted pairs. A restricted pair is either a *forced* pair that is required to be in the matching, or a *forbidden* pair that must not be in the matching. Both of these constraints are *lattice-linear* and therefore can be modeled in our system. The constrained shortest path problem asks for a rooted tree at the source node with the smallest cost at each vertex that satisfies additional constraints of the form "the cost of reaching node x is at least the cost of reaching node y", "the cost of reaching x must be equal to the cost of reaching y", and "the cost of reaching x must be within δ of the cost of reaching y". For the market clearing price problem, we consider constraints on the market clearing prices of the form that item i must be priced at least as much as item j, or the difference in prices for items i and j must not exceed δ.

Third, by applying a constructive version of Birkhoff's theorem on finite distributive lattices [Bir67, DP90], we give an algorithm that outputs a succinct representation of all feasible solutions. In particular, the join-irreducible elements [DP90] of the feasible sublattice can be efficiently determined (in polynomial time). For the constrained stable marriage problem, we get a concise representation of all stable marriages that satisfy given constraints. Thus, our method yields a more general version of the rotation posets [GI89] to represent all *constrained* stable marriages. Analogously, we get a concise representation of all constrained integral market clearing price vectors.

In the remainder of the chapter, we provide an introduction to algorithms, emphasizing their definition, efficiency analysis, and the mathematical basis. We explore what constitutes an algorithm and formalize efficiency with asymptotic notation.

1.1 What is an Algorithm?

An algorithm is a finite, unambiguous sequence of instructions that solves a problem or computes a function, transforming an input into an output in a finite number of steps. It must exhibit *finiteness* (terminates), *definiteness* (each step is precisely specified), and *effectiveness* (each operation is basic and executable). Algorithms are the foundation of computer science, solving problems from simple arithmetic to complex optimization.

We start with one of the earliest algorithms due to Euclid. Suppose that we are required to find the greatest common divisor (GCD) of two natural numbers a and b (integers greater than or equal to 1). If they are both equal, then we have the answer. Suppose that a is greater than b, then we claim that it is safe to reduce a to $a - b$ (prove it!). Similarly, if b is greater than a, then we reduce b to $b - a$. The algorithm terminates with all numbers identical and equal to the GCD.

For $a = 48$, $b = 18$, in the first step, we get a equals 30 and b equals 18. Since a is still greater than b, a becomes 12 and b is still 18. Now b is greater than a. So, in the next step b becomes $b - a$ which equals 6. Finally, a becomes 6. Now, both a and b are equal to 6 and the algorithm terminates. It is easy to see that the algorithm always terminates. All values are initially non-zero positive integers. They stay non-zero and integral and always decrease; therefore, the algorithm must terminate.

Algorithm GCD: GCD(a, b)

```
Input  : Two positive integers $a$, $b$ greater than or equal to 1
Output: The greatest common divisor of $a$ and $b$
while $a \neq b$ do
    if $(a > b)$ then $a := a - b$;
    if $(b > a)$ then $b := b - a$;
end
return $a$
```

In our algorithm, we have used subtraction to decrease the numbers. We leave it as an exercise for the reader to use the *mod* function to speed up the above algorithm.

Suppose that we want to compute the factorial of a non-negative integer n. Algorithm Factorial shows a method that uses *recursion*. A recursive approach requires specification of the *base case*, when n equals 0. For a higher value of n, we use the value of Factorial($n-1$) to compute Factorial(n).

Algorithm Factorial: Factorial(n)

```
Input  : A non-negative integer $n$
Output: $n!$
if $n = 0$ then return 1 ;
return $n \cdot Factorial(n-1)$
```

As a final simple example of an algorithm, suppose that we are required to find the largest element in an array A. Algorithm MaxElement uses a *for* loop to find this element.

Algorithm MaxElement: MaxElement(A)

```
Input  : Array $A$ of size $n$
Output: Maximum element in $A$
$max := A[0]$ ;
for $i = 1$ to $n-1$ do
    if $A[i] > max$ then $max := A[i]$ ;
end
return $max$
```

1.2 Asymptotic Notation (Big O, Big Omega, Big Theta)

Asymptotic notation quantifies runtime growth, abstracting constants and lower-order terms to focus on scalability. It is the cornerstone of algorithmic comparison.

- $T(n) = O(f(n))$ if $\exists c, n_0 > 0$ such that $T(n) \leq c \cdot f(n)$ for all $n \geq n_0$. For example, suppose that $T(n) = 2n + 5$. Since $2n + 5 \leq 3n$ for $n \geq 5$, we conclude that $T(n)$ equals $O(n)$.

- We say $T(n) = \Omega(f(n))$ if $\exists c, n_0 > 0$ such that $T(n) \geq c \cdot f(n)$ for all $n \geq n_0$. For example, suppose that $T(n) = n^2 - n$. Since $n^2 - n \geq 0.5n^2$ for $n \geq 2$, we conclude that $T(n)$ equals $\Omega(n^2)$.

- We say that $T(n) = \Theta(f(n))$ if $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$. For example, suppose that $T(n) = 4n^2 + 3n$, $3n^2 \leq 4n^2 + 3n \leq 5n^2$ for $n \geq 3$. We conclude that $T(n)$ equals $\Theta(n^2)$.

We first give an example, where $T(n)$ is a linear growing function of n. Let $T(n) = 7n + 10$. Then, it is easy to see that $T(n)$ equals $O(n)$: $7n + 10 \leq 8n$ for $n \geq 10$. In addition, $T(n)$ equals $\Omega(n)$: $7n + 10 \geq 7n$. Thus, $T(n)$ equals $\Theta(n)$.

As another example, suppose $T(n) = 2n^3 + 5n^2 + n$. Then $T(n)$ equals $O(n^3)$ because $2n^3 + 5n^2 + n \leq 3n^3$ for $n \geq 5$. Similarly, $T(n)$ equals $\Omega(n^3)$ because $2n^3 + 5n^2 + n \geq 2n^3$. Hence, $T(n)$ equals $\Theta(n^3)$.

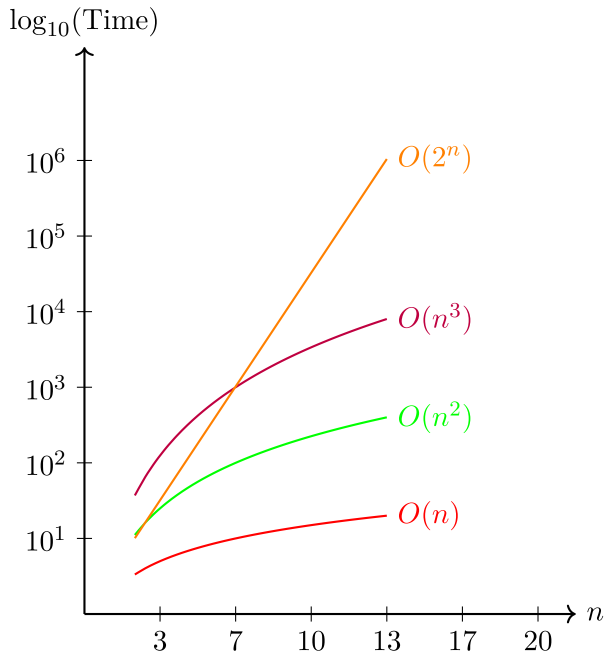


Figure 1.2: Comparison of Growth Rates for $n = 1$ to 20: Linear, Quadratic, Cubic, and Exponential (Log Scale)

1.3 Analyzing Algorithm Efficiency

Efficiency analysis evaluates the time and space complexity across best, worst, and average cases, providing a holistic view of performance. We start with the Algorithm LinearSearch.

The time complexity of the algorithm is as follows.

- *Best*: $O(1)$, key at $A[0]$.

- *Worst*: $O(n)$, key absent.

- *Average*: $\frac{n+1}{2} = O(n)$, assuming a uniform likelihood.

Algorithm LinearSearch: LinearSearch(A, key)

```
Input  : Array $A$ of size $n$, key to find
Output: Index of key or -1 if not found
for $i = 0$ to $n - 1$ do
    if $A[i] = key$ then return $i$ ;
end
return -1
```

Algorithm NestedSum: NestedSum(n)

```
Input  : Integer $n$
Output: Sum of products
$total := 0$ ;
for $i = 0$ to $n - 1$ do
    for $j = 0$ to $n - 1$ do
        $total := total + i \cdot j$ ;
    end
end
return $total$
```

We now consider the example of a *nested loop* shown in Algorithm NestedSum. The time complexity of the algorithm is $O(n^2)$. The space complexity of the algorithm is $O(1)$. The time and space complexity will always refer to the *worst case* unless otherwise specified.

Let us consider another example, in which our goal is to find the number of pairs in an array A that sum to the provided target sum. The reader should verify that the time complexity of the algorithm is $O(n^2)$.

Algorithm PairSum: PairSum(A, target)

```
Input  : Array $A$ of size $n$, target sum
Output: Number of pairs summing to target
$count := 0$ ;
for $i = 0$ to $n - 1$ do
    for $j = i + 1$ to $n - 1$ do
        if $A[i] + A[j] = target$ then $count := count + 1$ ;
    end
end
return $count$
```

We now give an example of an algorithm BinarySearch that speeds up Algorithm LinearSearch. We see recurrence $T(n) = T(n/2) + 1$. Later in this book, we show that $T(n) = O(\log n)$.

As another example, we describe the problem called Towers of Hanoi. The Towers of Hanoi problem involves three pegs and n disks of increasing size, initially stacked on one peg in decreasing size order. The goal is to move the entire stack to another peg, obeying two rules: only one disk can be moved at a

Algorithm BinarySearch: BinarySearch(A, key, low, high)

```
Input  : Array $A$, key, range $[low, high]$
Output: Index of key or -1
1 if $low > high$ then return -1 ;
2 $mid := \lfloor (low + high)/2 \rfloor$ ;
3 if $A[mid] = key$ then return $mid$ ;
4 else if $A[mid] > key$ then return $BinarySearch(A, key, low, mid - 1)$ ;
5 else return $BinarySearch(A, key, mid + 1, high)$ ;
```

time, and a larger disk cannot be placed on a smaller disk. The solution requires $2^n - 1$ moves, achieved recursively by moving $n-1$ disks to a spare peg, transferring the largest disk, then moving the $n-1$ disks onto the largest. Algorithm Hanoi shows a recursive solution to this problem. Here, the recurrence is: $T(n) = 2T(n-1) + 1$, $T(1) = 1$. On solving this recurrence, we get $T(n) = 2^n - 1$.

Algorithm Hanoi: Hanoi(n, source, aux, target)

```
Input  : Number of disks $n$, pegs
1 if $n = 1$ then Move disk from $source$ to $target$ ;
2 else
3     Hanoi$(n - 1, source, target, aux)$ ;
4     Move disk from $source$ to $target$ ;
5     Hanoi$(n - 1, aux, source, target)$ ;
6 end
```

Finally, we consider the problem of sorting an array A using a method called *MergeSort*. The algorithm uses recursion to sort an array using a method called divide-and-conquer. In this example, we have the recurrence $T(n) = 2T(n/2) + n$. We later show that the recurrence can be solved to determine that $T(n) = O(n \log n)$.

Algorithm MergeSort: MergeSort(A, low, high)

```
Input  : Array $A$, range $[low, high]$
1 if $low < high$ then
2     $mid := \lfloor (low + high)/2 \rfloor$ ;
3     MergeSort$(A, low, mid)$ ;
4     MergeSort$(A, mid + 1, high)$ ;
5     Merge$(A, low, mid, high)$ ;
6 end
```

1.4 Common Data Structures: Lists, Stacks, Queues, Binary Trees

This section briefly describes four fundamental data structures: linked lists, stacks, simple queues, and binary search trees, focusing on their main methods and time complexities.

Linked Lists

A linked list is a linear sequence of nodes, where each node contains data and a reference to the next node.

- **Insert** (at head/tail): $O(1)$
- **Delete** (given node): $O(1)$ with pointer; $O(n)$ to find
- **Search**: $O(n)$

Stacks

A stack is a Last-In-First-Out (LIFO) structure, typically implemented with an array or linked list.

- **Push** (add to top): $O(1)$
- **Pop** (remove from top): $O(1)$
- **Top** (access top): $O(1)$

Simple Queues

A simple queue is a First-In-First-Out (FIFO) structure, often using an array or linked list.

- **Enqueue** (add to rear): $O(1)$
- **Dequeue** (remove from front): $O(1)$ with linked list; $O(n)$ with array
- **Front** (access front): $O(1)$

Binary Search Trees

A binary search tree (BST) is a binary tree in which the left subtree of each node has smaller values and the right subtree has larger values.

- **Insert**: $O(h)$ (height h; $O(\log n)$ if balanced)
- **Delete**: $O(h)$ (height h; $O(\log n)$ if balanced)
- **Search**: $O(h)$ (height h; $O(\log n)$ if balanced)

1.5 Heaps

A heap is a complete binary tree satisfying the heap property: In a minimum heap, the value of each parent is less than or equal to the values of its children. We focus on min-heaps here, implemented as arrays.

Structure and Representation

Figures 1.3 and 1.4 show a min-heap with values 2, 4, 3, 7, 5.

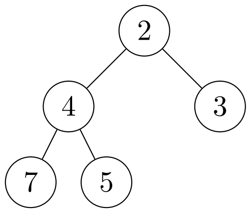


Figure 1.3: Min-heap as a binary tree

2	4	3	7	5
$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$

Figure 1.4: Min-heap as an array

Heapify

The Heapify algorithm ensures the min-heap property at a given node by comparing it with its children and swapping with the smallest if necessary, then recursively applying the process downward. It takes an array A, an index i, and a heap size n, running in $O(\log n)$ time due to the height of the tree.

Algorithm Heapify: Heapify (Min-Heap)

```
Input: Array $A$, index $i$, heap size $n$
$smallest := i$
$left := 2i + 1$
$right := 2i + 2$
if $left < n$ and $A[left] < A[smallest]$ then $smallest := left$ ;
if $right < n$ and $A[right] < A[smallest]$ then $smallest := right$ ;
if $smallest \neq i$ then
    Swap $A[i]$ and $A[smallest]$
    Heapify($A$, $smallest$, $n$)
end
```

Insert

The Insert algorithm adds a new value to the heap by placing it at the end of the array and sifting it up to its correct position, comparing it with its parent until the min-heap property is restored. It runs in $O(\log n)$ time, as it traverses up the tree's height.

Extract-Min

The Extract-Min algorithm removes the minimum element (root) by replacing it with the last element, reducing the heap size, and then calling Heapify to restore the min-heap property. It operates in $O(\log n)$

Algorithm HeapInsert: Heap-Insert

```
   Input: Array $A$, value $key$, heap size $n$
1  $n := n + 1$
2  $A[n-1] := \infty$
3  $i := n - 1$
4  while $i > 0$ and $A[\lfloor (i-1)/2 \rfloor] > key$ do
5      $A[i] := A[\lfloor (i-1)/2 \rfloor]$
6      $i := \lfloor (i-1)/2 \rfloor$
7  end
8  $A[i] := key$
```

time due to the heapification step.

Algorithm HeapExtractMin: Heap-Extract-Min

```
   Input: Array $A$, heap size $n$
1  if $n < 1$ then return $error$ ;
2  $min := A[0]$
3  $A[0] := A[n-1]$
4  $n := n - 1$
5  Heapify($A$, 0, $n$)
6  return $min$
```

Build-Heap

The Build-Heap algorithm constructs a min-heap from an unsorted array by iteratively applying Heapify from the last non-leaf node up to the root. It runs in $O(n)$ time, as the cumulative effect of heapifying all subtrees is linear in the number of nodes.

Algorithm BuildHeap: Build-Heap (Min-Heap)

```
   Input: Array $A$, size $n$
1  for $i := \lfloor n/2 \rfloor - 1$ downto 0 do
2      Heapify($A$, $i$, $n$)
3  end
```

The time complexity of various operations on a heap can be summarized as follows.

- **Heapify**: $O(\log n)$
- **Insert**: $O(\log n)$
- **Extract-Min**: $O(\log n)$
- **Build-Heap**: $O(n)$

1.6 Organization of the Book

Chapter 2 presents the stable marriage problem. Chapter 3 presents the general Lattice Linear Predicate algorithm. It shows how the stable marriage problem and many other problems can be solved using the LLP algorithm. In particular, the chapter gives a more general algorithm than the Gale-Shapley algorithm for the stable marriage. Chapter 4 discusses basic graph algorithms. These include various traversal techniques and the topological sort of a graph. Chapter 5 discusses various greedy algorithms. Chapter 6 discusses various algorithms for computing the shortest path in a graph. Chapter 7 describes algorithms for computing the minimum spanning tree in a weighted undirected graph. By considering different lattices on the set of edges, one can derive classical Kruskal's and Prim's algorithms. Chapter 8 discusses various sorting algorithms. Chapter 9 discusses the divide-and-conquer strategy to solve various problems. Chapter 10 discusses various dynamic programming problems. These include the longest increasing subsequence problem, the optimal binary search tree problem, and the knapsack problem. All of these problems are shown to be solvable using the LLP algorithm. Chapter 11 discusses the topic of computing max-flow in a directed graph with capacity on edges. This algorithm is useful in many applications. Chapter 12 describes one of these applications, bipartite matching, in detail. Chapter 13 describes the theory of NP-completeness for algorithms. This theory relates the difficulty of solving one problem to that of another. A standard course in algorithms may cover all the chapters up to this point. The remaining chapters may be covered depending on the interest of the instructor.

Chapter 14 discusses the housing allocation problem, which is a one-sided version of the stable marriage problem. Chapter 15 discusses the assignment problem. The problem, called the weighted bipartite matching problem, also has many applications. Chapter 16 discusses subclasses of predicates whose satisfiability can be evaluated efficiently. We then revisit the stable marriage problem in Chapter 17. In particular, we discuss many variants of the standard stable marriage problem. Chapter 18 discusses various number-theoretic algorithms from the lens of LLP algorithms. Chapter 19 describes linear programming, another general algorithm design technique. We show that many problems described using the LLP paradigm can also be solved using linear programming. Chapter 20 discusses the predicate detection problem in a poset. It explores special cases of LLP algorithms such as conjunctive predicates and the complexity of determining whether the given predicate is lattice-linear. Chapter 21 discusses methods to enumerate all the elements of a distributive lattice that satisfy a given predicate. Chapter 22 describes other classes of predicates called equilevel and solitary predicates that are also useful in designing or determining the complexity of algorithms.

1.7 Problems

1. Prove or disprove each of the following:
   - **(a)** $3^n = O(2^n)$
   - **(b)** If $f(n) = O(g(n))$, then $2^{f(n)} = O(2^{g(n)})$
   - **(c)** If $f(n) = O(g(n))$, then $f(n)^3 = O(g(n)^3)$
2. Prove the following statements.
   - **(a)** Prove that for any constant $d > 0$, $f(n) = O(d \cdot g(n))$ iff $f(n) = O(g(n))$.
   - **(b)** Prove that if $f(n) = O(h_1(n))$ and $g(n) = O(h_2(n))$ then $f(n) + g(n) = O(\max\{h_1(n), h_2(n)\})$.

(c) Prove that all logarithms are asymptotically equivalent. That is, prove that $\log_a(n) = O(\log_b(n))$ for any positive a and b.

3. Suppose that you have a min Heap of size n.
   (a) Give the pseudo-code to find and delete the smallest element in the Heap.
   What is the time complexity of your algorithm in terms of n ?
   (b) Give the pseudo-code to find and delete the k^{th} smallest element in the heap. What is the time complexity of your algorithm in terms of n and k ? Assume that you have a method available to you for a Heap that can insert any element in the heap in $O(\log n)$ time.

4. Suppose that we have an array A of n distinct integers in two forms: a min Heap H and a max heap K. We are given the following functions on the min-heap:
   $H.getMin()$ // return the least value in the heap H
   $H.deleteMin()$ // delete the least value from the heap H
   Similarly, we have analogous methods for the max-heap.

   Give pseudo code of an efficient program to return a value $A[i]$ in the array A such that there exists j where $A[i] + A[j]$ equals m. If there are no two such entries, then the program should return null.

   Give the tight asymptotic time complexity of your method.

1.8 Bibliographic Remarks

Heaps were introduced by J.W.J. Williams in 1964 as part of the heapsort algorithm, which leverages the heap structure for efficient sorting in $O(n \log n)$ time. The linear-time Build-Heap construction was later analyzed by Floyd, showing its $O(n)$ complexity. Heaps have since become foundational in priority queue implementations, with applications in scheduling, graph algorithms (e.g., Dijkstra's), and data compression (e.g., Huffman coding).

Chapter 2

The Stable Marriage Problem

2.1 Introduction

The Stable Matching Problem introduced by Gale and Shapley [GS62] has wide applications in economics, distributed computing, resource allocation, and many other fields. In the standard stable matching problem, there are n men and n women, each with their totally ordered preference list. The goal is to find a matching between men and women such that there is no instability, that is, there is no pair of a woman and a man such that they are not married to each other, but prefer each other over their partners.

We consider stable marriage instances with n men numbered $1, 2, \ldots, n$ and n women numbered $1, 2, \ldots n$. The variables $mpref$ and $wpref$ specify the men preferences and the women preferences, respectively. Thus, $mpref[i][k] = j$ iff woman j is k^{th} preference for man i. Fig. 2.1 shows an instance of the stable matching problem.

$mpref$				$wpref$			
m_1	w_2	w_3	w_1	w_1	m_1	m_3	m_2
m_2	w_1	w_2	w_3	w_2	m_2	m_1	m_3
m_3	w_3	w_1	w_2	w_3	m_1	m_2	m_3

Figure 2.1: Stable Matching Problem with men preference list ($mpref$) and women preference list ($wpref$).

In this chapter, we first give the Gale-Shapley algorithm in Section 2.3 which takes $O(n^2)$ time. We then consider a slight generalization of the problem in which the number of men and the number of women may be different, and the marriage is at least equal to the proposal vector (defined later). We assume that the number of women n_w is greater than or equal to the number of men n_m; otherwise, the roles of men and women can be switched. Section 2.4 presents the Algorithm α that takes any proposal vector to a stable proposal vector in $O(n_m^2 + n_w)$ time such that the resulting proposal vector has the least distance from the initial proposal vector of all stable proposal vectors that are greater than the initial proposal vector. This algorithm generalizes the GS algorithm, which assumes the initial proposal vector to be the top choices.

2.2 Proposal Vector Lattice

We use the notion of a *proposal vector* for our algorithms. A vector of (man) proposals, G, is of dimension n, the number of men. We view any vector G as follows: $(G[i] = k)$ if man i has proposed to his k^{th} preference, i.e., the woman given by $mpref[i][k]$. If $mpref[i][k]$ equals j, then $G[i]$ equals k corresponds to man i proposing to woman j. For convenience, let $\rho(G, i)$ denote the woman $mpref[i][G[i]]$. The vector $(1, 1, \ldots, 1)$ corresponds to the proposal vector in which every man has proposed his top choice. Similarly, $(n, n, \ldots, n)$ corresponds to the vector in which every man has proposed to his last choice. Our algorithms can also handle the case when the lists are incomplete, that is, a man prefers to stay alone to be matched to some women. However, for simplicity, we assume complete lists. It is clear that the set of all proposal vectors forms a distributive lattice under the natural less than order in which the meet and join are given by the component-wise minimum and the component-wise maximum, respectively. This lattice has n^n elements.

Given any proposal vector, G, there is a unique matching defined as follows: man i and $\rho(G, i)$ are matched in G if the proposal of man i is the best proposal for that woman in G or any proposal before G. The man i is unmatched otherwise. A woman q is unmatched in G if she does not receive any proposal in G or earlier.

A proposal vector G represents a *man-saturating matching* iff no woman receives more than one proposal in G. Formally, G is a man-saturating matching if $\forall i, j : i \neq j : \rho(G, i) \neq \rho(G, j)$. When the number of men equals the number of women, a man-saturating matching is a perfect matching (all men and women are matched). When the number of men is less than the number of women, then G is a man-saturating matching if every man is matched (but some women are unmatched). We say that a matching M_1 (or a marriage) is less than another matching M_2 if the proposal vector for M_1 is less than that of M_2.

A proposal vector G may have one or more blocking pairs. A pair of man and woman (p, q) is a *blocking pair* in G iff $\rho(G, p)$ is not q, man p prefers q to $\rho(G, p)$, and woman q prefers p to any proposal received in G. Observe that this definition works even when a woman q is unmatched, i.e. she has not received any proposals in G. In this case, woman q prefers p to staying alone, and p prefers q to $\rho(G, p)$.

A proposal vector G is a stable marriage (or a stable proposal vector) iff it is a man-saturating matching and there are no blocking pairs in G. The man-optimal marriage is the *least* stable matching in the proposal lattice, and the woman-optimal marriage is the *greatest* stable matching.

2.3 Gale-Shapley Algorithm

In this section, we first present an algorithm due to Gale and Shapley for this problem (also known as the deferred acceptance algorithm). In this algorithm, a free man proposes to women in his decreasing order of preferences, i.e., he first proposes to his top choice. Only when a man is rejected by his top choice would he move to his next top choice. We maintain the list of all men who are not engaged in the variable *mList*. Initially, all men are free (not engaged) and are on this list. For this section, recall that we assume that the number of men is equal to the number of women.

When any woman z receives a proposal from a man i, she always accepts it if she is not engaged. In this case, they both get engaged, and the variable $partner[z]$ is set to i. If the woman z is engaged, then she compares this proposal with her existing partner. If she prefers this proposal to her existing partner, then she breaks the engagement with her existing partner and makes i her partner. The previous partner joins *mList*, the list of free men. If the woman z prefers her existing partner, then the proposal of the man

i is rejected, and the man i goes back to *mList*. Algorithm Gale-Shapley shows the pseudocode for these steps.

Algorithm Gale-Shapley: Finding the man-optimal marriage

```
input: $mpref$, $rank$
output: man-optimal stable marriage
var
    $mList$: list of $1..n$ // list of men that are free, initially includes all men
    $partner$: array$[1..n]$ of $0..n$ initially $partner[i] = 0$ for all $i$ // Current fiance for woman $i$
    $G$: array$[1..n]$ of $0..n$ initially $G[i] = 0$ for all $i$// Number of proposals made by man $i$
$mList := [1..n]$ // Initialize all men as free
while $mList$ is nonempty do
    $i := mList.removeFirst()$
    $G[i] := G[i] + 1$ // Move to the next available top choice for man $i$
    $z := mpref[i][G[i]]$ // Woman corresponding to that choice
    if $partner[z] = 0$ then $partner[z] := i$;
    else if $rank[z][i] < rank[z][partner[z]]$ then
        $mList.add(partner[z])$
        $partner[z] := i$
    end
    else $mList.add(i)$ ;
end
return $G$
```

It is easy to verify the following properties of the algorithm.

Lemma 2.1 *All the following statements are valid.*

1. *As the algorithm progresses, the partner for a man can only worsen and the partner for a woman can only improve.*

2. *Once a woman is engaged, she stays engaged.*

Observe that once all men are engaged, the list *mList* is empty and the *while* loop in the algorithm terminates. Is it possible for all men to run out of all the choices and still not have a stable marriage? We claim that whenever a man i proposes to his last choice, the proposal is always accepted. The proposal is always accepted if the woman is not engaged. If the woman is engaged, we can deduce that all women are engaged because the man had previously proposed to all other women who are now engaged. However, this is a contradiction because all n women are engaged to different men and the man i is not engaged. Hence, we conclude that the last proposal of any man is always accepted.

How many times can the *while* loop iterate? Observe that in every iteration $G[i]$ increases by 1 for some i. Since $G[i]$ starts with 0 and cannot increase beyond n, the total number of iterations for the *while* loop is $O(n^2)$. Each iteration of the *while* loop takes $O(1)$ time. Hence, the time complexity of the Gale-Shapley Algorithm is $O(n^2)$.

2.4 Algorithm α: Upward Traversal

The algorithm α generalizes the Gale-Shapley algorithm in two fundamental ways.

- *Arbitrary Initial Proposal Vector*: Gale-Shapley algorithm always finds the man-optimal stable marriage. Suppose that we are interested in finding a stable marriage such that men can propose such that the proposal vector is at least I. For example, if $I = (3, 1, 2, 1)$, then the first man cannot propose to his two top choices, and the third man cannot propose to his top choice. The algorithm α works even when the initial proposal vector is arbitrary instead of the top choice for each man, that is, $I = (1, 1, \ldots, 1)$. Observe that the standard Gale-Shapley algorithm does not work as is when the starting proposal vector is arbitrary. The simple Gale-Shapley algorithm would require men to make proposals and women to accept the best proposals they have received so far. If the starting proposal vector is a perfect matching but not stable, then each woman gets a unique proposal. All women would accept the only proposal received, but the resulting marriage may not be stable.

- *Unequal number of men and women*: We consider the case where the number of women exceeds the number of men. If the number of men is greater than the number of women, then we can simply switch the roles in our algorithm.

If we started with the top choices of all men, then the Gale-Shapley algorithm would still return a man-optimal stable matching with the excess women unmatched. However, if we start from an arbitrary proposal vector, we can end up with all women getting unique proposals, but there may exist an unmatched woman who is preferred by some man over his current match. To address this problem, we first do a simple check on the initial proposal vector given by the following lemma. Let $numw(I)$ denote the total number of unique women that have been proposed in all vectors that are less than or equal to I.

Lemma 2.2 *Given any stable marriage instance with m men and the initial proposal vector I there is no stable marriage for any proposal vector $G \geq I$ whenever $numw(I)$ exceeds m.*

Proof: Consider any proposal vector $G \geq I$. Since the total number of men is m, there is at least one woman q who has been proposed in the past of G who does not have any proposal in G. Suppose that the proposal was made by man p. Then, man p prefers q to $\rho(G, p)$ and q prefers p to staying alone.

■

Hence, in our algorithm, we only consider I such that the total number of women proposed up to I is at most n_m.

Even when the number of men equals the number of women, the proposal vector may be a perfect matching but not stable. To address this problem, we first define $forbidden(G, i)$ as the predicate that there exists another man j such that (1) both i and j have proposed to the same woman in G and that woman prefers j, or (2) $(j, \rho(G, i))$ is a blocking pair in G.

We first show that

Lemma 2.3 *Let G be any proposal vector such that $numw(G) \leq n_m$. There exists a man i such that $forbidden(G, i)$ iff G is not a stable marriage.*

Proof: First, suppose that there exists i such that $forbidden(G, i)$. This means that there exists a man j such that j has proposed to the same woman and that woman prefers j or $(j, \rho(G, i))$ is a blocking pair in G. If both i and j have proposed to the same woman in G, then it is clearly not a matching. If $(j, \rho(G, i))$ is a blocking pair, then G is not stable.

Conversely, assume that G is not a stable marriage. This means that either G is not a perfect matching or there is a blocking pair for G. If G is not a perfect matching, then there must be some woman who has been proposed by multiple men. Any man i who is not the most favored in the set of proposals satisfies $forbidden(G, i)$. If G is a perfect matching but not a stable marriage, then there must be a blocking pair (p, q). If q has been proposed in G by man i, then $(p, \rho(G, i))$ is a blocking pair. Hence, $forbidden(G, i)$ holds. If q has not been proposed in G then we know at least $n_m + 1$ women who belong to $numw(G)$ which violates our requirement on G.

■

The algorithm UpwardTraversal exploits the $forbidden(G, i)$ function to find a stable marriage in the proposal lattice. The basic idea is that if a man i is forbidden in the current proposal vector G, then he must go down his preference list until he finds a woman who is unmatched or prefers him to her current match. The *while* loop at line (6) iterates until none of the men is forbidden in G. If the while loop terminates, then G is a stable marriage due to Lemma 2.3. At line (7), the man i advances on his preference list until his proposal is the most preferred proposal to the woman among all proposals that are made to her in any proposal vector less than or equal to G. If there is no such proposal, then there does not exist any $G \geq I$ such that G is stable and in line (8), the algorithm returns null. Otherwise, the man makes that proposal at line (9).

Algorithm UpwardTraversal: An Algorithm that returns the man-optimal marriage greater than or equal to the given proposal vector I.

```
1  input: A stable marriage instance, initial proposal vector $I$
2  output: smallest stable marriage greater than or equal to $I$ (if one exists)
3  $forbidden(G, i)$ holds if there exists another man $j$ such that either (1) both $i$ and $j$ have proposed
     to the same woman in $G$ and that woman prefers $j$, or (2) $(j, \rho(G, i))$ is a blocking pair for $G$.
4  if $numw(I) > n_m$ then return null // no stable matching exists ;
5  else $G := I$;
6  while there exists a man $i$ such that $forbidden(G, i)$ do
7      find the next woman $q$ in the list of man $i$ s.t. $i$ has the most preferred proposal to $q$ until $G$
8      if no such choice after $G[i]$ or the number of women proposed including $q$ exceeds $n_m$ then
         return null // no stable matching exists ;
9      else $G[i]$ := choice that corresponds to woman $q$;
10 end
11 return $G$
```

There are two main differences from the Gale-Shapley algorithm. The first difference is the simple check on the number of women who have been proposed up to G. We require $numw(G) \leq n_m$. Clearly, if the number of women is equal to the number of men, then $numw(G)$ cannot exceed n_m and this check can be dropped. Also, even if the number of women exceeds the number of men, but the initial proposal vector is $(1, 1, \ldots, 1)$, the number of women proposed cannot exceed n_m. However, this check is required when the number of women exceeds m and the initial proposal vector is arbitrary.

The second difference is in the definition of $forbidden(G, i)$. In the standard Gale-Shapley algorithm, a man advances on his preference list only when the woman he has proposed to is either matched with someone more preferable or receives a proposal from a more preferable man. Whenever the Gale-Shapley algorithm reaches a perfect matching, it is a stable matching. For any arbitrary I (for example, a perfect matching that is not stable), it is important to take blocking pairs into consideration as part of the forbidden predicate. The proof of correctness of the algorithm α is left as an exercise.

2.5 Problems

1. Show that the man-oriented Gale-Shapley algorithm returns the stable marriage, which is the worst stable marriage from the perspective of women.
2. Show that the set of stable marriages is closed under the meet operation and the join operation of the proposal lattice.
3. Give an algorithm in which all men start with their least favored choice and improve their choices in the search for a stable marriage.
4. Given an assignment vector, how will you check whether it forms a stable marriage?

2.6 Bibliographic Remarks

The stable matching problem has been studied extensively, with several books and survey articles devoted to the topic [GI89, Knu97b, RS92, IM08, Man13].

Chapter 3

Lattice-Linear Predicate Detection

3.1 Introduction

In this chapter, we discuss a general method called the Lattice-Linear Predicate (LLP) algorithm that can be used to solve a wide variety of problems including the stable marriage problem, the shortest path problem, the assignment problem, the minimum spanning tree problem, and the housing allocation problem.

Section 3.2 formally defines the lattice-linear predicates. These predicates are defined on a distributive lattice. The problem is framed as the search for an element in the lattice that satisfies the predicate. Generally, we are interested in the least element in the lattice that satisfies the predicate. Section 3.3 gives the notation that we use for programs based on lattice-linear predicates. It specifies the lattice that we are working on, the starting element in the lattice, the top element of the lattice, and the predicate *forbidden*, which allows us to advance in the lattice. Section 3.4 lists some desirable properties of the LLP algorithm. The algorithm is naturally *nondeterministic*. There are multiple ways in which the algorithm can advance in the lattice. The algorithm is *online*. It does not inspect any element higher than the current element of the lattice that is under consideration. Finally, the algorithm returns an optimal answer for all components.

3.2 Lattice-Linear Predicates

Let L be the lattice of all n-dimensional vectors of reals greater than or equal to zero vector and less than or equal to a given vector T where the order on the set of vectors is defined by the natural componentwise $\leq$. The minimum element of this lattice is the zero vector. The lattice is used to model the search space of the combinatorial optimization problem. For simplicity, we are considering the lattice of vectors of non-negative reals; later, we show that our results are applicable to any distributive lattice. The combinatorial optimization problem is modeled as finding the minimum element in L that satisfies a boolean *predicate* B, where B models *feasible solutions*. We will assume that the system maintains as its state the current candidate vector $G \in L$ in the search lattice. Even if we are using a sequential algorithm, we say that $G[i]$ is maintained in the process i. We call G, the global state, and $G[i]$, the state of process i.

Finding an element in the lattice that satisfies the given predicate B, is called *predicate detection problem*. Finding the *minimum* element that satisfies B (whenever it exists) is the combinatorial optimization problem. We now define *lattice-linearity*, which enables efficient computation of this minimum element. A key concept in the development of an efficient predicate detection algorithm is that of a *forbidden* state. Given a predicate B and a vector $G \in L$, a state $G[i]$ is *forbidden* (or equivalently, the index i is forbidden) if for any vector $H \in L$, where $G \leq H$, if $H[i]$ equals $G[i]$, then B is false for H. Formally,

Definition 3.1 (Forbidden State [CG98]) *Given any distributive lattice L of n-dimensional vectors of $\mathbf{R}_{\geq 0}$, and a predicate B, we define $forbidden(G, i, B) \equiv \forall H \in L : G \leq H : (G[i] = H[i]) \Rightarrow \neg B(H)$.*

We define a predicate B as *lattice-linear* with respect to a lattice L if, for any global state G, B is false in G implies that G contains a *forbidden state*. Formally,

Definition 3.2 (lattice-linear Predicate [CG98]) *A boolean predicate B is* lattice-linear *with the poly-time algorithm $\mathcal{A}$ with respect to a finite distributive lattice L generated from a poset P iff $\forall G \in L$: $\neg B(G) \Rightarrow (\exists i \in \mathcal{A}(G) : forbidden(G, i, B))$.*

The algorithm $\mathcal{A}$ is efficient in the size of the poset P. It tells us which indices are forbidden. From now on, the algorithm $\mathcal{A}$ would be implicit. We now give some examples of lattice-linear predicates.

1. **Job Scheduling Problem**: Our first example relates to scheduling of n jobs. Each job j requires time t_j for completion and has a set of prerequisite jobs, denoted by $pre(j)$, such that it can be started only after all its prerequisite jobs have been completed. Our goal is to find the minimum completion time for each job. We let our lattice L be the set of all possible completion times. A completion vector $G \in L$ is feasible iff $B_{jobs}(G)$ holds where $B_{jobs}(G) \equiv \forall j : (G[j] \geq t_j) \wedge (\forall i \in pre(j) : G[j] \geq G[i] + t_j)$. B_{jobs} is lattice-linear because if it is false, then there exists j such that either $G[j] < t_j$ or $\exists i \in pre(j) : G[j] < G[i] + t_j$. We claim that $forbidden(G, i, B_{jobs})$. Indeed, any vector $H \geq G$ cannot be feasible with $G[j]$ equal to $H[j]$. The minimum of all vectors that satisfy feasibility corresponds to the minimum completion time.

2. **Continuous Optimization Problem**: We are required to find minimum nonnegative x and y such that $B \equiv (x \geq y^2/4 + 5) \wedge (y \geq x - 4)$. We view this problem as finding the minimum (x, y) pair such that B holds. It is easy to verify that B is lattice-linear. If the first conjunct is false, then x is forbidden. Unless x is increased, the predicate cannot become true, even if other variables (y for this example) increase. If the second conjunct is false, then y is forbidden. In this case, $x = 6$ and $y = 2$ is the pointwise minimum solution. For some predicates, there may not be any solution. For example, when $B \equiv (x \geq 2y^2 + 5) \wedge (y \geq x - 4)$, there is no nonnegative (x, y) pair such that B holds (verify this!). The predicate B is still lattice-linear, but by advancing along x and y we go beyond any bounded (x, y).

3. **A Non Lattice-Linear Predicate** As an example of a predicate that is not lattice-linear, consider the predicate $B \equiv \sum_j G[j] \geq 1$ defined on the space of two-dimensional vectors. Consider the vector G equal to $(0, 0)$. The vector G does not satisfy B. For B to be lattice-linear, either the first index or the second index should be forbidden. However, none of the indices are forbidden in $(0, 0)$. The index 0 is not forbidden because the vector $H = (0, 1)$ is greater than G, has $H[0]$ equal to $G[0]$ but it still satisfies B. The index 1 is also not forbidden because $H = (1, 0)$ is greater than G, has $H[1]$ equal to $G[1]$ but it satisfies B.

The following lemma is useful in proving lattice-linearity of predicates.

Lemma 3.3 *Let B be any Boolean predicate defined on a lattice L of vectors.*
(a) Let $f : L \to \mathbf{R}_{\geq 0}$ be any monotone function defined on the lattice L of vectors of $\mathbf{R}_{\geq 0}$. Consider the predicate $B \equiv G[i] \geq f(G)$ for some fixed i. Then, B is lattice-linear.
(b) Let L_B be the subset of the lattice L of the elements that satisfy B. Then, B is lattice-linear iff L_B is closed under meets.
(c) If B_1 and B_2 are lattice-linear then $B_1 \wedge B_2$ is also lattice-linear.

Proof: (a) Suppose B is false for G. This implies that $G[i] < f(G)$. Consider any vector $H \geq G$ such that $H[i]$ is equal to $G[i]$. Since $G[i] < f(G)$, we get that $H[i] < f(G)$. The monotonicity of f implies that $H[i] < f(H)$ which shows that $\neg B(H)$.
(b) This is shown in [CG98]. Assume that B is not lattice-linear. This implies that there exists a global state G such that $\neg B(G)$, and $\forall i : \exists H_i \geq G : (G[i] = H_i[i])$ and $B(H_i)$. Consider $Y = \cup_i \{H_i\}$. All elements of $Y \in L_B$. However, $inf\ Y$ which is in G is not an element of L_B. This implies that L_B is not closed under the infimum operation. Conversely, let $Y = \{H_1, H_2, \ldots, H_k\}$ be any subset of L_B such that its infimum G does not belong to L_B. Since G is the infimum of Y, for any i, there exists $j \in \{1 \ldots k\}$ such that $G[i] = H_j[i]$. Since $B(H_j)$ is true for all j, it follows that there exists a G for which lattice-linearity does not hold.

(c) Follows from the equivalence of meet-closed predicates with lattice-linearity and that meet-closed predicates are closed under conjunction. For a more direct proof, suppose that $\neg(B_1 \wedge B_2)$. This implies that one of the conjuncts is false and therefore, from the lattice-linearity of that conjunct, a forbidden state exists.

■

For the job scheduling example, we can define B_j as $G[j] \geq max(t_j, max\{G[i] + t_j \mid i \in pre(j)\})$. Since $f_j(G) = max(t_j, max\{G[i] + t_j \mid i \in pre(j)\})$ is a monotone function, it follows from Lemma 3.3(a) that B_j is lattice-linear. The predicate $B_{jobs} \equiv \forall j : B_j$ is lattice-linear due to Lemma 3.3(c). Also note that the problem of finding the minimum vector that satisfies B_{jobs} is well-defined due to Lemma 3.3(b).

3.3 LLP Notation

We first go over the notation used in the description of our LLP algorithms. Fig. 3.1 shows parallel algorithms for job scheduling and shortest-path problems. We have a single variable G in all the examples shown in Fig. 3.1.

All other variables are derived directly or indirectly from G. There are three sections of the program.

The **init section** is used to initialize the state of the program. All parts of the program are applicable to all values of j. For example, the *init* section of the job scheduling program in Fig. 3.1 specifies that $G[j]$ is initially $t[j]$. Every thread j would initialize $G[j]$.

The **always section** defines additional variables that are derived from G. The actual implementation of these variables is left to the system. They can be viewed as macros.

The third section gives the desirable predicate, either by using the **forbidden** predicate or the **ensure** predicate. The *forbidden* predicate has an associated *advance clause* that specifies how $G[j]$ must be advanced whenever the forbidden predicate is true. For many problems, it is more convenient to use the complement of the forbidden predicate. The *ensure section* specifies the desirable predicates of the form $(G[j] \geq expr)$ or $(G[j] \leq expr)$. The statement *ensure* $G[j] \geq expr$ simply means that whenever the process j finds $G[j]$ to be less than $expr$; it can advance $G[j]$ to $expr$. Since $expr$ may refer to G, just by

```
$P_j$: Code for thread $j$
// common declaration for all the programs below
shared var $G$: array[$1..n$] of $0..maxint$;
job-scheduling:
    input: $t[j] : int$, $pre(j)$: list of $1..n$;
    init: $G[j] := t[j]$;
    forbidden: $G[j] < \max\{G[i] + t[j] \mid i \in pre(j)\}$;
    advance: $G[j] := \max\{G[i] + t[j] \mid i \in pre(j)\}$;

job-scheduling:
    input: $t[j] : int$, $pre(j)$: list of $1..n$;
    init: $G[j] := t[j]$;
    ensure: $G[j] \geq \max\{G[i] + t[j] \mid i \in pre(j)\}$;

shortest path from node $s$: Parallel Bellman-Ford
    input: $pre(j)$: list of $1..n$; $w[i,j]$: int for all $i \in pre(j)$
    init: if $(j = s)$ then $G[j] := 0$ else $G[j] :=$ maxint;
    ensure: $G[j] \leq \min\{G[i] + w[i,j] \mid i \in pre(j)\}$
```

Figure 3.1: LLP Parallel Program for (a) job scheduling problem using forbidden predicate (b) job scheduling problem using ensure clause and (c) the shortest path problems

setting $G[j]$ equal to *expr*, there is no guarantee that $G[j]$ continues to be equal to *expr* — the value of *expr* may change due to changes in other components. We use the *ensure* statement whenever *expr* is a monotonic function of G and therefore the predicate is lattice-linear.

3.4 Properties of the LLP Algorithm

The LLP algorithm has many useful properties. We list them here so that the reader can apply them to various problems studied in this book.

These properties are applicable to all the problems for which the LLP algorithm is used.

1. **Nondeterminism in Evaluation of Forbidden Predicate**: Given a global state G, there may be multiple indices j for which $G[j]$ is forbidden. The LLP algorithm is correct irrespective of the order in which these indices are updated. The efficiency of the algorithms may differ depending upon the order in which these indices are updated, but the correctness is independent of the order. For example, in the stable marriage problem, the final answer returned is independent of the order in which men propose. In the shortest path problem, the final answer returned is independent of the order in which the nodes update their estimates. It is important to note that the term *answer* is with respect to the LLP algorithm. For the shortest path problem, the lattice is with respect to the cost and not the actual paths. There may be multiple shortest paths to a vertex, and the order of evaluation of forbidden indices may result in the paths that are returned be different. However, they would all have the same cost.

2. **Parallel Evaluation of Forbidden Predicate**: Suppose that G is shared among different threads, such that thread j is responsible for evaluating $forbidden(G, j)$. While this thread is evaluating the

predicate other threads may have advanced on other indices, i.e., thread j may have old information of $G[i]$ for $i \neq j$. However, this would still keep the algorithm correct. For example, in the stable marriage problem, men can propose to women in parallel. In the shortest path algorithm, multiple vertices can update the estimate of their lower bounds and their parents in parallel.

3. **No Lookahead Required for evaluation of Forbidden Predicate**: The LLP algorithm determines whether an index j is forbidden depending upon only the current global state G. This means that these algorithms are applicable in online settings where the future part of the lattice is revealed only when a forbidden index needs to advance. For example, in the stable marriage problem, when we are computing the man-optimal stable marriage, a man may not reveal his preference list. Only when he is rejected (his state is forbidden), he needs to advance on his choices and therefore reveal the next woman on his list. The same reasoning applies to the housing allocation problem. For some problems, such as the shortest path problem, the weights on the edges is required to evaluate forbidden predicate. Therefore, the graph must be known and static for the LLP algorithm.

4. **The Optimal Feasible Global State is Optimal for Individual Nodes**: Suppose that for the stable marriage problem, one of the men, say m_1 is interested in finding the stable marriage in which he has the topmost choice possible for him. This man does not care how other men are paired. If we take the solution derived from the LLP algorithm, then the woman m_1 is paired with would be identical to the woman m_1 is paired with in any stable marriage that is optimal from just the perspective of m_1. More generally, let G be the global state that is feasible and optimal with respect to index i, i.e., for all feasible H, $G[i] \leq H[i]$. Let G_{llp} be the global state computed by the LLP algorithm, then $G[i] = G_{llp}[i]$. This follows from the meet-closure property of feasible states. If $G[i] < G_{llp}[i]$, then the global state given by $G \sqcap G_{llp}$ is also feasible and strictly smaller than G_{llp} contradicting that G_{llp} is the least global state that satisfies the feasibility predicate.

3.5 LLP Stable Matching Algorithm

We now derive the algorithm for the stable matching problem using Lattice-Linear Predicates. We let $G[i]$ be the choice number that man i has proposed to. Initially, $G[i]$ is 1 for all men. For convenience, let $\rho(G, i)$ denote the woman $mpref[i][G[i]]$.

Definition 3.4 *An assignment G is feasible for the stable marriage problem if (1) it corresponds to a perfect matching (all men are paired with different women) and (2) it has no blocking pairs.*

We show that the predicate "G is a stable marriage" is a lattice-linear predicate.

Lemma 3.5 *The predicate that a vector G corresponds to a stable marriage is lattice-linear.*

Proof: Let z be $\rho(G, j)$, the woman that corresponds to choice $G[j]$ for man j. We define j to be forbidden in G if there exists a man i such that z prefers man i to man j and either man i has also been assigned z in G or he prefers z to his current choice, i.e., man i and woman z would form a blocking pair in G. Formally, $forbidden(G, j)$ is defined as $(\exists i : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j]))$.

It is easy to see that G is not a stable marriage iff $\exists j : forbidden(G, j)$. If G is not a perfect matching then there must be at least one woman who is assigned to two men. In that case, the less preferred man is forbidden. If G is a perfect matching but has a blocking pair, then the partner of the woman in the blocking pair is forbidden. Conversely, $forbidden(G, j)$ implies that either G is not a perfect matching or has a blocking pair.

We only need to show that if $forbidden(G, j)$ holds, then there is no proposal vector H such that $(H \geq G)$ and $(G[j] = H[j])$ and H is a stable marriage.

Consider any H such that $(H \geq G)$ and $(G[j] = H[j])$. We show that H is not a stable marriage. Since $G[j]$ is equal to $H[j]$, $\rho(G, j)$ is equal to $\rho(H, j)$. Let i be such that $\exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j])$. Since $G \leq H$, $G[i] \leq H[i]$, we get that $\exists k \leq H[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j])$. Hence, $forbidden(H, i)$ also holds.

■

Lemma 3.5 immediately gives us the Algorithm LLP-ManOptimalStableMarriage. The **always** section defines variables which are derived from G. These variables can be viewed as macros. For example, for any thread $z = mpref[j][G[j]]$. This means that whenever $G[j]$ changes, so does z.

If man j is forbidden, it is clear that any vector in which man j is matched with z and man i is matched with his current or a worse choice can never be a stable marriage. Thus, it is safe for man j to advance to the next choice.

Algorithm LLP-ManOptimalStableMarriage: A Parallel Algorithm for Stable Matching

```
1 $P_j$: Code for thread $j$
2 input: $mpref[i, k]$: int for all $i, k$; $rank[k][i]$: int for all $k, i$;
3 init: $G[j] := 1$; // works for any initialization
4 always: $z = mpref[j][G[j]]$;
5 forbidden: $(\exists i : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j]))$
6     advance: $G[j] := G[j] + 1$;
```

Algorithm LLP-ManOptimalStableMarriage works for any initialization of G vector. If we know that G is initialized to $[1, 1, \ldots, 1]$, then we can simplify the forbidden condition to

$$(\exists i : (z = mpref[i][G[i]]) \wedge (rank[z][i] < rank[z][j])).$$

Observe that if we assume sequential implementation and if we implement a variable $partner[z]$ for any woman z, we can further simplify the condition to

$$(partner[z] \neq null) \wedge (rank[z][partner[z]] < rank[z][j]).$$

This is precisely the condition used in Gale-Shapley algorithm which is sequential and starts with the initial vector $[1, 1, \ldots, 1]$.

We now present an algorithm to find stable marriages that satisfy additional constraints. The following lemma proves lattice-linearity of many such constraints.

Lemma 3.6 *The following constraints are lattice-linear.*

1. *The regret of man i is at most that of the regret of man j.*

2. *Man i cannot be married to woman j.*

3. *The regret of man i is equal to that of man j.*

Proof: Let B be the predicate that G is a stable marriage and it satisfies the corresponding additional constraint.

1. Suppose that G is a stable marriage but it does not satisfy B. This means that regret of man i is more than the regret of man j. In this case, we have $forbidden(G, j)$, because unless $G[j]$ is advanced, the predicate cannot become true.

2. If $\rho(G, i) = j$, then $forbidden(G, i)$ holds.

3. This condition is a conjunction of two lattice-linear conditions of type in part (1).

■

We now enumerate advantages of the LLP algorithm.

1. It gives us a more general algorithm. Instead of starting from the initial vector, we can start from any vector and find a stable marriage greater than or equal to that vector, if one exists.

2. We can use the LLP algorithm for finding a stable marriage that satisfies additional constraint such as the regret of man i is at most that of the regret of man j.

3. We can automatically deduce that the intersection of two stable marriages is also a stable marriage.

4. The LLP algorithm is useful in the context of parallel computing. If two men are forbidden, then both of them can advance in parallel.

5. The LLP algorithm is also useful in the context of distributed computing. The man i may not have the most recent information about man j. Even then, the algorithm will terminate with the stable marriage that is the least in the proposal vector lattice.

3.6 Additional LLP Algorithms

For computing the GCD of an array of natural numbers A, we consider the lattice of natural numbers less than or equal to the largest value in A. The lattice is ordered decreasingly by a component-wise comparison of vectors. The algorithm terminates when all components are equal. Otherwise, the component j is forbidden if it is greater than some component i. In this case, we replace $G[j]$ by $G[i]$ if $G[i]$ divides $G[j]$. Otherwise, we replace $G[j]$ by $G[j]$ mod $G[i]$. When the algorithm terminates, all components of the G vector are equal to the greatest common divisor of A.

The Factorial algorithm for LLP considers the lattice of 2-dimensional vectors, where the first dimension will finally hold the result. The lattice is ordered in increasing order by the first component and decreasing order by the second component. The algorithm initializes $G[2]$ by n and $G[1]$ by 1. Whenever $G[2]$ is greater than 1, we multiply $G[1]$ by $G[2]$ and reduce $G[2]$ by 1. It is clear that, in $n - 1$ steps, $G[2]$ will become 1 and $G[1]$ will hold $factorial(n)$. In this example, there is no parallelism. We multiply the numbers, one at a time.

The MaxElement algorithm initializes G with $A[1]$. If for any index j, $A[j]$ is greater than G, we replace G by $A[j]$. Here the lattice is a single dimensional chain of n elements where n is the maximum entry in

```
GCD of an array $A$:
    $G$: array[$1..n$] of int initially $\forall i : G[i] = A[i]$;
    forbidden(j): $\exists i : G[j] > G[i]$
        advance(j): if $G[i]$ divides $G[j]$ then $G[j] := G[i]$;
            else $G[j] := G[j]\ mod\ G[i]$;
Factorial
    input: $n$: non-negative integer;
    $G$: array[$1..2$] of int initially $(G[1] = 1) \wedge (G[2] = n)$;
    forbidden(2): $G[2] > 1$
        advance: $G[1] := G[1] * G[2]; G[2] := G[2] - 1$;
MaxElement
    input: $A$: array of natural;
    $G$: int initially $G = A[1]$ ;
    forbidden(j): $A[j] > G$
        advance: $G := A[j]$;
LinearSearch
    input: $A$: array of integers; $key$: integer;
    $G$: int initially $G = -1$ ;
    forbidden(j): $(G == -1) \wedge (A[j] == key)$
        advance: $G := j$;
NestedSum
    input: $A$: array of integers;
    $G$: int initially $G = 0$ ;
    $done$: array[$0..n-1$, $0..n-1$] of int initially $\forall i, j : done[i, j] = 0$ ;
    forbidden(i,j): $(done[i, j] == 0)$
        advance: $G := G + i * j; done[i, j] := 1$;
PairSum
    input: $A$: array of integers; $target$: integer;
    $G$: int initially $G = 0$ ;
    $done$: array[$0..n-1$, $0..n-1$] of int initially $\forall i, j : done[i, j] = 0$ ;
    forbidden(i,j): $(i < j) \wedge (done[i, j] == 0) \wedge (A[i] + A[j] == target)$
        advance: $G := G + 1; done[i, j] := 1$;
BinarySearch
    input: $A$: array of integers;
    $G$: int initially $G = -1$ ;
    $low$ : int initially $low = 0$ ;
    $high$ : int initially $high = n - 1$ ;
    forbidden: $(low \leq high)$
        advance: $mid = \lceil low + high \rceil$;
            if $(A[mid] = target)$ then $low := mid; high := mid - 1; G = mid$;
            else if $(A[mid] < target)$ then $low := mid + 1$; else $high := mid - 1$;
```

Figure 3.2: LLP Programs for (a) GCD (b) Factorial (c) MaxElement (d) Linear Search (e) Nested Sum (f) PairSum and (g) Binary Search

the array A. The predicate B is $\forall j : (A[j] < G[j]$ For a sequential implementation, we need not traverse all indices j in any particular order.

The LinearSearch algorithm initializes G with -1. The goal is to replace G with the index j of the array A if $A[j]$ equals key. This is a nondeterministic program. If there are multiple j such that $A[j]$ equals key, the final answer could be any one of them.

The NestedSum algorithm is another nondeterministic algorithm. The variable $done[i, j]$ is 0 iff the entry $i * j$ has not been added to the variable G. Whenever all entries in the array $done$ become 1, the algorithm terminates.

The PairSum algorithm counts the number of tuples (i, j) where $i < j$ and $A[i]$ plus $A[j]$ equals $target$.

Finally, the BinarySearch algorithm returns the index j in $A[j]$ such that $A[j]$ equals key. It assumes that A is sorted. The algorithm continues until the range $[low \dots high]$ is nonempty. It finds the new range in A for searching based on the comparison of $A[mid]$ to $target$.

3.7 Dual of Lattice-Linearity

We briefly discuss detection of predicates when the search starts from the top of the lattice in addition to the bottom of the lattice. Many predicates, such as $B_{stableMarriage}$ satisfy not only the efficient advancement property but also its dual. Thus, the set of elements satisfying these predicates is closed not only for the meet operation but also for the join operation. In addition, whenever the predicate is false, we can efficiently determine the forbidden indices. If we are okay with returning either of the elements as our final answer, then searching for any of the elements in parallel may speed up the algorithm by a factor of the height of the lattice. The dual of the efficient advancement property can formally be defined as follows.

Definition 3.7 (dual of Efficient Advancement Property) *A Boolean predicate B has the dual of efficient advancement property with the polytime algorithm $\mathcal{A}$ with respect to a finite distributive lattice L generated from a poset P if*

$$\forall G \in L : \neg B(G) \Rightarrow \exists i \in \mathcal{A}(G) : dual\text{-}forbidden(i, G, B)$$

where $dual\text{-}forbidden(i, G, B)$ is defined as

$$\forall H \leq G : (H[i] = G[i]) \Rightarrow \neg B(H).$$

The above definition states that whenever B is false in G, the algorithm $\mathcal{A}(G)$ returns an index such that any global state H less than G that matches G on that index also has B false. When efficient advancement as well as dual are true, one can search for the satisfying element starting from both the bottom and top of the lattice as shown in Algorithm Solitary2.

Applying the idea to the stable marriage problem, one can search for the stable marriage starting from the top choices for all men (the $\perp$ element) of the lattice in parallel with the bottom choices for all men (the $\top$) element of the lattice. This algorithm will traverse the distance in a lattice given by the minimum of the distance of a stable marriage from the top or the bottom. Hence, depending upon the distance of man-optimal and man-pessimal stable marriage, it will return the stable marriage that is closer to the bottom or the top of the lattice.

In the above analysis, we ignore the 2 penalty factor that we incur because we run the algorithm from both the bottom and top of the lattice.

Algorithm Solitary2: An Algorithm for detecting Predicates with efficient advancement property and its dual.

```
var $G$: element of $L$ initially $\bot$ (the bottom element of $L$);
    $Z$: element of $L$ initially $\top$ (the top element of $L$);
while $\neg B(G)$ and $\neg B(Z)$ do
    forall $i$: forbidden($i$, $G$) do
        if $G$ cannot be advanced on $i$ then return false;
        else advance $G$ on $i$;
    end
    forall $j$: dual-forbidden($j$, $Z$) do
        if $Z$ cannot be retreated on $j$ then return false;
        else retreat $Z$ on $j$;
    end
end
if $B(G)$ then return $G$;
else return $Z$ ; // an optimal solution;
```

3.8 Problems

1. Let G be a n-dimensional vector of positive numbers. Let *pred* be a binary acyclic relation on $[n]$. Show that the predicate $B \equiv \forall(i,j) \in pred : G[j] \geq G[i] + 1$ is a lattice-linear predicate.

2. Let $(X, \leq)$ be a poset. A subset $Y \subseteq X$ is an order ideal if it satisfies

$$\forall u, v \in X : (v \in Y) \wedge (u \leq v) \Rightarrow (u \in Y)$$

   Show that the predicate $B(Y) \equiv$ "Y is an order ideal of $(X, \leq)$" is lattice-linear in the boolean lattice of all subsets of X.

3. Show that lattice-linearity is not closed under disjunction.

4. Show that lattice-linearity is not closed under negation.

5. Let L be any finite distributive lattice of global states. Let B be any Boolean predicate defined on L such that B is lattice-linear as well as dual lattice-linear. Let $M = \{M_1, M_2, \ldots, M_k\}$ be any subset of L of size k such that all elements of M satisfy B. Then, for each index m, we get a multiset $\{M_i[m] \mid 1 \leq i \leq k\}$. For any j, we construct the generalized j-median state, G^j as follows. For any m, $G^j[m]$ is given by the j^{th} element in the *sorted* multiset $\{M_i[m] \mid 1 \leq i \leq k\}$. Then, G^j also satisfies B.

3.9 Bibliographic Remarks

The lattice-linearity property is described in [CG98]. Its application to stable matching is shown first in [Gar17]. Its application to the design of parallel algorithms is in [Gar20].

Chapter 4

Basic Graph Algorithms

4.1 Introduction

Graph theory, a cornerstone of algorithmic design, is the foundation of numerous modern applications in diverse domains. In *social network analysis*, graphs model relationships where vertices represent individuals and edges denote interactions, enabling platforms like X to recommend connections or detect communities using algorithms such as breadth-first search or clustering. In *transportation and logistics*, directed graphs optimize routing: companies like Amazon use shortest-path algorithms (e.g., Dijkstra's algorithm) on road networks to minimize delivery times, while airlines use bipartite matching to schedule crews efficiently. *Computer networks* rely on graphs to represent topologies, with spanning tree protocols ensuring loop-free communication and centrality measures identifying critical nodes for cybersecurity. In *bioinformatics*, graphs map protein interactions or gene regulatory networks, where traversal algorithms identify functional pathways, helping drug discovery.

Graphs come in numerous varieties. They may be *directed* or *undirected*. They may be *simple* or with loops and parallel edges. They may be *weighted* or *unweighted*. They may be *acyclic* or not. For example, when modeling transportation networks, the nodes may represent important milestones in a city. A directed edge may represent a one-way street, and an undirected edge may represent a street that can be traversed in either direction. The weights may represent the distance between the nodes.

In this chapter, we cover some basic traversal algorithms in a simple directed graph. Section 4.4 gives an LLP algorithm to traverse a graph. Section 4.3 gives an LLP algorithm to construct a breadth-first-search (BFS) tree from a graph given a source vertex.

4.2 Graph Representations

We present an undirected graph and its directed counterpart, each with vertices v_0, v_1, v_2, v_3, shown with adjacency matrix and adjacency list representations (as linked lists).

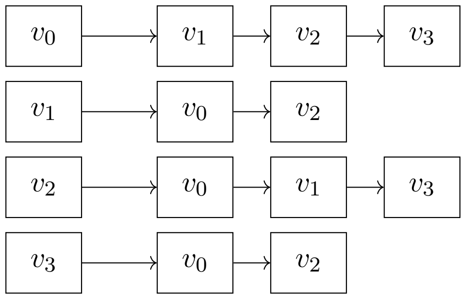


Figure 4.2: Adjacency representation of an undirected graph

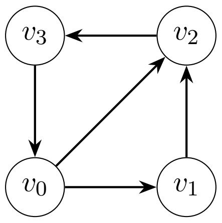


Figure 4.3: A Directed Graph

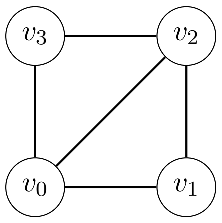


Figure 4.1: An undirected graph

The graph in Fig. 4.1 can be represented using an adjacency matrix as follows.

$$\left[\begin{array}{lcccc} & v_0 & v_1 & v_2 & v_3 \\ v_0 & 0 & 1 & 1 & 1 \\ v_1 & 1 & 0 & 1 & 0 \\ v_2 & 1 & 1 & 0 & 1 \\ v_3 & 1 & 0 & 1 & 0 \end{array}\right]$$

An entry of 1 indicates an edge; the matrix is symmetric. Such a representation allows one to answer any question of the form: is there an edge between v_i and v_j in constant time. However, it takes $O(n^2)$ space irrespective of the number of edges in the graph. When the graph is sparse, the adjacency list representation is less wasteful. Fig. 4.2 shows the adjacency list representation of the same graph.

We now consider the directed version of the graph in Fig. 4.3.

As before, we can have the adjacency matrix and adjacency list representation shown below.

$$\begin{tikzcd}
\boxed{v_0} \arrow[r] & \boxed{v_1} \arrow[r] & \boxed{v_2} \\
\boxed{v_1} \arrow[r] & \boxed{v_2} & \\
\boxed{v_2} \arrow[r] & \boxed{v_3} & \\
\boxed{v_3} \arrow[r] & \boxed{v_0} &
\end{tikzcd}$$

Figure 4.4: Adjacency representation of a directed graph

$$\left[\begin{array}{ccccc}
 & v_0 & v_1 & v_2 & v_3 \\
v_0 & 0 & 1 & 1 & 0 \\
v_1 & 0 & 0 & 1 & 0 \\
v_2 & 0 & 0 & 0 & 1 \\
v_3 & 1 & 0 & 0 & 0
\end{array}\right]$$

4.3 BFS and DFS Traversals in a Graph

In this section, we cover two of the most common ways of traversing a directed graph. The traversal is required for many reasons. We may be interested in searching for a node, finding the number of components in a graph, or checking for some user-specified property.

Breadth-First Search (BFS) Tree in a Directed Graph

Suppose that we are given one of the nodes v_0 in the graph. Our task is to find all the nodes that are reachable from v_0. Algorithm BFS-Traversal-Sequential can be used for this purpose. It will generate a tree rooted at v_0 such that every node is at the minimum distance from v_0. We maintain a variable *parent* that gives the parent of any node x in the graph. If x is reachable from v_0, then following the *parent* pointer from x to v_0, we will get a shortest path from v_0 to x. A node v_i has its parent set to v_j if v_j is along a path from v_0 to v_i and has distance one less than that of v_i.

Depth-First Search (DFS) Tree in a Directed Graph

Another way to traverse a graph is the depth-first search method. While the breadth-first search traverses the graph one layer at a time, the depth-first search continues to take a path as far as it can take, and then it backtracks to traverse the remaining graph (in the same fashion!).

Algorithm BFS-Traversal-Sequential: Algorithm BFS to find the Breadth-First-Search Tree in a Directed Graph

```
var $G$: vector of int initially $\forall i : G[i] = maxint$;
$parent$: vector of int initially $\forall i : parent[i] = -1$; // null parent
$S$: queue of indices;
$S.add(s)$;
$G[s] := 0$;
while $\neg S.empty()$ do
    $j := S.removeFirst()$; // remove an index from $S$
    forall $(k \in dep(j))$ do
        if $(G[k] > G[j] + 1)$ then
            $G[k] := G[j] + 1$;
            $parent[k] := j$;
            append $k$ to $S$;
        end
    end
end
return $G$; // the optimal solution
```

Algorithm DFS-Traversal: Sequential Algorithm DFS to find the Depth-First-Search Tree in a Directed Graph

```
var
    $G[0..n-1]$: int, initially 0
    $parent[0..n-1]$: int, initially -1
    $S$: stack, initially empty
    $discovered[0..n-1]$: int
    $finished[0..n-1]$: int
    $tick$: int, initially 1
$S.push(0)$
while $\neg S.empty()$ do
    $j := S.pop()$
    if $G[j] = 0$ then
        $G[j] := 1$
        $discovered[j] := tick$
        $tick := tick + 1$
        for $k \in edges(j)$ do
            if $G[k] = 0$ then
                $parent[k] := j$
                $S.push(k)$
            end
        end
        $finished[j] := tick$
        $tick := tick + 1$
    end
end
```

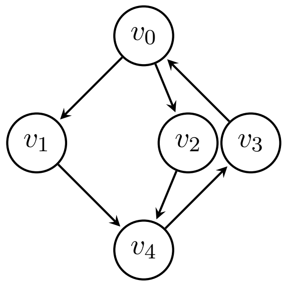


Figure 4.5: An Example of DFS-Traversal

Node	discovered	finished
v_0	1	10
v_1	2	7
v_4	3	6
v_3	4	5
v_2	8	9

Figure 4.6: The values of *discovered* and $finished$ for vertices with DFS on the graph in Fig. 4.5

Algorithm DFS-Traversal visits all the vertices reachable from the vertex s in a depth-first search manner. The variable G is used to mark all the vertices that are reachable from s. The variable S is used as a stack to store the vertices while traversing the graph. The variable *discovered* is used to store the time (tick) when each vertex in the graph is explored. The variable $finished$ is used to store the tick when the algorithm is finished exploring the vertex. The variable *parent* is used to store the depth-first tree. Anytime a vertex k is visited for the first time from a vertex j, the $parent[k]$ is recorded as j.

The variables *discovered* and $finished$ are used only to record *tick* when a vertex is starting to be explored and finished for exploration.

Fig. 4.5 shows the DFS traversal on a graph. Suppose that we start the DFS traversal from the node v_0. Then, the node v_1 is visited next. After v_1, we visit the node v_4. From node v_4, we visit node v_3. When v_3 is explored, we get to the node v_0 which has already been visited. We backtrack to the node v_4 which does not have any other outgoing edge. From v_4, we backtrack to v_1 and then to v_0. Now, from v_0, we can visit node v_2. From v_2, we can go to node v_4 which has already been visited. Hence, we backtrack to node v_0. At this point, all outgoing edges from v_0 have been traversed and the DFS traversal is done. The variables for the DFS traversal for this graph are shown in Fig. 4.6.

4.4 LLP Graph Traversal Algorithms

Given a graph (V, E) and a source vertex v_0, our goal is to find all the vertices that are reachable from v_0 . We can view this problem as the search for a subset of vertices of $W \subseteq V$ such that $v_0 \in W$ and $\forall x, y : (x \in W) \wedge (x, y) \in E$ implies $y \in W$. We model W using a Boolean array G such that $v_i \in W$ iff $G[i]$ is true.

The following LLP algorithm computes the least G (or the smallest W) that satisfies

$$B_{traverse} \equiv G[0] \wedge (\forall (v_i, v_j) \in E : G[j] \geq G[i])$$

The predicate B requires v_0 to be reachable from v_0 and for all edges (v_i, v_j) if v_i is reachable then so is v_j. It can be verified that $B_{traverse}$ is a lattice-linear predicate. It is a conjunctive predicate, and the first conjunct is a local predicate. We only need to show that the second conjunct is also lattice-linear. If the second conjunct is false, then there exists $(v_i, v_j) \in E$ such that $G[i]$ equals 1 and $G[j]$ equals 0. In this case, unless $G[j]$ is also set to 1, the predicate can never become true. Therefore, we get the following LLP algorithm.

Algorithm LLP-Traversal: Reachability Set

```
1 $P_j$: Process for vertex $j$
2 **input**: $edges(j)$: list of adjacent vertices
3 **init**: $G[j] := 1$ if $j = 0$, else 0
4 **forbidden**($j$): $\exists i : j \in edges(i) \wedge G[i] = 1 \wedge G[j] = 0$
5     **advance**($j$): $G[j] := 1$
```

LLP Algorithm is non-deterministic and one could use different strategies to compute forbidden indices. For example, one can maintain a set S that contains *frontier* vertices – set of vertices that are reachable but their outgoing edges have not been explored yet. Algorithm Traversal-Sequential gives a sequential implementation of the LLP algorithm with S as the set of frontier vertices. By using different strategies for removing an index from S, we can traverse the graph in different order. If S is maintained as a queue, the removal corresponds to removing from the head of the queue, and the addition corresponds to appending at the end of the queue, we traverse the graph in a *breadth-first* manner. If S is maintained as a stack such that removal corresponds to *pop*() and the addition corresponds to *push*() method on the stack, then we traverse the graph in a depth-first search manner.

Algorithm Traversal-Sequential: Finding the reachable set of vertices from v_0 in a Directed Graph

```
var
    $G$: vector of int initially $\forall i : G[i] = 0$
    $S$: set of indices
$S.add(0)$ // add 0 as the initial forbidden index
$G[0] := 1$
while $\neg S.empty()$ do
    $j := S.removeAny(G)$ // remove any index from $S$
    forall $(k \in dep(j))$ do
        if $(G[k] = 0)$ then
            $G[k] := 1$ // advance on $k$
            $S.add(k)$ // update frontier
        end
    end
end
return $G$ ;                                   // the optimal solution
```

In the previous section, we saw an algorithm to compute the set of reachable vertices from a given vertex. Now suppose that we are interested in computing the breadth-first-search tree rooted at any given node v_0. We first define the search lattice L to be the set of distance vectors where $G[i]$ is the distance of vertex i from v_0. Initially, $G[0]$ is zero for the vertex v_0 and ∞ for all other vertices. We define G to be feasible if

$$B_{bfs}(G) \equiv \forall j \neq 0 : \forall (i,j) \in E : G[j] \leq G[i] + 1.$$

Our goal is to maximize G subject to B_{bfs}.

A vertex j is defined to be forbidden if it has a predecessor i such that $G[j] > G[i] + 1$. If none of the vertices is forbidden, we get that $\forall j \neq 0 : \forall (i,j) \in E : G[j] \leq G[i] + 1$. Whenever a vertex j is forbidden, we advance the index j by setting $G[j]$ to $G[i] + 1$. This is equivalent to ensuring that $G[j]$ is at most $G[i] + 1$ for any $i \in pre(j)$.

Algorithm LLP-Traversal-BFS: Finding the reachable set of vertices using BFS.

```
1 Process $P_j$
2 input: $pre(j)$: list of $1..n$;
3 init($j$): if $(j = 0)$ $G[j] := 0$ else $G[j] := \infty$;
4 ensure($j$): $G[j] \leq min\{G[i] + 1 \mid i \in pre(j)\}$
```

The algorithm terminates when there is no forbidden vertex. Any vertex k such that $G[k]$ is infinite is not reachable from the vertex v_0.

We observe that Algorithm LLP-Traversal-BFS does not specify the order in which the algorithm (sequential or parallel) should check if the ensure condition is not met. For example, when the total number of threads is much smaller than the number of processes, then it may be worthwhile giving a priority to the indices that are checked for forbidden predicate. We use the clause **priority** with the LLP algorithm to put an order in which the indices should be checked for the *forbidden* or *ensure* clause. Algorithm LLP-Traversal-BFS-priority shows an algorithm in which the priority of forbidden indices is given by $G[j]$.

Algorithm LLP-Traversal-BFS-priority: Finding the reachable set of vertices using BFS.

```
1 Process $P_j$
2 input: $pre(j)$: list of $1..n$;
3 init($j$): if $(j = 0)$ $G[j] := 0$ else $G[j] := \infty$;
4 ensure($j$): $G[j] \leq min\{G[i] + 1 \mid i \in pre(j)\}$
5 priority: $G[j]$
```

Algorithm LLP-Traversal-BFS is based on using the predecessor information for each vertex. Alternatively, we can use the adjacency list for each vertex. The set S keeps the set of reachable indices. Initially, only v_0 is the reachable vertex. We explore all the outgoing edges from any vertex $v_j \in S$. The set T stores all the vertices that become reachable when vertices in S are explored. If any vertex k adjacent to v_j gets its $G[k]$ reduced because of the edge (v_j, v_k), it is added to the set T. This exploration is continued until there are no unexplored vertices.

4.5 Layering of a Directed Acyclic Graph

Suppose that we are given a directed graph with no cycles. Such a graph can be "layered" as follows . Every vertex i in the graph is assigned a number $G[i]$ such that if there is a directed edge from i to j, then the number assigned to i is strictly less than that of j, i.e., for all edges $(i, j) \in E$, $G[i]$ is less than $G[j]$. An application of this concept is the prerequisite structure of courses in a University and the integer assigned to the course corresponds to the earliest semester in which the course can be taken by a student.

In this application, our lattice consists of vectors of natural numbers. We are determining the least vector, G, that satisfies the predicate

$$B_{layer} \equiv \forall (i, j) \in E : G[i] < G[j].$$

The predicate is lattice-linear because if it is false then there exist $(i, j) \in E$ such that $G[i] \geq G[j]$. In this case, the index j is forbidden and unless $G[j]$ is advanced B_{layer} can never become true. For efficiency, we define a predicate

$$fixed(j) \equiv \forall (i, j) \in E : G[i] < G[j].$$

Now, the predicate B_{layer} can be rewritten as

$$B_{layer} \equiv \forall j \in [n] : fixed(j).$$

We will use $fixed[j]$ as a variable in our program. Algorithm shown in Fig. LLP-Layering gives the layering of a directed acyclic graph. We use the boolean array $fixed$ to mark the vertices whose level number in G are final. Initially, all vertices that do not have any predecessors are $fixed$ and labeled with 0. A vertex is forbidden if it is not fixed and all its predecessors are fixed. Whenever a vertex j is forbidden, we mark its level as 1 more than any of its predecessors. In this algorithm, $G[j]$ gives the length of the longest path from any initial vertex to j.

Algorithm LLP-Layering: Layering of a Directed Acyclic Graph.

```
input: $pre(j)$: list of $1..n$;
init(j): $G[j] := 0$;
    if $pre(j) = \{\}$ then $fixed[j] := true$ else $fixed[j] := false$;
forbidden(j): $\neg fixed[j] \wedge \forall i \in pre(j) : fixed[i]$
    advance(j): $G[j] := \max_{i \in pre(j)} G[i] + 1$; $fixed[j] := true$;
```

The algorithm takes parallel time equal to the length of the longest path in the directed acyclic graph. Its work complexity is $O(n + m)$.

In many applications, we are interested in coming up with a *total order* on all vertices such that for all edges (i, j), $G[i]$ is strictly less than $G[j]$. This operation is called *topological sort* of a directed acyclic graph. The reader is invited to modify LLP-Layering for such applications.

4.6 Problems

1. Algorithm LLP-Layering gives a level number $G[i]$ for each vertex i such that for any edge (i, j), we have that $G[i]$ is strictly less than $G[j]$. Modify the algorithm to generate a total order on all vertices.

2. Algorithm LLP-Layering gives us the least integral labels satisfying that for all edges (i, j), $G[i]$ is strictly less than $G[j]$. Such a property is possible only for acyclic graphs. Modify the algorithm to output "error" whenever there is a cycle in the input graph.

4.7 Bibliographic Remarks

The reader is referred to [CLRS01] for basic graph algorithms.

Chapter 5

Greedy Algorithms

5.1 Introduction

Many problems can be solved by making multiple choices such that at the end of these choices, we have the solution to the problem. The greedy algorithms start with the solution to the trivial size problem, generally of size 0 or 1. Then it continues to increase the size of the problem until we get the original problem back after n steps, for some $n > 0$. For many applications, this strategy results in a solution that is a global optimal or close to a global optimal. Prim's Minimum Spanning Tree (MST) algorithm is as follows. Given a connected graph, we need to find a spanning tree of that graph which is minimal in the total edge weight. The greedy approach is as follows. Start with an arbitrary node and always add the smallest edge that connects any node in the tree to a node outside the tree. Similarly, Dijkstra's Shortest Path Algorithm to find the shortest path from the source to all vertices in the given graph is also greedy. It always picks the next vertex with the shortest known distance.

This chapter is organized as follows. Section 5.2 describes a greedy algorithm for the interval scheduling problem. In this problem, we schedule a maximum number of jobs on a processor. Section 5.3 describes a greedy algorithm for the problem of scheduling intervals using as few rooms as possible so that no two overlapping intervals are assigned in the same room. Section 5.4 discusses the problem of scheduling n jobs so that maximum lateness of jobs is minimized. Section 5.5 discusses the construction of a Huffman tree that minimizes the expected length of the binary code for a symbol. Section 5.6 describes the LLP versions of these algorithms.

5.2 Interval Scheduling Problem

We start with a simple greedy algorithm. Suppose that we have m intervals with start times s_i and finish times f_i for jobs $i = 1..m$, where $s_i < f_i$. We assume that activity i occurs during the open interval $[s_i, f_i)$. Two intervals i and j are *compatible* if $f_i \leq s_j$ or $f_j \leq s_i$. Our goal is to select a maximum-sized subset of mutually compatible intervals. We assume that the intervals are sorted by their finish times. We assume that jobs with the same finish times are sorted according to their start times. Furthermore, no two intervals have identical start and finish times.

Let us begin with a sequential algorithm for this problem shown in Fig. IntervalSequential. Let G be the set of mutually compatible intervals represented as a Boolean array such that $G[i]$ is 1 iff the activity i is in the subset of mutually compatible intervals that can be selected. It is clear that the first activity (with the least finishing time) is always in G. (Why?) We now traverse over all intervals as follows. We use the variable *cur* to denote the current activity under consideration and the variable *last* as the last activity that was included in G. We initialize *last* to 0. The current activity is included in G if and only if its start time is greater than or equal to the finish time of the last activity. If it is, our current activity becomes the last activity, and we explore the next activity.

Algorithm IntervalSequential: Sequential Interval Scheduling

```
Input: $s$: array[0..$n-1$] of int // Start times
Input: $f$: array[0..$n-1$] of int // Finish times
Output: $G$: array[0..$n-1$] of 0..1 initially 0 // Selected intervals
$G[0] := 1$;                                   // First interval always selected
int $last := 0$;                               // Index of last selected interval
for int $i := 1$ to $n-1$ do
    if $s[i] \geq f[last]$ then
        $G[i] := 1$;                           // Interval $i$ is compatible
        $last := i$
    end
end
```

The sequential algorithm takes $O(n)$ time (assuming that the intervals are provided in a sorted manner based on the finish times) and is optimal.

Theorem 5.1 *For the Activity Selection Problem, selecting activities in increasing order of their finish times yields an optimal solution, i.e., the maximum number of mutually compatible activities.*

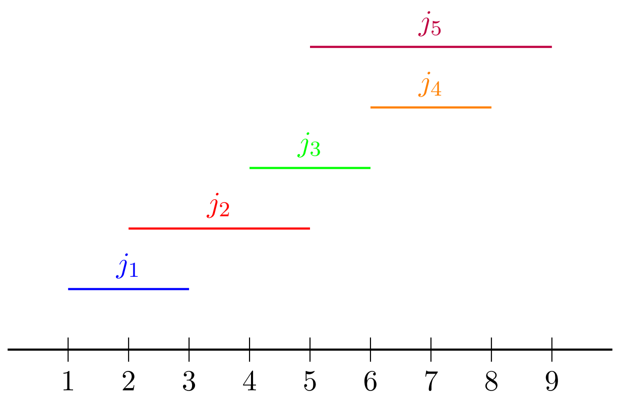
Proof: We prove the theorem using the **exchange argument**. Let G be the set of activities chosen by the greedy algorithm, which selects the activity with the earliest finish time first. Let O be an optimal solution with the maximum number of activities. The greedy algorithm selects the activity with the earliest finish time, say A_1. Suppose O selects a different first activity A_k. Since activities are sorted by finish time, $f_1 \leq f_k$. Replacing A_k with A_1 in O still allows the remaining selections in O to proceed without reducing the total count. By repeatedly applying this argument for the next selected activity, we replace each activity in O with the corresponding activity of G without decreasing the number of selected activities. After at most n exchanges, O is transformed into G. Since O was an optimal solution and G has the same size, G is also optimal.

■

Suppose that we are given five jobs, each with a start time and an end time. The goal is to select the maximum number of non-overlapping jobs.

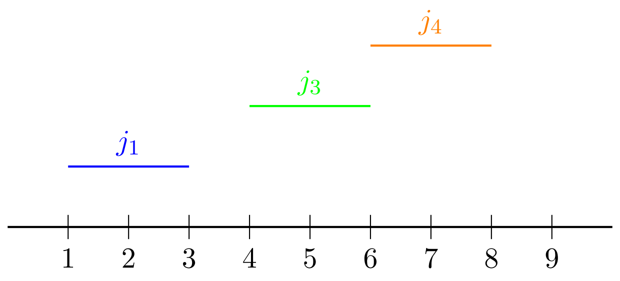
Job	Start Time	End Time
j_1	1	3
j_2	2	5
j_3	4	6
j_4	6	8
j_5	5	9

Table 5.1: List of jobs with their start and end times.



An optimal schedule selects the maximum number of non-overlapping jobs. The optimal solution is:

$$\{j_1, j_3, j_4\}$$



5.3 Interval Partition Problem

Consider a university that offers multiple courses for students. Every course i has a fixed slot $[s_i, f_i)$ where s_i indicates the start time of the lecture in the course i and f_i indicates the finish time for that lecture. There are n courses in total. The university wants to use as few classrooms as possible for lectures. Our task is to assign a room to each course so that if two lectures overlap, then they must be assigned different rooms. Since each lecture is modeled as an interval, this problem is called the *interval partition problem*.

For this problem, we first look at the lower bound on the number of rooms required. Suppose that there are m overlapping lectures. Then it is clear that any room assignment must use at least m rooms.

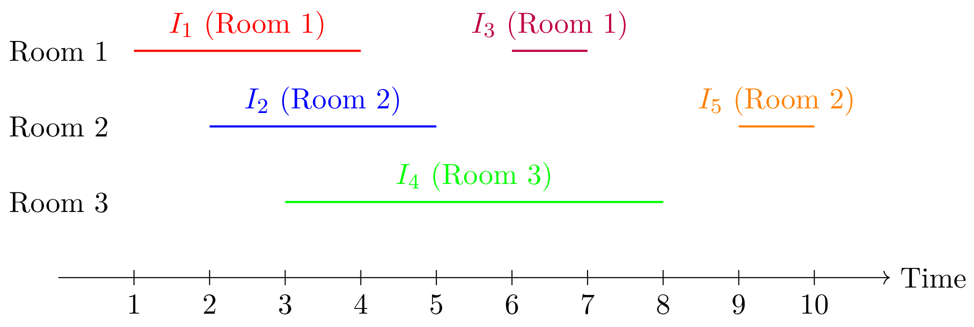
We now show a simple greedy algorithm that achieves the lower bound on the number of rooms. We sort the courses based on the start times of the lectures. We assign courses to rooms one at a time. The first course is assigned the room 1. Now suppose that k courses have been assigned their rooms. We assign the course $k+1$ as follows: assign it to the least numbered room that is available at the start time of the lecture. It is clear that a course is assigned a *new* room only if there are overlapping lectures for all small numbered rooms. Thus, our greedy algorithm achieves the minimum number of rooms assigned.

Given the intervals:

$$I_1 = [1,4], \quad I_2 = [2,5], \quad I_3 = [6,7], \quad I_4 = [3,8], \quad I_5 = [9,10]$$

We sort them based on the start times.

$$(1,4),(2,5),(3,8),(6,7),(9,10)$$



The minimum number of rooms required is equal to the maximum number of overlapping intervals, which is **3**.

Theorem 5.2 *For the Interval Partitioning Problem, assigning intervals to the first available room in order of their start times produces an optimal solution, that is, it minimizes the number of rooms required.*

Proof: The greedy algorithm sorts the intervals by start times and assigns each interval to the first available room. If no room is available, it assigns the interval to a new room. Let G be the number of rooms used by this algorithm, and let O be the minimum number of rooms required in an optimal solution. At any time t, let $D(t)$ be the number of overlapping intervals. Since at least $D(t)$ rooms are necessary at that time, the minimum number of rooms required is at least $\max D(t)$. The greedy algorithm assigns a new room only when all current rooms are occupied, ensuring that it never exceeds $\max D(t)$ rooms. Thus, $G \leq O$ and, since O must also use at least $\max D(t)$ rooms, we conclude that $G = O$. Therefore, the greedy algorithm produces an optimal solution.

■

Now, let us determine the time complexity of our algorithm. The sorting of intervals based on the start times can be done in $O(n \log n)$ time. Suppose that we keep *available* rooms in a heap ordered by the room number. Whenever we need a room for a lecture, we simply remove the minimum available in the heap. This heap operation takes $O(\log n)$ time. Thus, we take $O(n \log n)$ time for the entire algorithm.

5.4 Minimizing Maximum Lateness of Jobs

Suppose that we have n jobs such that each job j takes time t_j to execute and should ideally be completed before its deadline d_j. Our task is to schedule these jobs on a single processor. Suppose that the job j is started at time s_j. Then, the task ends at time $s_j + t_j$. If $s_j + t_j$ is less than or equal to the deadline d_j, then this job is not late. Otherwise, this job has *lateness* equal to $s_j + t_j - d_j$. Our task is to schedule these jobs so that the maximum lateness is minimized.

Let us start with a sequential algorithm. Should we consider these jobs in the order of their processing times, their *slack* times $(d_j - t_j)$), or their deadlines d_j? It turns out that, for this problem, we should consider jobs in the order of their deadlines.

Algorithm MinMaxLateness: A Sequential Program for the Minimizing Maximum Lateness problem

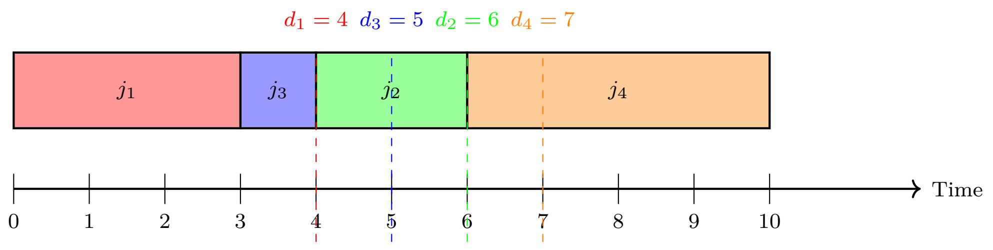
```
Input: $t$: array[$1 \ldots n$] of int; // completion times
Input: $d$: array[$1 \ldots n$] of int ; // deadlines
var $G$: array[$1 \ldots n$] of int initially 0;
int $last := 0$;
int $lateness := 0$;
for $int\ i := 1\ to\ n$ do
    $G[i] := last$;
    $last := G[i] + t[i]$;
    if $(last > d[i])$ then $lateness := \max\{lateness, (last - d[i])\}$;
end
```

Algorithm MinMaxLateness is quite simple. It processes jobs in the order of increasing deadlines. The variable $G[i]$ denotes the starting time of the job i. The algorithm takes $O(n)$ time when the input is provided as sorted on the deadlines.

Consider the following example.

Job j_i	**Processing Time** t_i	**Deadline** d_i
j_1	3	4
j_2	2	6
j_3	1	5
j_4	4	7

The schedule Using Earliest Deadline First (EDF) is shown next.



The job j_2 finishes at time 6 but had a deadline of 4. The job j_4 finishes at time 10 but had a deadline of 7. Using earliest deadline first, this schedule minimizes *maximum lateness*, which is 3 units.

Theorem 5.3 *For the problem of scheduling n jobs with known processing times and deadlines on a single machine to minimize the maximum lateness, the Earliest Deadline First (EDF) algorithm produces an optimal solution.*

Proof:

The EDF algorithm schedules jobs in increasing order of their deadlines. Let G be the schedule produced by the EDF algorithm, and let O be an optimal schedule that minimizes maximum lateness. Suppose for contradiction that O is different from G and has a different order of jobs while achieving a smaller maximum lateness.

Consider the first position in which O and G differ. Let job j be scheduled earlier in O than job i, while G places job i before job j. Since EDF sorts jobs by increasing deadlines, we have $d_i \leq d_j$. Since both schedules are contiguous, swapping i and j in O does not delay the completion time of any job before i, nor does it improve the lateness of any job. Instead, it ensures that the job with an earlier deadline finishes sooner, which prevents an increase in lateness. By iterative application of this argument, we can transform O into G without increasing the maximum lateness. Since O was assumed to be optimal, it follows that G is also optimal. Therefore, EDF produces an optimal schedule.

■

5.5 Huffman Tree

The goal of the Huffman tree problem is to find an efficient encoding of a set of symbols. We are given n symbols. For each symbol i, the value $p[i]$ gives the probability that it appears in the given text such that

$$\sum_i p[i] = 1$$

Our goal is to design a *prefix* code such that the given text of the symbols can be translated to a string so that the text of the symbols can be recovered from the string. We will create a binary tree so that the leaves correspond to the symbols and the path from the root to the leaf corresponds to the *code* for the symbol. Note that only the leaves of the tree correspond to the symbol. Also note that leaves may be at different levels, and therefore this method returns a variable length code for symbols rather than a fixed length code.

Given n symbols with their probabilities of occurrence, the Huffman coding algorithm constructs an optimal prefix code using the following greedy approach:

1. Insert all symbols into a min-priority queue sorted by probability.
2. While there is more than one node in the queue:
   - Remove the two nodes with the smallest probabilities.
   - Merge them into a new node whose probability is the sum of the two.
   - Insert the new node back into the priority queue.
3. The final remaining node is the root of the Huffman tree.
4. Assign binary codes by traversing the tree: assign '0' to the left branch and '1' to the right branch.

Algorithm Huffman Coding: Huffman Coding

```
Input: Set of symbols $S = \{s_1, s_2, \ldots, s_n\}$ and frequencies $f(s_1), f(s_2), \ldots, f(s_n)$
Output: Huffman tree $T$
1  $Q$ := priority queue (min-heap) of nodes, initialized with $S$
2  for each $s_i \in S$ do
3      Create a leaf node $n_i$ with frequency $f(s_i)$ and symbol $s_i$
4      Insert $n_i$ into $Q$
5  end
6  for $i = 1$ to $n - 1$ do
7      $x$ := extract-min($Q$) ;            // Remove node with smallest frequency
8      $y$ := extract-min($Q$) ;            // Remove next smallest
9      $z$ := new internal node with children $x$ and $y$
10     $f(z) := f(x) + f(y)$ ;               // Sum frequencies
11     Insert $z$ into $Q$
12 end
13 $T$ := extract-min($Q$) ;                // Final tree (root)
14 return $T$
```

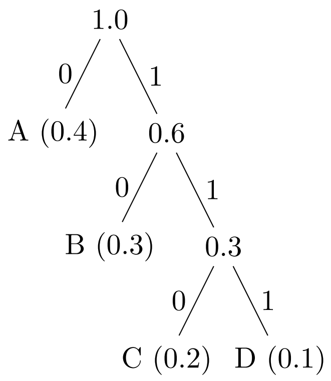
Consider the following symbols and their probabilities:

Symbol	Probability
A	0.4
B	0.3
C	0.2
D	0.1

- Merge (C, D) to create node (CD) with probability 0.3.
- Merge (CD) with B to create node (BCD) with probability 0.6.
- Merge (BCD) with A to create the root node with probability 1.0.

The resulting Huffman codes are shown below.

Symbol	Huffman Code
A	0
B	10
C	110
D	111



The time and space complexity of the Huffman coding algorithm depend on the use of a min-heap for the priority queue Q.

- **Time Complexity:**
  - *Initialization*: Creating n leaf nodes and inserting them into Q takes $O(n \log n)$, as each insertion into a min-heap of size up to n is $O(\log n)$.
  - *Main Loop*: The loop runs $n - 1$ times. Each iteration performs two extract-min operations (each $O(\log n)$) and one insertion ($O(\log n)$), totaling $O(3 \log n)$ per iteration. Thus, the loop takes $O((n - 1) \cdot 3 \log n) = O(n \log n)$.
  - *Final Extraction*: Extracting the root is $O(\log n)$, negligible compared to the loop.
  - *Total*: $O(n \log n) + O(n \log n) = O(n \log n)$.
- **Space Complexity:**
  - *Priority Queue*: Q stores at most n nodes initially, reducing to 1, requiring $O(n)$ space.
  - *Tree Nodes*: The Huffman tree has n leaf nodes and $n - 1$ internal nodes, totaling $2n - 1$ nodes, each storing a frequency and pointers (or symbol for leaves), using $O(n)$ space.
  - *Total*: $O(n) + O(n) = O(n)$.

Thus, the algorithm runs in $O(n \log n)$ time and uses $O(n)$ space, where n is the number of symbols.

5.6 LLP Versions of Greedy Algorithms

In this section, we provide LLP versions of greedy algorithms.

LLP: Interval Scheduling Algorithm

We maintain a linked list of all the jobs under consideration. These jobs are in the order of their finish times. The algorithm will maintain a Boolean $G[j]$ for each job j. The variable $G[j]$ is initially false and is set to true only if it is guaranteed to be in the maximal set of jobs that the algorithm chooses. We call a job j *fixed* if $G[j]$ is true. We now make the following observations.

1. If any job j has its start time greater than the finish time of the previous job, then that job is always included in the final set. Thus, the job j can be fixed.

2. For any job, if the previous job in the linked list is fixed and the start time of the current job is less than or equal to the finish time of the previous job, then this job can be deleted.

It follows from the above two rules that eventually all jobs will be fixed or deleted. The algorithm terminates when all jobs are fixed. The set of all fixed jobs is a maximal set of jobs that can be completed.

The LLP algorithm for the activity selection problem is shown in Fig. IntervalLLP.

Algorithm IntervalLLP: LLP Program for the Interval Scheduling problem

```
// Assume that all jobs are in a doubly linked list
// The variable $prev$ denotes the previous node in the linked list
input:
    $s[j]$ : $int$, $f[j]$: int;
var:
    $G[j]$: boolean;
init: if $(j = 1)$ then $G[j]$ := true; else $G[j]$:= false;
forbidden: $\neg G[j] \wedge f[prev] < s[j]$
    advance: $G[j] := true$;
forbidden: $(j > 1) \wedge G[prev] \wedge s[j] < f[prev]$
    advance: delete this node from the linked list;
```

For example, consider the following four jobs: $job_1 = [1, 4), job_2 = [2, 5), job_3 = [4, 5), job_4 = [5, 7)$. The first job gets fixed because the first job is always fixed. The fourth job also gets fixed because its starting point is bigger than the finishing time of the previous job. Since the first job is fixed and the second job starts before the first job is finished, it is deleted from the list. Now, the third job is also fixed because it starts after the first job is finished.

The sequential algorithm takes $O(n)$ time.

LLP: Interval Partition Algorithm

We now explore an LLP version of the algorithm for Interval Partition. Let $G[i]$ denote the room assigned to course i (where courses are sorted based on start times). We initialize $G[i]$ to 1 for all i. This room assignment is feasible iff there are no overlapping intervals. We now define the feasibility of G for any set of intervals. An assignment G is feasible if for any pair of intervals i, j whenever they overlap, they are in different rooms. Formally, let $B(k)$ be defined as

$$\forall i, j : (i \leq k) \wedge (j \leq k) \wedge s_j \leq f_i \Rightarrow room(i) \neq room(j)$$

$B(k)$ denotes that the room assignment is feasible for the courses $1 \ldots k$. Then G is feasible if it satisfies $B(n)$. Observe that $B(0)$ holds trivially.

The predicate $B(n)$ may not be true for the initial assignment when there are overlapping intervals. Also, note that if we had assigned a separate room for each interval, then the predicate is trivially true. Our interest is in finding the least assignment to the rooms that makes the predicate true. We consider the lattice of n dimensional vectors of natural numbers. The bottom element of this lattice is the vector with all the ones. We use $fixed(i)$ to denote that the course i has been assigned a room that will not be changed. Let $pre(i)$ denote all courses numbered less than i that start prior to course i or at the same time as course i and finish later than s_i.

We say that P_k is forbidden if all courses in $pre(k)$ are fixed and $\neg fixed(k)$. Whenever P_k is forbidden, it can be advanced as follows. Let r be the least room that has not been assigned to any overlapping course prior to k. Then $room(k)$ is set to r and $fixed(k)$ is set to true.

Algorithm IntervalPartition: An LLP Program for the Interval Partition Problem

```
P_j: Code for thread j
input:
    s[j] : int, f[j]: int;
    pre[j]: set of 1..n initially $pre[j] = \{i \mid (i < j) \wedge (s[i] \leq s[j]) \wedge (s[j] < f[i])\}$;
var:
    G[j]: int initially 1;
    fixed[j]: boolean initially false;
forbidden: $(\forall i \in pre(j) : fixed(i)) \wedge \neg fixed(j)$
    advance: $G[j] := \min_{k \in pre(j)} \{r \mid r \neq G[k]\}$;
        fixed[j] := true;
```

As before, we assume that the intervals are provided to us, sorted by their start times. Each index j can keep track of the number of intervals in $pre[j]$ that have been fixed. Whenever all prerequisite intervals are fixed, the index j is forbidden. We have to compute the minimum room that is available when all prerequisite intervals are fixed.

LLP: Minimizing Maximum Lateness of Jobs

We now consider the LLP version of this algorithm. The variable $G[j]$ denotes the completion time for the job j.

Algorithm MinLateLLP: LLP Program for the Minimum Lateness problem

```
// Assume that all jobs are in a doubly linked list
// The variable prev denotes the previous node in the linked list
input:
    t[j]: int; // time required for job j
    d[j]: int; // deadline for job j
var:
    G[j]: int initially 0;
    fixed[j]: boolean initially false;
forbidden: $\neg fixed[j] \wedge ((j = 1) \vee (fixed[j-1])$
    advance: $fixed[j] = true$; if $(j > 1)$ then $G[j] := G[j-1] + t[j]$; else $G[j] := t[j]$
```

LLP: Huffman Coding

We now give an LLP algorithm for Huffman coding.

Algorithm LLP-HuffmanTree1: Finding Huffman Code

```
1 input: $p$:array of real;// frequency of each symbol
2 init: $G[i,j] = 0\ \forall i \neq j$;
3     $G[i,i] = p[i]$;
4 always: $s(i,j) = \sum_{k=i}^{k=j} p[i]$
5 ensure$(i,j)$:
6     $G[i,j] \geq \min_{i \leq k < j} G[i,k-1] + s(i,j) + G[k+1,j]$
```

5.7 Problems

1. A network processes requests that require a fixed bandwidth for a given time interval. Find the minimum number of servers required to handle all requests without exceeding bandwidth limits.

2. A railway station needs to schedule maintenance for the incoming trains before their next departure. Each train has a maintenance duration and a deadline. Find the optimal order of maintenance to minimize the worst delay.

3. Given n intervals and an integer k, find the maximum subset of intervals such that at most k intervals overlap at any time.

4. Each task has a required start time and finish time, but there must be a mandatory idle gap of at least g units between consecutive tasks in the same room. Find the minimum number of rooms required.

5. Each task has a start and finish time, but also requires a *specific type of machine*. Different machine types may be limited in number. Find the minimum number of machines needed.

5.8 Bibliographic Remarks

This chapter on greedy algorithms covers several classic problems: Interval Scheduling, Interval Partitioning, Minimizing Lateness, and Huffman Tree construction, each with well-established foundations in the literature. The Interval Scheduling Problem and Interval Partitioning Problem are covered in [CLRS09], and [KT06]. For the Minimizing Lateness problem, scheduling jobs by earliest deadline (EDF) to minimize maximum lateness is a well-known greedy solution, as shown in Algorithm MinMaxLateness. The optimality of this algorithm, proven via an exchange argument, is a standard result in scheduling theory. [LRK76] provides one of the first rigorous treatments of this problem, establishing the effectiveness of the EDF rule for scheduling a single machine with deadlines. The Huffman tree construction, which builds an optimal prefix code by greedily merging the least probable symbols, is a cornerstone of data compression. Introduced by [Huf52], this algorithm's greedy nature and optimality for variable-length coding are foundational to information theory. Our sequential description mirrors the classic priority-queue implementation in $O(n \log n)$ time, as detailed in [CLRS09].

Chapter 6

The Shortest Path Problem

6.1 Introduction

The single source shortest path (SSSP) problem has wide applications in transportation, networking, and many other fields. The problem takes as input a weighted directed graph with n vertices and e edges. We are required to find $cost[x]$, the minimum cost of a path from the *source* vertex v_0 to all other vertices x where the cost of a path is defined as the sum of edge weights along that path. Dijkstra's algorithm (or one of its variants) is the most popular single source shortest path algorithm used in practice.

Dijkstra's algorithm and the LLP algorithm assume that all edge weights are positive. Section 6.7 derives Johnson's algorithm from a lattice-linear predicate to convert the problem of a graph with negative weights (but no negative cost cycle) to a problem with non-negative weights. Section 6.3 discusses Bellman-Ford algorithm. Bellman-Ford algorithm works on any graph so long as there are no negative cost cycles.

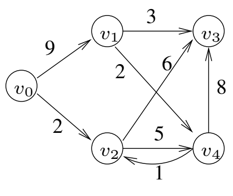
In this chapter, we first describe Dijkstra's algorithm in Section 6.2. We then formulate the shortest path problem as the search for an appropriate element in a lattice and derive an LLP algorithm for the shortest path in Section 6.5. Section 6.7 describes the techniques that we can use when the directed graph has edges with negative weights. In particular, it describes Johnson's algorithm that converts a weighted directed graph to another weighted graph that preserves the shortest paths, and the resulting graph does not have negative weight edges. Section 6.3 describes the Bellman-Ford algorithm that can be used to find the cost of the shortest path when the graph may have negative edge weights. We then consider the problem of finding the costs of shortest paths for all pairs of vertices. Section 6.8 starts with an algorithm for matrix multiplication. Then, it extends the algorithm to the problem of computing the transitive closure of a binary matrix. Section 6.8 extends this problem to non-binary matrices by viewing it as a computation of the cost of the shortest path for all pairs of vertices.

6.2 Dijkstra's Algorithm

We consider a directed weighted graph (V, E, w) where V is the set of vertices, E is the set of directed edges, and w is a map from the set of edges to positive reals (see Fig. 6.1 for a running example). To avoid trivialities, we assume that the graph is loop-free and that every vertex x, except the source vertex v_0, has at least one incoming edge.

Algorithm Dijkstra: Finding the shortest costs from v_0 .

```
var $dist$: array[$0 \ldots n-1$] of integer initially $\forall i : dist[i] = \infty$;
    $fixed$: array[$0 \ldots n-1$] of boolean initially $\forall i : fixed[i] = false$;
    $H$: binary heap of $(j, d)$ initially empty;// $j$ is the vertex and $d$ is the cost
$dist[0] := 0$;
$H$.insert((0,dist[0]));
while $\neg H.empty()$ do
    $(j, d) := H$.removeMin();
    if $(fixed[j])$ then continue;
    $fixed[j] := true$;
    forall $k$: $\neg fixed(k) \wedge (j, k) \in E$ do
        if $(dist[k] > dist[j] + w[j, k])$ then $dist[k] := dist[j] + w[j, k]$;
        $H$.add $(k, dist[k])$ ;
    end
end
```



(a)

Figure 6.1: A Weighted Directed Graph

Dijkstra's algorithm maintains $dist[i]$, which is a cost to reach v_i from v_0. Each vertex x in the graph has initially $dist[x]$ equal to ∞. Whenever a vertex is discovered for the first time, its $dist[x]$ becomes less than ∞. We use the predicate $discovered(x) \equiv dist[x] < \infty$. The variable *dist* decreases for a vertex whenever a shorter path is found due to edge relaxation.

In addition to the variable *dist*, a Boolean array *fixed* is maintained. Thus, every discovered vertex is either *fixed* or *non-fixed*. The invariant maintained by the algorithm is that if a vertex x is *fixed* then $dist[x]$ gives the final shortest cost from vertex v_0 to x. If x is *non-fixed*, then $dist[x]$ is the cost of the shortest path to x that only goes through fixed vertices.

A heap H keeps all vertices that have been discovered but are not fixed along with their distance estimates *dist*. We view the heap as consisting of tuples of the form $(j, dist[j])$ where the heap property is with respect to *dist* values. The algorithm has one main *while* loop that removes the vertex with the minimum distance from the heap with the method H.removeMin(), say v_j, and marks it as fixed. It then *explores* the vertex v_j by relaxing all its adjacent edges going to non-fixed vertices v_k. The value of $dist[k]$ is updated to the minimum of $dist[k]$ and $dist[j] + w[j, k]$. This step is called *edge relaxation.* If v_k is not in the heap, then it is inserted; else if $dist[k]$ has decreased, then the label associated with vertex k is inserted in the heap. We abstract this step as the method H.add$(k, dist[k])$. Since the vertex may already be on the heap, the insertion may cause a vertex to be present in a heap with different distances. Hence, when we remove a vertex from the heap, if it is already fixed, we simply go to the next vertex in the heap. The algorithm terminates when the heap is empty. At this point there are no discovered non-fixed vertices and *dist* reflects the cost of the shortest path to all discovered vertices. If a vertex j is not discovered then $dist[j]$ is infinity reflecting that v_j is unreachable from v_0.

Observe that every vertex goes through the following states. Every vertex x is initially *undiscovered* (i.e., $dist[x] = \infty$). If x is reachable from the source vertex, then it is eventually *discovered* (i.e., $dist[x] < \infty$). A discovered vertex is initially *non-fixed*, and is therefore in the heap H. Whenever a vertex is removed from the heap it is a *fixed* vertex. A fixed vertex may either be *unexplored* or *explored.* Initially, a fixed vertex is unexplored. It is considered explored when all its outgoing edges have been relaxed.

The following lemma simply summarizes the well-known properties of Dijkstra's algorithm. We leave them as an exercise.

Lemma 6.1 *The outer loop in Dijkstra's algorithm satisfies the following invariants.*
(a) For all vertices x: $fixed[x] \Rightarrow (dist[x] = cost[x])$.
(b) For all vertices x: $dist[x]$ *is equal to cost of the shortest path from* v_0 *to* x *such that all vertices in the path before* x *are fixed.*

For complexity analysis, let n be the number of vertices and m be the number of edges. We analyze the version of Dijkstra's algorithm in which, instead of adjusting the key in the heap for a vertex, we simply insert the vertex in the heap. As a result, the heap may have a vertex multiple times with different keys. When a vertex is removed, we check if it has already been fixed. If it is fixed, then we do not need to explore it, and we can remove the next vertex in the heap. Since a vertex can be inserted in the heap only when an edge is relaxed, we get that there are at most m insertions from the heap. We can also conclude that there are at most m deletions from the heap. Since an insertion or a deletion from a heap takes $O(\log m) = O(\log n)$ time, we get the overall time complexity of $O(m \log n)$.

Here is a walk-through of Dijkstra's algorithm on the graph, starting from vertex v_0.

1. Set v_0 as the source vertex and assign it a distance of 0. Assign all other vertices a tentative distance of infinity. We insert the source vertex v_0 in the heap H.

2. Since the heap is not empty, we remove the vertex at the top of the heap. It is v_0 with a distance of 0. We mark that vertex as *fixed*. We examine its neighbors, v_1 and v_2. For $v_0 \to v_1$, the existing distance to v_1 is infinity. The distance 0 (distance to v_0) + 9 (edge weight from v_0 to v_1) is less than infinity, so we update the distance to v_1 to 9 and add it to the heap. For $v_0 \to v_2$, the existing distance to v_2 is infinity. The distance 0 + 2 is less than infinity, so update the distance to v_2 to 2 and add v_2 to the heap.

3. We remove the vertex at the top of the heap. The smallest distance among the unvisited vertices is 2 (for v_2), so set v_2 as the current vertex. We examine its neighbors, v_3 and v_4. For $v_2 \to v_3$, the existing distance to v_3 is infinity. Since 2 (distance to v_2) + 6 (edge weight from v_2 to v_3) is less than infinity, we update the distance to v_3 to 8. For $v_2 \to v_4$, the existing distance to v_4 is infinity. Since $2 + 5$ is less than infinity, we update the distance to v_4 to 7.

4. Continuing in this manner until the heap becomes empty, we get the final distances as: v_0: 0, v_1: 9, v_2: 2, v_3: 8, v_4: 7.

6.3 Bellman-Ford Algorithm

In this section, we give an algorithm that computes the shortest paths from any given vertex even when edge weights are negative. Why could we not apply Dijkstra's algorithm for this problem? Consider the case where the node that can be reached from the source node on one edge with a minimum cost w has a cost of x. Let this node be v. Dijkstra's algorithm assumes that this is the shortest path to node v, because any other path would already have incurred the cost of w by going to some other vertex. However, in the presence of negative weights, there may be a longer path to v by incurring more cost initially but then traversing an edge with negative cost.

When there are negative cost edges, we have to be careful that there are no negative cost cycles. In the presence of negative cost cycles, the cost of going from s to some vertex v may be decreased in an unbounded manner by going through the cycle. For now, we will assume that the graph may have negative cost edges but no negative cost cycles.

In the 1950s, Bellman and (independently) Ford gave an algorithm for graph with negative weights. The algorithm uses dynamic programming. To establish a recurrence relation, we need to come up with an appropriate variable with an index k such that its base case when k is small is easy and when k is large, we reach our goal. We let $d^k[v]$ be the cost of reaching v from the source vertex s with a path of at most k edges. It is clear that the base case of k equal to 0 is quite simple. The value $d^0[v]$ equals 0 when v equals s and ∞ otherwise. When k equals $n-1$, we claim that $d^k[v]$ is exactly equal to the shortest path from s to v. Why could there not be a path with n edges or more, with a lower cost? If there is a path with n edges, then at least some vertex w appears more than once. The path from w to itself forms a cycle. From our assumption, there are no negative cost cycles. If the cycle is not negative, then we get a smaller path with equal or lower cost by removing the cycle.

The recurrence relation is as follows: $d^0[v] = 0$ if v equals s and ∞, otherwise.

$$d^k[v] = \min\{d^{k-1}[v], \min_{(u,v)\in E} d^{k-1}[v] + w(u,v)\}$$

We let $p[v]$ denote the predecessor of the node v in the shortest path. Initially, $p[v]$ is null for all vertices.

Fig. Bellman-Ford Algorithm shows the procedure used when the edge weights in a directed graph may be negative.

Algorithm Bellman-Ford Algorithm: Finding the shortest costs from v_0 when edge weights may be negative .

```
Input: Graph $G = (V, E)$, source vertex $s$, edge weights $w(u, v)$
Output: Distance array $d[]$, Predecessor array $p[]$
forall vertex $v \in V$ do
    $d[v] := \infty$;
    $p[v] :=$ null;
end
$d[s] := 0$;
for $k = 1$ to $|V| - 1$ do
    forall edge $(u, v) \in E$ do
        if $d[u] + w(u, v) < d[v]$ then
            $d[v] := d[u] + w(u, v)$;
            $p[v] := u$;
        end
    end
end
forall edge $(u, v) \in E$ do
    if $d[u] + w(u, v) < d[v]$ then return "Negative cycle detected" ;
end
return $d[], p[]$;
```

The first *for* loop initializes the value of d and p for the base case corresponding to k equal to zero. We now compute d^k for k equal to $1 \ldots n-1$. Instead of computing the recurrence explicitly for each vertex, we *relax* each edge to check if $d^k[v]$ can be reduced by some edge (u, v). Finally, we check if there is a negative cost cycle in the graph. If there is a negative cost cycle that can be reached from the source vertex, then there would be an edge (u, v) in the graph such that $d[v]$ can be reduced further by using that edge even after k has reached the value $n - 1$.

The time complexity of Bellman-Ford is $O(|V| \cdot |E|)$, where $|V|$ is the number of vertices and $|E|$ is the number of edges, making it less efficient than Dijkstra's algorithm for graphs with non-negative weights but more versatile due to its handling of negative weights.

In our algorithm, we have iterated k from 1 to $n - 1$. What if the values of d do not change in some iteration of k? The reader is invited to change the algorithm that checks for this condition and terminates it sooner.

6.4 All Pairs Shortest Path Algorithm

Suppose that $A[i, j]$ represents the weight of the direct edge from i to j denoting the cost of going from i to j using at most one edge. We assume that all diagonal entries are zero and that there is no negative weight cycle. Our goal is to find a matrix $G[i, j]$ such that $G[i, j]$ is the least cost to go from i to j using any number of edges. We apply the idea of dynamic programming again. However, now we define $d^k(i, j)$ as the cost of going from vertex i to vertex j by using intermediate vertices from $1..k$. When k is zero, it is clear that we cannot use any intermediate vertex. Thus,

$$d^0[i,j] = 0 \ if \ (i = j), w(i,j) \ otherwise.$$

Observe that if there is no edge from i to j when $i \neq j$, $w(i,j)$ is ∞. Now, consider the case when $k \geq 1$. Then, we get the following recurrence:

$$d^k[i,j] = \min\{d^{k-1}[i,j], d^{k-1}[i,k] + d^{k-1}[k,j]\}$$

Thus, we get the Floyd-Warshall algorithm Floyd-Warshall.

Algorithm Floyd-Warshall: All pairs shortest path Algorithm

Input : A weighted graph represented by an $n \times n$ cost matrix $G[i,j]$, where $G[i,j]$ is the cost from vertex i to j (possibly ∞ if no edge exists).

Output: Updated matrix $G[i,j]$ containing shortest path costs between all pairs of vertices.

```
1 for $k = 1$ to $n$ do
2     for $i = 1$ to $n$ do
3         for $j = 1$ to $n$ do
4             if $G[i,k] + G[k,j] < G[i,j]$ then
5                 $G[i,j] := G[i,k] + G[k,j]$
6             end
7         end
8     end
9 end
```

The reader is invited to make two changes to the algorithm. First, modify the algorithm to return an error message when there is a negative cost cycle in the graph. Second, add a variable $p[u][v]$ to the algorithm to return the predecessor of v on the path from u to v.

It is clear that the time complexity of Floyd-Warshall's algorithm is $O(n^3)$.

6.5 Constrained Single Source Shortest Path Algorithm

In this section, we assume that all edge weights are *strictly positive*. We are required to find the minimum cost of a path from a distinguished *source* vertex v_0 to all other vertices, where the cost of a path is defined as the sum of edge weights along that path. For any vertex v, let $pre(v)$ be the set of vertices u such that (u,v) is an edge in the graph. To avoid trivialities, assume that every vertex v (except possibly the source vertex v_0) has nonempty $pre(v)$ and that all nodes in the graph are reachable from the source vertex.

As the first step of the predicate detection algorithm, we define the lattice for the search space. We assign to each vertex v_i, $G[i] \in \mathbf{R}_{\geq 0}$ with the interpretation that $G[i]$ is the cost of reaching the vertex v_i. We call G, the *assignment vector*. The invariant maintained by our algorithm is: for all i, the cost of any path from v_0 to v_i is greater than or equal to $G[i]$. The vector G only gives the lower bound on the cost of a path, and there may not be any path to the vertex v_i with cost $G[i]$. To capture that an assignment is feasible, we define *feasibility* that requires the notion of a *parent*. We say that v_i is a parent of v_j in G (denoted by the predicate $parent(j,i,G)$) iff there is a direct edge from v_i to v_j and $G[j]$ is at least $(G[i] + w[i,j])$, i.e., $(i \in pre(j)) \wedge (G[j] \geq G[i] + w[i,j])$. A node may have multiple parents.

In Fig. 6.2, let G be the vector $(0,2,3,5,8)$. Then, v_0 is a parent of v_2 because $G[2]$ is greater than $G[0]$ plus $w[0,2]$($i.e.$, $3 \geq 0+2$). Similarly, v_1 is a parent of v_4 because $G[4] \geq G[1] + 2$. A node may have multiple parents. The node v_2 is also a parent of v_4 because $G[4] \geq G[2] + 5$.

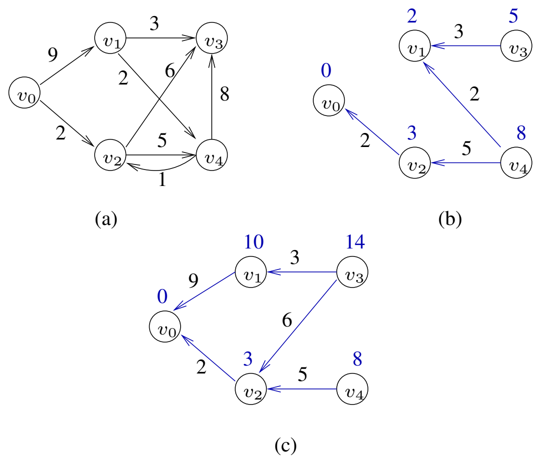


Figure 6.2: (a) A Weighted Directed Graph (b) The parent structure for $G = (0, 2, 3, 5, 8)$ (c) The parent structure for $G = (0, 10, 3, 14, 8)$. Since every non-source node has at least one parent, G is feasible.

Since $w[i, j]$ are strictly positive, there cannot be a cycle in the parent relation. Now, the feasibility can be defined as follows.

Definition 6.2 (Feasible for paths) *An assignment G is* feasible for paths *iff every node except the source node has a parent. Formally, $B_{path}(G) \equiv \forall j \neq 0 : (\exists i : parent(j, i, G))$.*

Hence, an assignment G is feasible iff one can go from any non-source node to the source node by following any of the parent edges. We now show that feasibility satisfies lattice-linearity.

Lemma 6.3 *For any assignment vector G that is not feasible, $\exists j : forbidden(G, j, B_{path}(G))$.*

Proof: Suppose G is not feasible. Then, there exists $j \neq 0$ such that v_j does not have a parent, i.e., $\forall i \in pre(j) : G[j] < G[i] + w[i, j]$. We show that $forbidden(G, j, B_{path}(G))$ holds. Pick any H such that $H \geq G$. Since for any $i \in pre(j)$, $H[i] \geq G[i]$, $G[j] < G[i] + w[i, j]$ implies $G[j] < H[i] + w[i, j]$. Therefore, whenever $H[j] = G[j]$, v_j does not have a parent.

■

Since B_{path} is a lattice-linear predicate, it follows from Lemma 3.3(c) that the set of feasible assignment vectors are closed under meets (the component-wise min operation). Hence, we can use the LLP algorithm with $\alpha(G, j, B_{path}) = \min\{G[i] + w[i, j] \mid i \in pre(j)\}$.

For an unweighted graph (that is, each edge has weight equal to 1), the above parallel algorithm requires a time equal to the distance of the farthest node from the root. However, the LLP algorithm derived from B_{path}, may take a time that depends on the weights because the progress along a forbidden process may be small.

We now give an alternative feasible predicate that results in an algorithm that takes larger steps. We first define a node j as *fixed* in G if it is the source node, or it has a parent that is a fixed node, i.e. $fixed(j, G) \equiv (j = 0) \vee (\exists i : parent(j, i, G) \wedge fixed(i, G))$.

Observe that node v_0 is always fixed. Any node v_j such that one can reach from v_j to v_0 using parent relation is also fixed. We now define another feasible predicate, B_{rooted}, as $B_{rooted}(G) \equiv \forall j : fixed(j, G)$.

Even though it may first seem that the predicate B_{rooted} is strictly stronger than B_{path}, the following Lemma shows otherwise.

Lemma 6.4 *$B_{path}(G)$ iff $B_{rooted}(G)$.*

Proof: If G satisfies B_{rooted}, then every node other than v_0 has at least one parent by definition of $fixed$, hence $B_{path}(G)$. Conversely, suppose that every node except v_0 has a parent. Since parent edges cannot form a cycle, by following the parent edges, we can go from any node to v_0.

■

It follows that the predicate B_{rooted} is also lattice-linear. Then, the following threshold $\beta(G)$ is well defined whenever the set of edges from the fixed vertices to non-fixed vertices is nonempty.

$$\beta(G) = \min_{(i,j):i \in pre(j)} \{G[i] + w[i,j] \mid fixed(i,G), \neg fixed(j,G)\}.$$

If the set of such edges is empty then no non-fixed vertex is reachable from the source. We call these set of edges *Heap* (because we would need the minimum of this set for advancement).

We now have the following result in advancement of G.

Lemma 6.5 *Suppose $\neg B_{rooted}(G)$.*
Then, $\neg fixed(j,G) \Rightarrow forbidden(G, j, B_{rooted}, \beta(G))$.

Proof: Consider any assignment vector H such that $H \geq G$ and $H[j] < \beta(G)$. We show that H is not B_{rooted}. In particular, we show that j is not fixed in H. Suppose j is fixed in H. This implies that there is a path W from v_0 to v_j such that all nodes in that path are fixed. Let the path be the sequence of vertices $w_0, w_1, \ldots w_{m-1}$, where $w_0 = v_0$ and $w_{m-1} = v_j$. Let $w_l = v_k$ be the first node in the path that is not fixed in G. Such a node exists because w_{m-1} is not fixed in G. Since w_0 is fixed, we know that $1 \leq l \leq m-1$. The predecessor of w_l in that path, w_{l-1} is well-defined because $l \geq 1$. Let $w_{l-1} = v_i$.

We show that $H[k] \geq \beta(G)$ which contradicts $H[j] < \beta(G)$ because $H[k] \leq H[j]$ as the cost can only increase going from k to j along the path W. We have $H[k] \geq H[i] + w[i,k]$ because i is a parent of k in H. Therefore, $H[k] \geq G[i] + w[i,k]$ because $H[i] \geq G[i]$. Since i is fixed in G and k is not fixed in G, from the definition of $\beta(G)$, we get that $\beta(G) \leq G[i] + w[i,k]$. Hence, $H[k] \geq \beta(G)$.

■

By using the advancement Lemma 6.5, we get the algorithm LLP-ShortestPath shown in Fig. LLP-Shortest Path. In this algorithm, in every iteration we find all nodes that are forbidden (not fixed) and advance them. All nodes are advanced to $\alpha(G,j)$ that combines $\beta(G)$ with $\min\{G[i] + w[i,j] \mid i \in pre(j)\}$. Note that if a node is fixed, its parent is fixed and therefore any algorithm that advances $G[j]$ only for non-fixed nodes j maintains that once a node becomes fixed it stays fixed.

By removing certain steps, we get Dijkstra's algorithm from the algorithm LLP-ShortestPath. It is clear that the algorithm stays correct if $\alpha(G,j)$ uses just $\beta(G)$ instead of $\max\{\beta(G), \min\{G[i]+w[i,j] \mid i \in pre(j)\}\}$. Secondly, the algorithm stays correct if we advance G only on the node j such that (i,j) are in Heap edges and the node j minimizes $G[i] + w[i,j]$. Finally, to determine such a node and $\beta(G)$, it is sufficient to maintain a min-heap of all non-fixed nodes j, along with the label that equals $\min_{i \in pre(j), fixed(i,G)} G[i] + w[i,j]$. On making all these changes to LLP-Shortest Path, we get Dijkstra's algorithm (modified to run with a heap).

It is illustrative to compare the algorithm LLP-Shortest Path with Dijkstra's algorithm. In Dijkstra's algorithm, the nodes become fixed in the order of the cost of the shortest path to them. In the LLP-Shortest Path algorithm, a node may become fixed even when nodes with shorter cost have not been discovered. In Fig. 6.2, node v_1 becomes fixed earlier than nodes v_3 and v_4. This feature is especially useful when we are interested in finding the shortest path to a single destination and that destination becomes fixed sooner than it would have been in Dijkstra's algorithm.

Dijkstra's algorithm maintains a distance vector *dist* such that it is always feasible, i.e., for any vertex v there exists a path from source to v with cost less than or equal to $dist[v]$. We maintain the invariant that the cost of the shortest path from source to v is guaranteed to be at least $G[v]$. Therefore, in Dijkstra's algorithm, $dist[v]$ is initialized to ∞ whereas we initialize G to 0. Dijkstra's algorithm and indeed many algorithms for combinatorial optimization, such as simplex, start with a feasible solution and march towards the optimal solution, our algorithm starts with an extremal point in search space (even if it is infeasible) and marches towards feasibility. Also note that in Dijkstra's algorithm, $dist[v]$ can only decrease during execution. In our algorithm, G can only increase with execution.

The *ShortestPath* algorithm has a single variable G. All other predicates and functions are defined using this variable.

Algorithm LLP-Shortest Path: Finding the minimum cost assignment vector.

```
input: $pre(j)$: list of $1..n$; $w[i,j]$: positive int for all $i \in pre(j)$
init: $G[j] := 0$;
always: $parent[j,i] = (i \in pre(j)) \wedge (G[j] \geq G[i] + w[i,j])$;
        $fixed[j] = (j = s) \vee (\exists i : parent[j,i] \wedge fixed[i])$
        $Heap = \{(G[i] + w[i,k]) | (i \in pre(k)) \wedge fixed(i) \wedge \neg fixed(k)\}$;
forbidden: $\neg fixed[j]$
advance: $(G[j] := \max\{\min Heap, \min\{G[i] + w[i,j] \mid i \in pre(j)\}\})$
```

We get the following structural result on the assignment vectors that are feasible.

Lemma 6.6 *Let G and H be two assignment vectors such that they satisfy B_{rooted}, then $min(G, H)$ also satisfies B_{rooted}.*

Proof: Follows directly from Lemma 3.3, because B_{rooted} is a lattice-linear predicate.

■

The set of feasible assignment vectors is not closed under the join operation. In Fig. 6.2, the vectors $(0, 10, 3, 14, 8)$ and $(0, 9, 10, 12, 11)$ are feasible, but their join $(0, 10, 10, 14, 11)$ is not feasible.

We now consider the generalization of the shortest path algorithm with constraints. We assume that all constraints specified are lattice-linear. For example, consider the constraint that the cost of vertex i is at most cost of vertex j. The predicate $B \equiv G[j] \geq G[i]$ is easily seen to be lattice-linear. If any cost vector G violates B, then the component j is forbidden (with $\alpha(G, j)$ equal to $G[i]$). The predicate $(G[i] = G[j])$ is also lattice-linear because it can be written as a conjunction of two lattice-linear predicates $(G[i] \geq G[j])$ and $(G[j] \geq G[i])$. The predicate $B \equiv (G[i] \geq k) \Rightarrow (G[j] \geq m)$ is also lattice-linear. If any cost vector violates B, then we have $(G[i] \geq k) \wedge (G[j] < m)$. In this case, the component j is forbidden with $\alpha(G, j)$ equal to m. Again, the algorithm *LLP* can be used to solve the constrained shortest path algorithm by combining *forbidden* and α for constraints with B_{rooted}. An application of the constrained shortest path problem is as follows. Suppose that there are n dispatch trucks that start from the source vertex at time 0. Let $w[i, j]$ denote the time it takes for a truck to go from node i to node j. An assignment vector G is feasible if it is possible to design a tree rooted at the source vertex such that the path from the source vertex to vertex i takes $G[i]$ units of time and G satisfies specified constraints. In Fig. 6.2, the vector $(0, 9, 2, 8, 7)$ is feasible, but does not satisfy the constraint that $G[1]$ equals $G[2]$. The least vector that satisfies this additional constraint is $(0, 9, 9, 12, 11)$. LLP algorithm can be used to find this vector. When the set of additional constraints is empty, we get back the standard shortest path problem.

An example of a predicate that is not lattice-linear is $B \equiv G[i] + G[j] \geq k$. If the predicate is false for G, then we have $G[i] + G[j] < k$. However, neither i nor j may be forbidden. The component i is not forbidden because if $G[i]$ is fixed but $G[j]$ is increased, the predicate B can become true. Similarly, j is also not forbidden.

Algorithm LLP-ShortestPath2: An Implementation of Algorithm LLP-Shortest Path.

```
var $G$: array[0..$n-1$] of real initially $\forall i : G[i] = 0$;
while $\exists i : \neg fixed(i, G)$ do
    $E' := \{ (i, k) \in E \mid fixed(i, G) \wedge \neg fixed(k, G)\}$;
    if $(E' = \phi)$ then return "non-fixed nodes not reachable";
    Let $(i^*, j^*) \in E'$ minimize $G[i] + w[i, j]$;
    $G[j^*] := \beta(G)$;
    forall $j \neq j^*$ such that $forbidden(G, j)$ do
        $G[j] := \alpha(G, j)$;
    end
end
return $G$;
$parent(j, i, G) \equiv (i \in pre(j)) \wedge (G[j] \geq G[i] + w[i, j])$
$fixed(j, G) \equiv (j = 0) \vee (\exists i : parent(j, i, G) \wedge fixed(i, G))$
$\beta(G) = \min\{G[i] + w[i, j] \mid (i, j) \in E'\}$
$forbidden(G, j) \equiv \neg fixed(j, G)$
$\alpha(G, j) = \max\{\beta(G), \min\{G[i] + w[i, j] \mid i \in pre(j)\}\}$
```

6.6 LLP-Bellman-Ford Algorithm

Consider the distance of other vertices from s. To apply LLP algorithm, we view finding the optimal distance vector as finding the largest $d[x]$ for every vertex $x \neq v_0$ which satisfies the following feasibility predicate:

$$\text{for all edges } (i, j) \in E : d[j] \leq d[i] + w[i, j]$$

If we define B_e for any edge $e = (i, j)$ as $d[j] \leq d[i] + w[i, j]$. The feasibility predicate B can be written as

$$B \equiv \wedge_{e \in E} B_e$$

We will view the search for optimal distance vector as searching for d vector in a lattice defined as follows. The lattice has the bottom element as $(M, M, \ldots, M)$ where M equal $n - 1$ times the largest weight edge in the graph. Since we are looking for the largest distance vector satisfying B, it can never be greater than

M. In this lattice $d \leq_L d'$ iff $\forall i : d[i] \geq d'[i]$. Thus, finding the least vector in this lattice corresponds to finding the largest vector d that satisfies B.

We now claim that

Theorem 6.7 *B is a lattice linear predicate.*

Proof: It is sufficient to show that B_e is lattice linear because any conjunction of linear predicates is also lattice linear. B_e for $e = (i, j)$ is equivalent to $d[j] \leq d[i]+w[i, j]$. This is equivalent to $d[j] \geq_L d[i]+w[i, j]$. Since the right hand side is a monotonic function of d, we get that B_e is lattice linear.

■

Algorithm LLP-Edge-Relaxation: Finding the minimum distance vector satisfying B

```
input: $pre(j)$: list of $1..n$; $w[i,j]$: int for all $i \in pre(j)$
init: if $(j = s)$ then $d[j] := 0$ else $d[j] :=$ maxint;
ensure: $d[j] \leq \min\{d[i] + w[i,j] \mid i \in pre(j)\}$
```

Algorithm LLP-Edge-Relaxation gives the correct answer; however, it may result in large computation complexity if d's are lowered in a non-disciplined manner. We now optimize this algorithm. Algorithm LLP-BellmanFord chooses forbidden indices more carefully. Whenever there is a forbidden vertex, all forbidden vertices are advanced. The *while* loop checks if there is any forbidden vertex. The *forall* loop advances all forbidden vertices.

Algorithm LLP-BellmanFord: Finding the minimum distance vector satisfying B

```
$P_j$: Code for thread $j$
var $d$: array$[0..n-1]$ of real initially $(d[0] = 0) \wedge (\forall i \neq 0\colon\ d[i] = \infty)$
always
    $forbidden(d, j) \equiv \exists i \in pre(j) : d[j] > d[i] + w[i, j]$
while $\exists j : forbidden(d, j)$
    forall $j : forbidden(d, j)$ do
        $d[j] := \min\{d[i] + w[i, j] \mid i \in pre(j)\}$;
```

Let us analyze the sequential time complexity of Algorithm LLP-BellmanFord. There are at most n iteration of the *while* loop. The *forall* loop can be implemented with $O(m)$ time complexity giving us the overall sequential time complexity of $O(mn)$. For parallel time complexity, it is sufficient to observe that the *forall* loop can be implemented in $O(\log n)$ parallel time. Thus, the parallel time complexity is $O(n \log n)$.

Earlier, we had assumed that the graph does not have any negative cost cycle. What if the user input is such that it has a negative cost cycle? In that case, the algorithm LLP-BellmanFord will never terminate. The problem 2 asks you to modify the algorithm so that it detects if the graph has a negative cost cycle and then outputs such a message.

6.7 Graphs with Negative Weights: Johnson's Algorithm

We now consider directed graphs which have negative weight edges. For such graphs, the shortest path may not be defined if there is a cycle with negative cost. By going around the cycle, one can arbitrarily decrease the cost of the path. Therefore, we assume that, even though there are edges with negative costs, there are no negative cost cycles. Our strategy for finding the shortest path in such a graph X would be to convert it into another graph Y on the same set of vertices such that Y has all nonnegative edges and preserves all shortest paths, i.e., a path is shortest in X iff it is also a shortest path in Y. The graph Y has the same set of vertices and edges as X. The weight of any edge (i, j) is updated as follows:

$$w'[i,j] = w[i,j] + p[j] - p[i] \tag{6.1}$$

where p is a *price* vector associated with vertices. A price vector p is a non-negative vector such that when we compute new costs of edges, called *reduced* costs, we get that the new cost of every edge is at least 0. The advantage of updating the weights using Equation 6.1 is that it preserves the shortest paths.

Lemma 6.8 *Let s and t be any two vertices in the graphs. The weight of any path on the graph Y from s to t equals the weight in the graph X plus $(p[t] - p[s])$.*

Proof: Let the path be v_0, v_1, v_2, v_k where $v_0 = s$ and $v_k = t$. As we compute the cost of the path in Y, we add the price of every intermediate vertex once and subtract it once. Only the price of v_0 is subtracted exactly once, and the price of v_k is added exactly once, giving us the lemma.

■

Since the costs of all paths between s and t change by the same amount, it follows that any shortest path in X is a shortest path in Y and vice versa. Now our task is reduced to finding a price vector such that $w'[i, j]$ is at least 0 for all edges. We use LLP algorithm to find such a vector. Our feasibility predicate B for the pricing vector is

$$\forall(i,j) \in E : w[i,j] + p[j] - p[i] \geq 0$$

Furthermore, we require $p[i] \geq 0$ for all i. We first show that B is lattice linear.

Lemma 6.9 *Let X be any graph such that the edge (i, j) has weight $w[i, j]$ and every vertex i has price $p[i]$. Consider the lattice of all non-negative price vectors. Then, the predicate*

$$B \equiv \forall(i,j) \in E : w[i,j] + p[j] - p[i] \geq 0$$

is lattice linear.

Proof: Since the lattice linearity is closed under conjunction, it suffices to show that $B_e \equiv w[i,j] + p[j] - p[i] \geq 0$ is lattice linear for an arbitrary edge $e = (i, j)$. B_e can be rewritten as $p[j] \geq p[i] - w[i, j]$. The right-hand side of this inequality is a monotone function on p and hence, from the Key Lemma of lattice-linearity, we get that B_e is lattice linear.

■

By applying the LLP algorithm, we get the following method to find the price vector.

Since we have not provided the stopping point in Algorithm LLP-Johnson, the method may not terminate. Indeed, if there is a cycle in the graph with negative cost, the above algorithm will not terminate. We show

Algorithm LLP-Johnson: Finding the minimum price vector.

```
1 input: $pre(j)$: list of $1..n$; $w[i,j]$: int for all $i \in pre(j)$
2 init: $p[j] := 0$;
3 ensure: $p[j] \geq \max\{p[i] - w[i,j] \mid i \in pre(j)\}$
```

Lemma 6.10 *Algorithm LLP-Johnson terminates iff there is no negative cost cycle in the graph X.*

Proof:

We first show that if there is a negative cost cycle, then the algorithm never terminates. Considering s and t to be the same vertex, Lemma 6.8 implies that the cost of any cycle remains unchanged after relabeling. Hence, if there is any negative cost cycle in X, then for all pricing vectors the cost of the cycle is negative in Y. For any cycle to have a negative cost, at least one edge must be negative. Hence, for all pricing vectors, there exists i, j such that $w'[i,j] = w[i,j] + p[j] - p[i] < 0$. This implies that there exists j such that $p[j] < p[i] - w[i,j]$. Hence, $forbidden(p,j)$ holds.

Now, assume that there is no negative cost cycle in the graph. Since the LLP algorithm finds the least feasible element of the lattice that satisfies the feasibility predicate B, it is sufficient to show that there exists at least one feasible solution. Consider an additional vertex z added to the graph such that z has a directed edge to every node in the graph with cost 0. Since there is no incoming edge to z, adding z cannot introduce a negative cost cycle in the graph. Since there are no negative cost cycles, there are a finite number of paths with the least cost from z to any vertex x. The cost of the shortest path from z to any vertex is 0 or smaller because there is a direct edge from z to any vertex with weight 0. Let $d[z,x]$ be the distance of any vertex x from z. We define $p[x]$ as $-d[z,x]$ and claim that $w[i,j] + p[j] - p[i] \geq 0$ for all edges (i,j). If not, we get $w[i,j] - d[z,j] + d[z,i] < 0$. However, this implies $d[z,j] > d[z,i] + w[i,j]$ which violates that $d[z,j]$ and $d[z,i]$ correspond to the cost of the shortest paths from z since by first going through i, the cost of reaching j can be reduced.

■

The proof of Lemma 6.10 shows that there always exists a price vector that makes edge weights non-negative. From the Key Lattice-Linearity Lemma, we know that the set of all feasible price vectors is closed under meets. It can also be observed that if the price vector p is feasible, and that if we add a fixed positive constant to all prices, then all prices stay positive and all reduced weights stay the same. This observation also implies that the least price vector p that satisfies B must have at least one component that is zero. Otherwise, we can subtract the least price from all prices and still get a feasible price vector that contradicts p being the least feasible price vector.

We now bound the number of iterations in the LLP algorithm required to find the price vector.

Lemma 6.11 *The LLP algorithm requires at most n iterations before termination.*

Proof: We claim that, after i iterations of the LLP algorithm, $p[x]$ is negative of the cost of any path from any vertex y to x with at most i edges. Initially, when i equals 0, $p[x]$ equals 0 which is equal to the negative of the shortest path (from x to itself). Assume that the claim holds for the $i-1$ iterations. In the i^{th} iteration, we show that $p[x]$ is updated appropriately. Consider any shortest path of length i with y as the predecessor vertex on that path. If this path is shorter than any path of length $i-1$, we have

$-p[y] + w[y, x] \leq -p[x]$. However, this condition is equivalent to $p[x] < p[y] - w[y, x]$ that guarantees that $p[x]$ is forbidden and increased so that $p[x] = p[y] - w[y, x]$.

Since any shortest path can have at most length n, the lemma follows.

■

Since each LLP iteration can be computed in $O(m)$ time, we get that the graph can be "reweighted" in $O(mn)$ time. Now we can use Dijkstra's algorithm to find the shortest path in the transformed graph for a single source shortest path problem. Since the algorithm for reweighting the graph dominates Dijkstra's shortest path algorithm, we get an algorithm with $O(mn)$ complexity to find shortest paths from a vertex to all vertices in a weighted directed graph possibly with negative costs. Furthermore, all pairs shortest path problem can be solved by using Dijkstra's algorithm from every vertex after reweighting the graph. This gives us an all pairs shortest path algorithm with the time complexity of $O(mn + n(m + n \log n) = O(mn + n^2 \log n)$. This algorithm has a lower time complexity than $O(n^3)$ Floyd-Warshall algorithm whenever the graph is not dense.

6.8 LLP based All Pair Shortest Path Algorithm

We start with the problem of matrix multiplication of two $n \times n$ square matrices A and B, i.e., we want to compute $C = AB$. The entry $C[i, j]$ can be computed as

$$C[i, j] = \sum_k A[i, k] * B[k, j].$$

This algorithm takes $O(n^3)$ work. Here, we have used the standard sequential algorithm for matrix multiplication. By using more advanced algorithms, one can reduce the exponent for the work. For example, by using Coppersmith and Winograd's algorithm, the work can be reduced to $O(n^{2.376})$. However, for simplicity, we will continue to use the standard matrix algorithm in this chapter.

Let $A[i, j]$ be a boolean matrix of size n by n. This can also be viewed as a directed graph on n vertices such that there is an edge from i to j iff $A[i, j]$ equals 1.

We now define the reflexive transitive closure of a matrix as

$$A^* = A^0 + A^1 + A^2 + \ldots A^n$$

The entry $A^*[i, j]$ equals 1 iff there is any path from i to j. Our goal is to compute the reflexive transitive closure of A. We can apply the technique of *repeated squaring* to compute the transitive closure. The ideas is to compute A^{2^k} by multiplying $A^{2^{k-1}}$ with itself. Now, it is a simple matter to compute the reflexive transitive closure. We first compute a matrix $C = I + A$. This matrix simply corresponds to adding self-loops in the graph at every vertex. Now, it is easy to show that $A^* = C^n = C^{2^{\log n}}$.

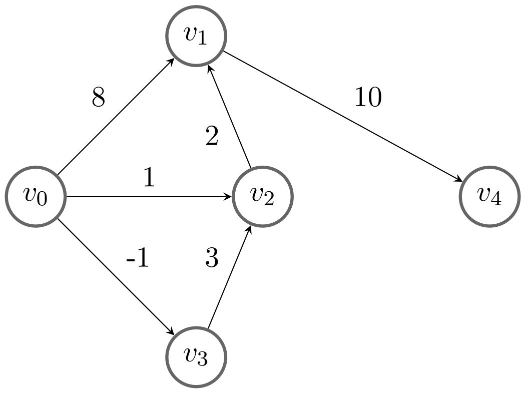
We now show that the problem of reflexive transitive closure can also be solved using the LLP technique. We view the problem as as searching for the least vector G indexed by tuple (i, j) where $i, j \in [0..n-1]$ such that

$$B_{closure} \equiv \forall i, j : G[i, j] \geq \max(A[i, j], \max\{G[i, k] \wedge G[k, j] \mid k \in [0..n-1]\}).$$

Since the right hand side of the inequality is a monotone function of G, it follows that $B_{closure}$ is lattice-linear. We can apply *LLP* algorithm to compute the transitive closure. We have a boolean lattice of all vectors of size $O(n^2)$. Our goal is to find the least vector in this lattice that satisfies $B_{closure}$. The corresponding *LLP* algorithm is shown in Fig. LLP-Transitive-Closure

Algorithm LLP-Transitive-Closure: Computing the transitive closure of a matrix

```
1 input: $A$: matrix of boolean; // $A[i,j]$ is 1 if there is an edge from $i$ to $j$;
2 var $G$: matrix of boolean initially $\forall i,j : G[i,j] = A[i,j]$;
3 forbidden$(i,j)$: $(\exists k : G[i,j] < G[i,k] \wedge G[k,j])$
4     advance: $G[i,j] := 1$;
```



We now extend the algorithm to nonbinary matrices. Suppose that $A[i,j]$ represents the weight of the direct edge from i to j denoting the cost of going from i to j using at most one edge. We assume that all diagonal entries are zero and that there is no negative weight cycle. Our goal is to find a matrix $G[i,j]$ such that $G[i,j]$ is the least cost of going from i to j by using any number of edges.

Let us formulate this problem as finding the least vector of size n^2 in a distributive lattice. We initialize G as A. This is the bottom element of our distributive lattice. Our goal is to find the least vector that satisfies the following constraint:

$$B_{Floyd-Warshall} \equiv \forall i,j : i \neq j : G[i,j] \leq \min_k \{G[i,k] + G[k,j]\}.$$

When i equals j, $G[i,j]$ equals 0 and will stay fixed through the execution of the algorithm. The cost of going from i to j must be smaller than the cost of going from i to k and k to j. Fig. LLP-Floyd-Warshall gives a high-level LLP algorithm for all pairs shortest path.

Algorithm LLP-Floyd-Warshall: Computing the shortest cost matrix for a directed graph

```
1 input: $A$: matrix of real; // $A[i,j]$ is the weight of the edge from $i$ to $j$;
2 var $G$: matrix of real initially $\forall i,j : G[i,j] = A[i,j]$;
3 forbidden$(i,j)$: $(\exists k : G[i,j] > G[i,k] + G[k,j])$
4     advance: $G[i,j] := \min_k \{G[i,k] + G[k,j]\}$;
```

6.9 Problems

1. Prove Lemma 6.1.

2. Modify LLP-BellmanFord so that it detects if the input graph has a negative cost cycle and then outputs such a message.

6.10 Bibliographic Remarks

The single source shortest path problem has a rich history. For the history of Dijkstra's algorithm, the reader is referred to the book by [Eri19]. One popular research direction is to improve the worst case complexity of Dijkstra's algorithm by using different data structures. For example, by using Fibonacci heaps for the min-priority queue, Fredman and Tarjan [FT87] gave an algorithm that takes $O(e + n \log n)$. There are many algorithms that run faster when weights are small integers bounded by some constant γ. For example, Ahuja et al [AMOT90] gave an algorithm that uses Van Emde Boas tree as the priority queue to give an algorithm that takes $O(e \log\log\gamma)$ time. Thorup [Tho00] gave an implementation that takes $O(n + e \log\log n)$ under special constraints on the weights. Raman [Ram97] gave an algorithm with $O(e + n\sqrt{\log n \log\log n})$ time. The LLP algorithm for the shortest path is taken from [AKG20]. Bellman and Ford's algorithm is from [Bel58] and [For56].

Chapter 7

The Minimum Spanning Tree Problem

7.1 Introduction

Suppose that we have an undirected weighted graph on n vertices with m edges. Our goal is to find the minimum spanning tree (MST). We assume that all edge weights are distinct. It is known that if the graph is connected and all edge weights are distinct, then there is a unique minimum spanning tree. If the graph is not connected, then there is a unique minimum spanning forest. For simplicity of exposition, we will assume that the underlying graph is connected.

Section 7.2 describes the notion of a *fragment* of a minimum spanning tree. This notion is useful in understanding all the spanning tree algorithms that appropriately extend a fragment (or fragments).

The problem of finding the minimum spanning tree is a canonical problem for which a greedy approach works. In this chapter, we first present a greedy sequential algorithm called Kruskal's algorithm in Section 7.3. Kruskal's algorithm chooses the least-cost edge that is feasible as its next edge. It is sequential in nature, because it picks one edge at a time. We then show a parallel version of Kruskal's algorithm in which every processor is responsible for either choosing or rejecting one edge.

We then present another greedy sequential algorithm: Prim's algorithm in Section 7.4. Prim's algorithm is greedy in a different way. It chooses the next vertex that can be added to the existing tree with the least cost. Next, Section 7.5 discusses an LLP version of Prim's algorithm. In particular, instead of adding one vertex to the tree at a time, we show a strategy that allows multiple vertices to be added and explored.

In Section 7.6, we discuss Boruvka's algorithm, which chooses multiple edges at a time. Boruvka's algorithm maintains multiple *fragments* and adds edges to all of them in every iteration.

7.2 Fragments

The notion of a *fragment* is crucial in understanding MST algorithms. A fragment is simply a subtree of the MST. Consider the graph in Fig. 7.1. The minimum spanning tree in this graph corresponds to the edges $\{2, 3, 4, 7\}$. The subtree formed by the edges 3 and 4 is a fragment with three vertices $\{a, b, c\}$ and two edges $\{(a, c), (b, c)\}$.

A crucial property of the MST is as follows.

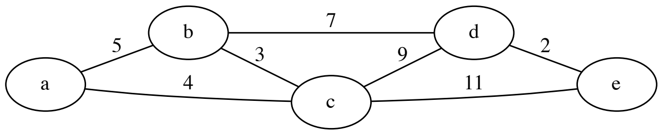


Figure 7.1: An undirected weighted graph

Lemma 7.1 *Let F be a fragment. Let e be the edge with the minimum weight that is outgoing from F. Then $F \cup \{e\}$ is also a fragment.*

Proof: In the minimum spanning tree T, there must be at least one edge going out of the fragment F. Let that edge be f. If we add e to T and remove f, we get another tree T' with a lower weight than T, a contradiction because we assumed that T is the minimum spanning tree.

■

In the fragment formed by edges with weights 3 and 4, there are three outgoing edges — edges with weights 7, 9, and 11. The edge 5 is not outgoing, as it connects vertices that are part of the fragment. According to Lemma 7.1, the edge with weight 7 can be added to the edges with weight 3 and 4 to grow the fragment.

7.3 Kruskal's Algorithm

Kruskal's algorithm is a canonical greedy algorithm for the minimum-spanning tree problem. It chooses edges one at a time in a greedy fashion. Let T be the set of edges chosen by the algorithm at any stage. The algorithm maintains the invariant that T does not have any cycle. Initially T is empty and therefore trivially satisfies the invariant. The algorithm considers the edges in increasing order of weights. For this reason, it is sometimes also known as the *shortest-edge-next* algorithm. Suppose that the algorithm has chosen the edges in T so far. To grow T, it finds the edge with the least weight that does not form a cycle with existing edges in T. If any edge e forms a cycle with T, it is rejected, and the algorithm considers the least weight edge of the remaining edges.

In Algorithm Kruskal, the variable T keeps the set of all edges chosen, and the variable *Rejected* keeps the set of all edges rejected. Consider the graph shown in Fig. 7.1. Kruskal's algorithm will first choose the edge with weight 2 since it has the least weight. Then, it chooses the edges with weight 3 and 4 because these edges do not form any cycle. The next edge has weight 5. However, this edge should be rejected because it forms a cycle with the already chosen edges with weights 3 and 4. The algorithm then chooses the edge with weight 7 and terminates.

The algorithm is dominated by the cost of sorting the edges: $O(m \log n)$. Checking whether the edge e forms a cycle with T can be done efficiently using the find-union data structure.

The Find-Union data structure maintains a collection of disjoint sets, supporting two primary operations:

Algorithm Kruskal: Computing Minimum Spanning Tree

```
1  Input: Undirected Weighted Graph: $(V, E, w)$.
2  Output: Minimum Weight Spanning Tree
3  var
4      $T, Rejected$: set of edges initially $\{\}$;
5  while $|T| < n - 1$ do
6      if $T \cup Rejected = E$ then return null // no spanning tree in the graph ;
7      $e :=$ the least weight edge that is not in $T \cup Rejected$;
8      if $e$ forms a cycle in $T$ then $Rejected := Rejected \cup \{e\}$;
9      else $T := T \cup \{e\}$;
10 end
11 return $T$
```

- **Find**: Determines which set a particular element belongs to by returning its representative (or "root"). With path compression, this operation is nearly constant time, $\mathcal{O}(\alpha(n))$, where $\alpha(n)$ is the inverse Ackermann function.

- **Union**: Merges two sets into one by linking their roots, typically using rank or size heuristics to keep trees balanced, also achieving $\mathcal{O}(\alpha(n))$ amortized time.

Initially, each element is in its own set. As Kruskal's algorithm processes edges, Union merges sets of vertices connected by selected edges, while Find checks if adding an edge would form a cycle (i.e., if its vertices are already in the same set).

Here is the basic implementation of the find-union data structure.

Algorithm Find: Finding the root of x's set with path compression

```
1  Input: Vertex $x$
2  Output: Root of the set containing $x$
3  var
4      $parent$: array of int
5  if $parent[x] \neq x$ then $parent[x] := Find(parent[x])$ //path compression ;
6  return $parent[x]$
```

Algorithm Union: Computing Union by rank of sets containing x and y

```
Input: Vertices $x$, $y$
Output: Unified set containing $x$ and $y$
var
    $parent$: array of int
    $rank$: array of int
$rootX := Find(x)$;
$rootY := Find(y)$;
if $rootX = rootY$ then return// already in same set, no union needed ;
if $rank[rootX] < rank[rootY]$ then $parent[rootX] := rootY$ ;
else if $rank[rootX] > rank[rootY]$ then $parent[rootY] := rootX$ ;
else $parent[rootY] := rootX$
$rank[rootX] := rank[rootX] + 1$ // increment rank if equal ;
```

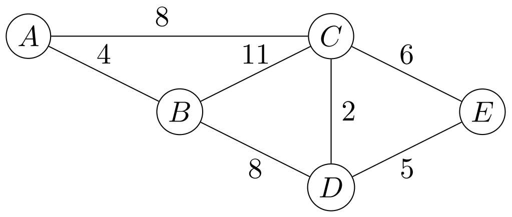
Kruskal's algorithm sorts all edges by weight, then iteratively adds the smallest edge to the MST if it does not form a cycle. The Find-Union structure ensures cycle detection:

1. Sort edges in a non-decreasing order of weight.
2. For each edge (u, v), if Find$(u) \neq$ Find(v), include the edge and perform Union(u, v).
3. Repeat until the MST has $n - 1$ edges, where n is the number of vertices.

Consider a graph with five vertices (A, B, C, D, E) and the following edges with weights:

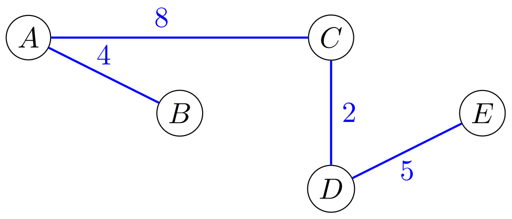
- $A - B : 4$, $A - C : 8$, $B - C : 11$, $B - D : 8$, $C - D : 2$, $C - E : 6$, $D - E : 10$

We apply Kruskal's algorithm, using Find-Union to build the MST for the following graph.



1. **Sort Edges**: $C - D(2)$, $A - B(4)$, $D - E(5)$, $C - E(6)$, $A - C(8)$, $B - D(8)$, $B - C(11)$.
2. **Initialize**: Each vertex in its own set: $\{A\}, \{B\}, \{C\}, \{D\}, \{E\}$.
3. **Edge** $C - D(2)$: Find$(C) = C$, Find$(D) = D$. Union: $\{A\}, \{B\}, \{C, D\}, \{E\}$.
4. **Edge** $A - B(4)$: Find$(A) = A$, Find$(B) = B$. Union: $\{A, B\}, \{C, D\}, \{E\}$.
5. **Edge** $D - E(5)$: Find$(D) = C$, Find$(E) = E$. Union: $\{A, B\}, \{C, D, E\}$.
6. **Edge** $C - E(6)$: Find$(C) = C$, Find$(E) = C$. Same set, skip.
7. **Edge** $A - C(8)$: Find$(A) = A$, Find$(C) = C$. Union: $\{A, B, C, D, E\}$. MST complete.
8. **Reject Others**: $B - D(8)$, $B - C(11)$ form cycles.

The MST includes edges $C - D(2)$, $A - B(4)$, $D - E(5)$, $A - C(8)$, with total weight 19.



The Find-Union data structure enables Kruskal's algorithm to efficiently construct an MST by managing vertex sets and avoiding cycles. With optimizations like path compression and union by rank, it achieves near-linear time complexity

We now give an LLP version for Kruskal's algorithm based on the simple Boolean lattice of all edges. Our distributive lattice is the Boolean lattice formed from all edges. We assume that the edges are given to us in increasing order. Then, an edge e_j between vertices v_k and v_l is in the minimum spanning tree iff there is no path between vertices v_k and v_l using edges $e_1 \ldots e_{j-1}$. Note that when j is 1, e_j corresponds to the lightest edge and is always in the minimum spanning tree. An LLP version is presented as Algorithm LLP-Kruskal.

Algorithm LLP-Kruskal: Finding the minimum spanning tree in a graph.

```
var $G$: array$[1..m]$ of $\{0, 1\}$;
// Edges $G$ is assumed to be in increasing order of weights
init $\forall j : G[j] := 0$;
forbidden(j):
    $(G[j] = 0)$ and there exists no path from $v_k$ to $v_l$ with edges $1..j - 1$ for the edge $e_j = (v_k, v_l)$
advance(j): $G[j] := 1$;
```

7.4 Prim's Algorithm

Prim's algorithm is also a greedy algorithm. It builds the minimum spanning tree by increasing the size of a single fragment by adding the minimum weight outgoing edge of the fragment. It simply exploits the lemma 7.1 to increase the size of the fragment until it becomes the MST. At any stage, Prim's algorithm has a fragment F. It finds the minimum outgoing edge of that fragment e. This edge can be viewed as the edge from the fragment to its nearest neighbor. Therefore, this algorithm is sometimes also known as the *nearest-neighbor-next* algorithm. To find the nearest neighbor, every vertex v maintains a label d that corresponds to the cost of adding v to the fragment. At each iteration, the algorithm chooses the vertex v with the minimum value d and adds it to the fragment. The array $fixed$ keeps track of the vertices in the fragment. Whenever a new vertex v is *fixed* and added to the fragment, the d values for any adjacent vertex v' are updated as follows. We check whether the weight of the edge (v, v') is lower than the previous value of $d[v']$. If this is true, then $d[v']$ is updated to $w[v, v']$. We also use a *parent* pointer with each node which keeps track of the node v that is responsible for the d value of v'.

Consider again the graph shown in Fig. 7.1. For Prim's algorithm, we start from a fixed node. Suppose that we start from the vertex a. Then, the nearest neighbor is c with a cost of 4. The next nearest neighbor to the fragment with vertices $\{a, c\}$ is the vertex b. The cost of adding b is 3. At this point, we have the vertices $\{a, b, c\}$ in the fragment. The cost of adding vertex d is 7 and the cost of adding vertex e is 11.

Algorithm Prim: Finding a Minimum Spanning Tree (MST)

```
Input: Undirected Weighted Graph: $(V, E, w)$.
Output: Minimum Weight Spanning Tree
var
    $d$: array[0..$n-1$] of real initially $\infty$; //$d[v]$ is the cost to add vertex $v$
    $parent$: array[0..$n-1$] of int initially $-1$; // $parent[v]$ is the node that corresponds to $d[v]$ cost
    $fixed$ : array[0..$n-1$] of boolean initially $false$; //vertices whose $d$ value is fixed
    $T$: set of edges initially {};
$d[0] := 0$;
while $|T| < n-1$ do
    $v := arg\ min_i\{d[i] \mid \neg fixed[i]\}$ ;
    if $d[v] = \infty$ then return null // no spanning tree in the graph;
    if $parent[v] \neq -1$ then add $(v, parent[v])$ to $T$ ;
    $fixed[v] := true$;
    forall $(v, v') \in E$ do
        if $w[v, v'] < d[v']$ then
            $d[v'] := w[v, v']$;
            $parent[v'] := v$;
        end
    end
end
```

We add the vertex d to our fragment with the cost 7. Finally, e is added with the cost 2. Hence, the edges are added to the tree in the order $4, 3, 7, 2$. Note that the set of edges chosen is identical to Kruskal's algorithm, even though the order in which they are chosen is different. This is not surprising because there is a unique minimum spanning tree when all edge weights are unique (Problem 1).

The step of finding the vertex v with a minimum value d can be performed by simply traversing the array d or maintaining d in a heap. If we simply traverse the array d to find the minimum, the work complexity of the above algorithm is $O(n^2 + e)$. In every iteration of the while loop, we perform $O(n)$ work to find the minimum, and there are $O(n)$ iterations of the loop. The work for processing edges over all iterations is $O(e)$ because every edge is processed at most once. If we use a heap to store d values of all vertices that are not fixed, we require $O(m)$ insertions in the heap resulting in $O(m \log n)$ work complexity. This approach results in better work complexity when the graph is sparse and m equals $O(n)$.

7.5 An LLP based algorithm for Prim

We now show an LLP algorithm to find a minimum spanning tree rooted at a fixed vertex v_0. As before, we assume that all edge weights are unique. Every node other than v_0 has to choose an edge that we call the G edge. A node keeps all its edges in the sorted order and begins by choosing its least edge as its G edge. For the graph in Fig. 7.1, with root as the vertex a, we get the following choices of edges for other vertices.

```
b: 3, 5, 7
```

```
c: 3, 4, 9, 11
d: 2, 7, 9
e: 2, 11
```

We let $G[i]$ be the edge chosen by node i. Thus, initially, the vector G is $\{3, 3, 2, 2\}$. Observe that the set of all possible combinations of edges forms a distributive lattice. Since every node except v_0 has exactly one outgoing edge, following the chosen edge from every node, we either reach v_0, or end up in a cycle. We define a vertex as *fixed* if traversing the path starting from the edge proposed by that vertex leads to v_0. The vertex v_0 is trivially fixed. It is clear that the edge proposed by a non-fixed vertex can only lead to a non-fixed vertex, and the edge proposed by a fixed vertex can only lead to a fixed vertex. Thus, any G partitions the set of vertices into *fixed* and *non-fixed* vertices. In our example, the initial vector $\{3, 3, 2, 2\}$ makes the vertex a fixed and all other vertices $\{b, c, d, e\}$ non-fixed.

We define the set of edges between these two partitions as follows.

$$E'(G) := \{(i, k) \in E \mid fixed(i, G) \wedge \neg fixed(k, G)\}$$

In our example, we get the following edges as $E'(G) = \{(a, b), (a, c)\}$.

Let $x = (i, j)$ be the edge in E' with minimum $w[i, j]$. For our example, it is (a, c) with an edge weight of 4. We claim that the process j is forbidden in G. Consider any H such that $H[j] < w[i, j]$. Suppose, if possible, that H forms the minimum spanning tree. Any spanning tree must have an edge y between the set $fixed(G)$ and *non-fixed*(G). We claim that H cannot be the minimum spanning tree. This claim holds because the edge set $H - \{y\} + \{x\}$ is also a spanning tree and has a lower weight than H. Therefore, unless the process j advances to the edge (i, j), G cannot be the minimum spanning tree.

When we advance $G[j]$ to that edge, v_j becomes fixed. Observe that additional nodes may become fixed if their proposed edge leads to v_j directly or indirectly. The algorithm continues to advance G until all vertices become fixed or the edge set $E'(G)$ is empty. In the latter case, the graph is not connected and there is no minimum spanning tree.

Algorithm LLP-Prim shows the forbidden condition and the advance statement.

Algorithm LLP-Prim: Finding the minimum spanning tree in a graph .

```
1 var G: array[1..n − 1] of real initially $\forall i : G[i]$ = minimum edge adjacent to $i$;
2 always
3     $fixed(j, G) \equiv$ there exists a directed path from $v_j$ to $v_0$ using edges in $G$
4     $E' := \{ (i, k) \in E \mid fixed(i, G) \wedge \neg fixed(k, G)\}$;
5 forbidden(j)$\equiv \exists i : (i, j) \in E'$ such that it has minimum weight $w[i, j]$ of all edges in $E'$
6 advance(j) $G[j] := min\{w[i, j] \mid (i, j) \in E'\}$
```

7.6 Boruvka's Algorithm

In Prim's algorithm, we started with a trivial fragment that included only the vertex v_0. We kept increasing the size of the fragment until it became a spanning tree. In Boruvka's algorithm, we may have more than one fragment. We increase the size of all fragments by adding the minimum outgoing edge for each fragment.

Algorithm BoruvkaSeq: Finding Minimum Spanning Tree using Boruvka's Algorithm

```
Input: Undirected connected Weighted Graph: $(V, E, w)$.
Output: Minimum Weight Spanning Tree
var
T: { set of edges } initially {};
cid: array[1..n] of 0..n initially all 0;
mwe: array[1..n] of edge initially all null;
dist: array[1..n] of 0..n initially all $\infty$;
while $|T| < n - 1$ do
    visited: array[1..n] of boolean initially all false;
    for $i := 1$ to $n$ do
        if ($\neg visited[i]$) then // do a BFS in the graph $(V, T)$ from vertex $i$ setting cid of every
          visited vertex to $i$
        $BFS(i)$; ;
    end
    for $(i, j) \in E$ such that $(cid[i] \neq cid[j])$ do
        if  $w[i, j] < dist[cid[i]]$ then  $dist[cid[i]] = w[i, j]$
        $mwe[cid[i]] = (i, j)$ ;
        if $w[i, j] < dist[cid[j]]$ then  $dist[cid[j]] = w[i, j]$
        $mwe[cid[j]] = (i, j)$ ;
    end
    forall $i$ do
        $T := T \cup mwe[cid[i]]$
    end
end
return T
```

Algorithm BoruvkaSeq presents the sequential Boruvka's algorithm to find the MST. We use T to denote the set of tree edges. Initially, T is empty. When we determine the components in (V, T), we find that there are n components, since each vertex is a component by itself when T is empty. The algorithm finds the minimum weight outgoing edge for each component as follows. In any iteration, we use BFS to find the least numbered vertex to which any vertex is connected in the graph (V, T). This vertex serves as the identifier for the component of the node i, and we use the variable $cid[i]$ to store it. Once we have determined the component identity of all nodes, we move to the next step of determining the minimum weight outgoing edge for each component. We traverse all edges, and for each edge that connects two different components, we check whether it is cheaper than the previously known outgoing edge for the component on either side. Once we have determined all minimum weight edges for each component, we add these to T and start the next iteration.

For example, consider the graph in Fig. 7.1. Initially, T is empty and there are 5 components. We compute mwe for each component and get the edges $4, 3, 3, 2, 2$ as the minimum weight edges of a, b, c, d, e, respectively. Once these edges are added, we have two components: $\{a, b, c\}$ and $\{d, e\}$. We then find mwe of these two components as the edge 7. On adding this edge, we have chosen $(n-1)$ edges, and the algorithm terminates with the edges $\{2, 3, 4, 7\}$.

After every iteration, the number of connected components in (V, T) is reduced by at least a factor of two. This is because every component gets attached to some other component. The worst case is when every least weight edge chosen by any component is also chosen as the least weight edge by the component on the other side of the edge. Hence, the algorithm takes at most $O(\log n)$ iterations of the while loop. It is easy to see that the least weight edge outgoing from each component is found in $O(m)$ work. Thus, the algorithm takes $O(m \log n)$ work.

7.7 Problems

1. Show that there is a unique minimum spanning tree in any weighted undirected graph when all edge weights are distinct.

2. Give an LLP version of Boruvka's algorithm.

7.8 Bibliographic Remarks

Sequential algorithms for the minimum spanning tree can be found in [CLRS01]. Parallel Boruvka algorithm is available in [SMDD19].

Chapter 8

Sorting Algorithms

8.1 Introduction

In this chapter, we consider some basic ideas in the design of algorithms that are crucially based on the notion of *order* in a set of elements of size n. We will assume that the order is total, that is, for every two elements x and y, either x is less than or equal to y, or y is less than or equal to x in that order.

Before we discuss sorting algorithms, let us consider the problem of searching for an element in a sorted array. Suppose that we are given a sorted array A of size n and an element x. We are interested to find out if there exists i such that $A[i]$ equals x. On a single processor, we can accomplish this using binary search in $O(\log n)$ time. The idea is to compare x with the middle element of the array. If the middle element is equal to x, we are done; otherwise, the range of indices of A that can possibly have x is divided by a factor of 2.

This chapter is organized as follows. Section 8.2 discusses sorting algorithms based on the swapping of consecutive entries that are out of order. This is a general class of algorithms that are very natural. However, these algorithms take $O(n^2)$ sequential time in the worst case. Section 8.3 discusses *MergeSort*. Section 8.4 discusses *QuickSort*. Since MergeSort and QuickSort are based on divide-and-conquer paradigm, they are revisited in Chapter 9. All these sorting algorithms are based on the comparison of two entries in the array. Finally, Section 8.5 describes a sorting algorithm that is not based on the comparison of two entries.

8.2 Algorithms based on Swapping Consecutive Entries

Let A be an array of distinct integers. Our goal is to sort the array. At face value, this problem does not appear to be searching for a satisfying element in a lattice. However, with a little effort, it can be viewed from that angle. Observe that the problem of sorting the array is the same as finding a permutation π such that, applying that permutation to the array, we get a sorted array. For example, if the input array is $[45, 12, 15]$, then we know that the permutation $[3, 1, 2]$ will sort the array once the entry i in the array A goes to $\pi(i)$. There are many different ways to represent a permutation. Let us define the *inversion* number of any entry i as the number of entries less than i that appear to the right of i in the permutation. Thus, the permutation $[3, 1, 2]$ can be equivalently written using the concept of inversions as $[2, 0, 0]$ because there are two numbers less than 3 that appear to the right of 3, zero numbers less than 1 that appear to the right of 2 and zero numbers less than 2 that appear to the right of 2. The reader can verify that there is a 1-1 correspondence between the set of inversions and the set of permutations.

We now consider the set of all inversions. Note that the last entry on an inversion table is always zero because there cannot be any inversion after the last entry in the array. The first entry can have $0 \ldots n-1$ inversions, the second entry can have $0 \ldots n-2$ inversions, etc. Thus, the total number of possible inversion vectors is $n!$. We define an order between two inversion vectors based on the component-wise order. The set of all inversions forms a finite distributive lattice under this order. The bottom element is the zero vector, which corresponds to the identity permutation. When the zero-inversion vector is applied to any array, we get the same array back. Our goal is to find the inversion vector that in its application gives us the sorted array. G is initialized to the zero inversion vector. Let us formalize the definition of a feasible inversion vector. Suppose that we are at an inversion vector G such that it is less than the unique inversion vector that sorts A. When G is applied to A, the array is not sorted. This means that there exists an index i that still has nonzero inversions, i.e. there exists $j > i$ such that $A[j] < A[i]$. It is clear that any inversion vector in which the number of inversions for i is not increased cannot result in the sorted array. To check for forbidden indices, instead of checking for all inversions, in our first algorithm, we will only check for *immediate inversions*. If $A[i] > A[i+1]$ on applying G, then clearly i is forbidden in G. Swapping $A[i]$ and $A[i+1]$ will advance the vector G by incrementing $G[i]$.

Instead of first finding the inversion vector and then applying it, we will continue to apply inversions as we traverse the lattice searching for the optimal inversion vector. In fact, we will not even maintain the inversion vector; we will only keep the effect of applying the inversion vector to the original array.

Algorithm LLP-Sort1: Sorting via Adjacent Swaps

```
1 $P_j$: Code for thread $j$
2 var
3     $A[1..n]$: array of int ;                              // Input array
4     $G[1..n]$: array of $0..n-1$ initially 0 ;              // Inversion vector (abstract)
5 forbidden($j$): $A[j] > A[j+1]$ ;                          // $G[j]$ misses an inversion
6     advance($j$): $swap(A[j], A[j+1])$ ;                   // Increment $G[j]$
```

Algorithm LLP-Sort1 is a nondeterministic algorithm, since multiple j may be forbidden at any point. One can create many instances of deterministic algorithms with different orders of evaluating forbidden indices.

The algorithm Bubble-Sort checks the forbidden indices from 1 to $n-1$ in round-robin order until no forbidden index is found. We leave it as an exercise to show that the repeat loop is executed at most n times, giving us the sequential time complexity of $O(n^2)$.

Another schedule is to increase the size of the sorted array one at a time. Suppose that $A[1..i-1]$ is sorted. This means that the inversion vector is a zero vector for $A[1..i-1]$. We now consider the array $A[1..i]$. Since there is at most one entry added at the end, the inversion number for any entry can increase by at most one. We simply have to check for inversion with respect to the new entry.

The time complexity is clearly $O(n^2)$ due to nested *for* loops. We leave it to the reader to show how the second loop can be cut short.

We now prove that any comparison-based sorting algorithm that restricts itself to comparing consecutive entries (adjacent elements) in an array of size n requires at least $\Omega(n^2)$ comparisons in the worst case. This

Algorithm Bubble-Sort: Bubble Sort

```
var
    $A[1..n]$: array of int;
    $found$: boolean;
repeat
    $found := false$
    for $j = 1$ to $n - 1$ do
        if $A[j] > A[j + 1]$ then
            $found := true$;
            $swap(A[j], A[j + 1])$;
        end
    end
until $\neg found$
```

Algorithm Insertion-Sort: Insertion Sort

```
var
    $A[1..n]$: array of int;
for $i = 1$ to $n$ do
    for $j = i - 1$ downto $1$ do
        if $A[j] > A[j + 1]$ then $swap(A[j], A[j + 1])$ ;
    end
end
```

applies to algorithms such as Bubble Sort or Insertion Sort, where comparisons are limited to pairs of elements at positions i and $i+1$.

Theorem 8.1 *Consider an array $A = [a_1, a_2, \ldots, a_n]$ of n distinct elements. A sorting algorithm that uses only consecutive comparisons requires at least $\Omega(n^2)$ comparisons.*

Proof: An adjacent comparison between a_i and a_{i+1} can identify or resolve an *inversion* - a pair where $a_i > a_{i+1}$ in the current permutation, which must be swapped to achieve sorted order ($a_1 < a_2 < \cdots < a_n$). Each comparison either: (1) confirms $a_i < a_{i+1}$ (no swap needed), or (2) detects $a_i > a_{i+1}$, potentially leading to a swap (e.g., in Bubble Sort).

The maximum number of inversions in a permutation of n elements is achieved by the reverse order $[n, n-1, \ldots, 2, 1]$, with:

$$\text{Number of inversions} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Each adjacent comparison can resolve at most one inversion (e.g., swapping a_i and a_{i+1}), so a permutation with $\frac{n(n-1)}{2}$ inversions requires at least that many comparisons to sort fully.

■

8.3 Merge Sort

Merge Sort is the classic example of divide-and-conquer method in algorithms design. To sort an array, A, assume that the left half and the right half of the arrays are sorted. Then we are simply left with the task of merging these two sorted halves. How do we sort the left and right halves? By using recursion with the base case, when the halves have only single elements and are already sorted. So, it is sufficient to consider the following problem: we have two sorted arrays B and C, each of size n. We would like to merge them into another array D so that D is sorted.

Algorithm MergeSort-Seq: Sequential Merge Sort

```
1 MergeSort($A, low, high$)
2 if $low < high$ then
3     $mid := \lfloor (low + high)/2 \rfloor$;
4     $B :=$ MergeSort($A, low, mid$);
5     $C :=$ MergeSort($A, mid, high$);
6     return MergeTwo($B, C$)
7 end
8 else return $A[low]$ ;
```

It is easy to design a sequential algorithm that merges two arrays B and C into D. We simply keep two indices i and j in the arrays B and C, respectively. In any step of the algorithm, we compare $B[i]$ and $C[j]$. If $B[i]$ is smaller than $C[j]$, then we copy $B[i]$ into the next available slot in D and advance the index i. If $C[j]$ is smaller than $B[i]$, then we copy $C[j]$ into the next available slot in D and advance the index j. This algorithm takes $O(n)$ time.

Sequentially, $MergeTwo$ takes $O(m+n)$ time. For n total elements, the recurrence of the merge sort is $T(n) = 2T(n/2) + O(n)$, resulting in $O(n \log n)$ time.

Algorithm MergeTwo: Merging Two Sorted Arrays

```
var
    $B[1..m], C[1..n]$: arrays of int ;                    // Sorted inputs
    $D[1..m+n]$: array of int ;                            // Output
$i, j, k := 1, 1, 1$
while $i \leq m$ and $j \leq n$ do
    if $B[i] < C[j]$ then
        $D[k] := B[i]$
        $i := i + 1$
    end
    else
        $D[k] := C[j]$
        $j := j + 1$
    end
    $k := k + 1$
end
while $i \leq m$ do
    $D[k] := B[i]$; $i := i + 1$; $k := k + 1$
end
while $j \leq n$ do
    $D[k] := C[j]$; $j := j + 1$; $k := k + 1$
end
```

8.4 Quicksort

Quicksort (due to C.A.R. Hoare) is one of the fastest sorting algorithms on sequential computers. It has $O(n^2)$ worst case time complexity but requires $O(n \log n)$ time on average. The algorithm is based on partitioning the array into two parts using a *pivot*. Once we have the property that the lower side of the array is less than or equal to pivot, and the upper side of the array has elements greater than pivot, then we can recurse on each side. Since sorting each of the sides is independent, they can be sorted in parallel.

There are multiple methods to choose a pivot to partition the array A. Choosing an element at random to be the pivot is an easy method that will result in approximately equal sized partitions on average. This gives us the average case sequential time complexity of $O(n \log n)$. In the worst case, the recursion may reduce the range by only 1, resulting in the sequential time complexity of $O(n^2)$.

Algorithm QuickSort-Seq: Sequential QuickSort

```
QuickSort($A, low, high$)
if $low < high$ then
    $pivot := choosePivot()$;
    $(p, q) :=$ Partition($A, pivot, low, high$);
    QuickSort($A, low, p$);
    QuickSort($A, q, high$);
end
```

We describe a slight variant of the Quicksort algorithm in which we partition the array into three parts. Algorithm Three-Way-Partition takes array A, *low*, and *high* as input parameters. The part of the array this method sorts is given by

$$\{A[i] \mid low \leq i < high\}$$

Note that when *low* equals *high*, the range is empty.

The method *partition* returns two indices p and q. All elements in the range $[lo \ldots p)$ are strictly less than the pivot, in the range $[p \ldots q)$ are equal to the pivot, and in the range $[q \ldots high)$ are greater than the pivot. We only need to recurse on the first and the third parts. Depending upon the pivot, any (or both!) of the first and third parts may be empty.

Once we have a pivot, how do we partition the array into three parts: the first partition with all elements less than the pivot, the second one with all elements equal to the pivot, and the third partition with elements strictly greater than the pivot. Algorithm Three-Way-Partition is due to Dijkstra, who called this problem the Dutch National Flag problem. The *while* loop maintains the invariant that the entries $[low \ldots p)$ are less than the pivot, $[p \ldots q)$ are equal to the pivot and $[k \ldots high)$ are greater than the pivot. The algorithm initializes p and q to *low*, therefore the first two ranges are empty initially and trivially satisfy the invariants. It also initializes k to *high*, making the range $[k \ldots high)$ empty and thereby ensuring that the invariant holds initially. The range $[q..k)$ corresponds to the initial input and initially contains entries that may be less than, equal to, or greater than the pivot. In each iteration of the *while* loop, this range is reduced by one by increasing q or decreasing k. Depending on the comparison between $A[q]$ and the pivot, the entry $A[q]$ is placed in the appropriate range keeping the invariants.

It is easy to verify that the algorithm takes $O(n)$ time when the range has n elements.

We show that any comparison-based sorting algorithm for an array of n distinct elements requires at least $\Omega(n \log n)$ comparisons in the worst case. This result applies to algorithms using unrestricted pairwise

Algorithm Three-Way-Partition: Three-Way Partition (Dutch National Flag)

```
Partition($A, pivot, low, high$) returns (int, int)
$p, q := low$
$k := high$
while $q < k$ do
    if $A[q] < pivot$ then
        $swap(A[p], A[q])$
        $p := p + 1$
        $q := q + 1$
    end
    else if $A[q] > pivot$ then
        $k := k - 1$
        $swap(A[q], A[k])$
    end
    else $q := q + 1$ ;
end
return $(p, q)$
```

comparisons (e.g., Merge Sort), contrasting with the $\Omega(n^2)$ bound for consecutive-only comparisons. We use a decision tree model.

Theorem 8.2 *Given an array $A = [a_1, a_2, \dots, a_n]$ of n distinct elements, a comparison-based sorting algorithm uses binary comparisons ($a_i < a_j$) to sort A into $a_{\pi(1)} < a_{\pi(2)} < \cdots < a_{\pi(n)}$, where π is a permutation. The minimum number of comparisons needed to sort any input permutation in the worst case is $\Omega(n \log n)$.*

Proof: We model the sorting algorithm as a decision tree, where internal nodes represent comparisons, and leaves correspond to permutations. The worst-case number of comparisons is the tree's height.

For n distinct elements, there are $n!$ possible permutations. The algorithm must distinguish all $n!$ permutations, so the decision tree requires at least $n!$ leaves.

In a comparison-based algorithm, each node compares two elements (a_i vs. a_j), branching on the less-than operator ($<$) or the greater-than operator $>$. With distinct elements, each node has two children (binary tree). After k comparisons, the maximum number of leaves is 2^k.

To cover all permutations:

$$2^k \geq n!.$$

Taking the base-2 logarithm:

$$k \geq \log_2(n!).$$

The value $\log_2(n!)$ represents the minimum comparisons needed. For large n, it is well-known that:

$$\log_2(n!) = \Omega(n \log n),$$

since $n!$ grows factorially, and its logarithm reflects the information required to order n elements. This can be seen intuitively: sorting halves the problem repeatedly (e.g., in Merge Sort), requiring $\log n$ levels, each with $O(n)$ comparisons.

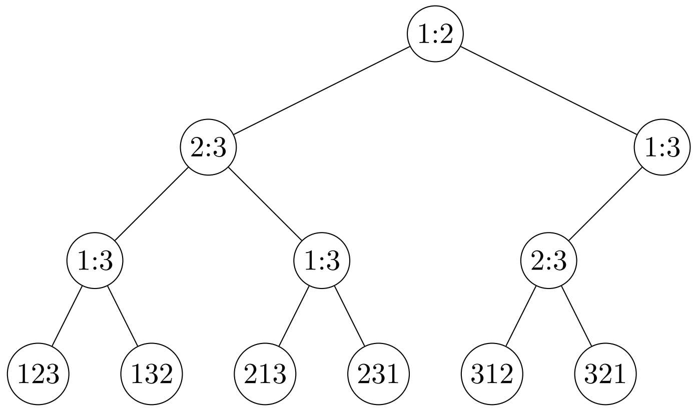


Figure 8.1: A comparison tree for any sorting algorithm.

For $n = 3$, there are $3! = 6$ permutations. A decision tree might start with $a_1 < a_2$. Depending upon the result of the comparison, it may compare other elements. A decision tree for such an algorithm is shown in Fig. 8.1.

Since $k \geq \log_2(n!)$, and $\log_2(n!) = \Omega(n \log n)$:

$$k = \Omega(n \log n).$$

This is the minimum number of comparisons in the worst case, as the tree must be tall enough to reach $n!$ leaves.

■

8.5 Radix Sort

All our sorting algorithms, so far, have been based on comparison. Any comparison-based sorting algorithm must do at least $O(n \log n)$ work. We now show an algorithm that works when the sorting keys have a fixed number of digits (in any radix r, generally a power of 2). In our following examples, we simply use digits at the base 10, as they are easy for us humans. Suppose that we need to sort the following list. $[85, 72, 94, 45, 13, 12, 61, 81]$. Here, our keys have two digits. The idea of radix sorting is to sort numbers one digit at a time. When sorting on a single digit, we use a simple linear time sorting, exploiting the fact that there are only r possibilities for each digit. Upon getting any number, we can simply add it to the pile associated with that value. The number of passes that we will make on the array is equal to the number of digits.

It may appear that it is easier to sort starting from the most significant digit (msd) first. If we used this strategy, we would get $[13, 12, 45, 61, 72, 85, 81, 94]$ after the first pass. The second pass would consist of sorting all subarrays with the same msd, and we would get the array $[12, 13, 45, 61, 72, 81, 85, 94]$. The problem with this approach is that we are forced to maintain different subarrays, one for each digit after the first pass. With every subsequent pass, it becomes even more cumbersome. Hence, somewhat counterintuitively, we will employ the least significant digit first strategy. After the first pass, we get $[61, 81, 72, 12, 13, 94, 85, 45]$. We do not need to remember any sublists that are created during the first pass. Now, we sort by the second least significant digit. We need to ensure that if two numbers have the same digit, then their order is preserved. In other words, we require our sorting algorithms at each pass to be *stable*. After the second pass, we get the sorted array $[12, 13, 45, 61, 72, 81, 85, 94]$. Since 12 appeared before 13 after the first pass, the order is preserved after the second pass.

Thus, the sequential algorithm is simply stated as Algorithm RadixSort.

Algorithm RadixSort: Radix Sort for integers with k digits

```
1 RadixSort($A[1..n]$)
2 for $i = 1$ to $k$ do
3     StableSort($A$, digit $i$)
4 end
```

The time complexity of this algorithm is $O(kn)$ assuming that the stable sort is achieved in $O(n)$ time. For $r = 10$, sorting $[85, 72, 94, 45, 13, 12, 61, 81]$ yields $O(kn)$ time (here, $k = 2$, $O(n)$ per pass).

8.6 Problems

1. Implement the binary search algorithm discussed in Section 8.1.
2. Show that any algorithm that is based on comparison of consecutive entries must take $O(n^2)$ comparisons in the worst case.
3. Give an algorithm to merge k sorted arrays of size n into a single sorted array.
4. We have partitioned the array into three parts in the QuickSort algorithm. Give a version of the QuickSort algorithm in which the array is partitioned only in two parts: entries that are less than *pivot* and the entries that are greater than or equal to the pivot.
5. Give a randomized version of Quicksort in which the pivot is chosen at random. Give the expected running time of your algorithm.

8.7 Bibliographic Remarks

Quicksort is a divide-and-conquer algorithm that was invented by Tony Hoare in 1962 [Hoa61] It is one of the most widely used sorting algorithms and has a time complexity of $O(n \log n)$ in the average case. Merge sort is another divide-and-conquer algorithm that was first described by John von Neumann in 1945. Radix sort is a non-comparison-based sorting algorithm that was first described by Herman Hollerith in 1887.

Chapter 9

Divide and Conquer

9.1 Introduction

A useful strategy to solve a problem is to divide the problem into subproblems, solve the subproblems, and then merge the solutions to the subproblems to obtain the solution of the original problem. When the subproblem is of trivial size, then it can simply be solved by case analysis. This strategy is called the divide-and-conquer approach. We give many examples of this approach.

Section 9.2 revisits the Merge Sort using this approach. When we apply the divide-and-conquer paradigm, the time complexity of solving the problem is usually analyzed using a recurrence relation on the size of the input. Section 9.3 gives a technique for analyzing the time complexity of such methods. Section 9.4 discusses the problem of determining nearest neighbors in the Euclidean plane. Section 9.5 discusses the problem of counting inversions in an array. Section 9.6 describes a solution to the problem of computing the convex hull of a polyhedron. Section 9.7 describes a method for multiplying two natural numbers using the divide-and-conquer approach. Finally, Section 9.8 gives a method of multiplying two matrices based on this idea.

9.2 Merge sort: Revisited

To sort an array of size n, we divide it into two subarrays of size $n/2$ and sort each of the parts. The main trick is to determine how to merge these two sorted arrays.

Now, we merge the two sorted subarrays to get the sorted version of the original array. Let us first look at the sequential time complexity. We have the following recurrence

$$T(n) = 2T(n/2) + O(n); T(1) = O(1)$$

Solving this recurrence, we get the sequential time complexity as $T(n) = O(n \log n)$.

Algorithm MergeSort: MergeSort

```
Input: Array $A$, indices $low$, $high$
Output: Sorted array $A[low..high]$
MergeSort($A$, $low$, $high$)
if $low < high$ then
    $mid := \lfloor (low + high)/2 \rfloor$;
    $B$ := MergeSort($A$, $low$, $mid$);
    $C$ := MergeSort($A$, $mid + 1$, $high$);
    return $MergeTwo(B, C)$;
end
else
    return $A$;
end
```

9.3 The Master Theorem

The Master Theorem provides a method for solving recurrence relations of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^c)$$

where:

- $a \geq 1$ is the number of subproblems,
- $b > 1$ is the factor by which the problem size is reduced,
- $O(n^c)$ (with $c \geq 0$) is the cost of work done outside the recursive calls.

The time complexity is given by:

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } c < \log_b a \\ O(n^c \log n) & \text{if } c = \log_b a \\ O(n^c) & \text{if } c > \log_b a \end{cases}$$

Proof of Correctness

We prove the Master Theorem using recursion tree expansion.

Step 1: Expand the Recursion

$$\begin{aligned} T(n) &= aT(n/b) + O(n^c) \\ &= a\left(aT(n/b^2) + O((n/b)^c)\right) + O(n^c) \\ &= a^2T(n/b^2) + O(a(n/b)^c) + O(n^c) \end{aligned}$$

For k levels:

$$T(n) = a^k T(n/b^k) + \sum_{i=0}^{k-1} O(a^i (n/b^i)^c)$$

The sum is as follows:

$$\sum_{i=0}^{k-1} a^i (n/b^i)^c = n^c \sum_{i=0}^{k-1} \left(\frac{a}{b^c}\right)^i$$

Step 2: Stopping Condition

Recursion stops when $n/b^k = 1$, so $k = \log_b n$:

$$T(n) = a^{\log_b n} T(1) + O\left(n^c \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^c}\right)^i\right)$$

Since $a^{\log_b n} = n^{\log_b a}$ and $T(1) = O(1)$, the first term is $O(n^{\log_b a})$.

Step 3: Analyze the Sum

For $r = a/b^c$:

- If $c < \log_b a$, then $r > 1$, the sum is $O(n^{\log_b a})$, and $T(n) = O(n^{\log_b a})$.
- If $c = \log_b a$, then $r = 1$, the sum is $O(\log n)$, and $T(n) = O(n^c \log n)$.
- If $c > \log_b a$, then $r < 1$, the sum is $O(n^c)$, and $T(n) = O(n^c)$.

Example 1: Merge Sort (Case 2)

$$T(n) = 2T(n/2) + O(n)$$

$a = 2$, $b = 2$, $c = 1$. Since $\log_2 2 = 1 = c$, case 2 applies:

$$T(n) = O(n \log n)$$

Example 2: Binary Search (Case 1)

$$T(n) = T(n/2) + O(1)$$

$a = 1$, $b = 2$, $c = 0$. Since $\log_2 1 = 0 = c$, case 2 applies (noting $O(n^0) = O(1)$):

$$T(n) = O(\log n)$$

Example 3: Strassen's Matrix Multiplication (Case 1)

$$T(n) = 7T(n/2) + O(n^2)$$

$a = 7$, $b = 2$, $c = 2$. Since $\log_2 7 \approx 2.81 > 2$, case 1 applies:

$$T(n) = O(n^{2.81})$$

Example 4: Karatsuba's Algorithm (Case 1)

$$T(n) = 3T(n/2) + O(n)$$

$a = 3$, $b = 2$, $c = 1$. Since $\log_2 3 \approx 1.58 > 1$, case 1 applies:

$$T(n) = O(n^{1.58})$$

Example 5: (Case 3)

$$T(n) = 2T(n/2) + O(n^2)$$

$a = 2$, $b = 2$, $c = 2$. Since $\log_2 2 = 1 < 2$, case 3 applies. Thus:

$$T(n) = O(n^2)$$

Additional Examples

(a) $T(n) = 2T(n/2) + O(n^2)$

$a = 2$, $b = 2$, $c = 2$. Since $\log_2 2 = 1 < 2$, case 3 applies.

$$T(n) = O(n^2)$$

(b) $T(n) = 9T(n/3) + O(n)$

$a = 9$, $b = 3$, $c = 1$. Since $\log_3 9 = 2 > 1$, case 1 applies:

$$T(n) = O(n^2)$$

(c) $T(n) = T(2n/3) + O(1)$

$a = 1$, $b = 3/2$, $c = 0$. Since $\log_{3/2} 1 = 0 = c$, case 2 applies:

$$T(n) = O(\log n)$$

9.4 Nearest Neighbors in the Euclidean Space

We are given n points in the two-dimensional Euclidean space. Our goal is to find a pair of points with the smallest distance between them. That is, determine two points (p_i, p_j) such that the Euclidean distance between them is minimized:

$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

The easiest algorithm would be to compute the distance between every pair of points and choose a pair with the least distance. This approach requires $O(n^2)$ sequential time. Our goal is to reduce the sequential time complexity for this problem.

The divide-and-conquer approach for this problem is as follows. We divide the set of points into two sets S_1 and S_2. Suppose that we have the nearest neighbors in S_1 and S_2. Now, if we had nearest neighbors such that one point is in S_1 and the second point is in S_2, then we simply have to choose the smallest of these three distances: nearest neighbors in S_1, nearest neighbors in S_2, and nearest neighbors across the partition. Suppose that the distance of the nearest neighbors in S_1, or in S_2 is δ. Then, we only need to consider the nearest neighbors across the partition if they are closer than δ. Let S_y be the set of points within δ of the dividing line sorted in the y-coordinate. We claim that if $s, s' \in S$ are such that $d(s, s') < \delta$, then s and s' are within 15 positions of each other in the sorted list S. Let L be the vertical line that divides the set of points according to the coordinate x. We consider boxes of size $\delta/2$ around L. We first show that there cannot be two points within a box. The largest distance within a box is $\sqrt{2}(\delta/2)$. This is equal to $\delta/\sqrt{2}$, which is less than δ.

We next claim that there cannot be two points that are sixteen positions apart in S_y and still have distance less than δ. Since the points are sixteen positions apart and at most one point in each box, we get that there are at least three rows of boxes between these two points. However, this means that the distance between them is at least $3 * \delta/2$, which is greater than δ.

Hence, it is sufficient to consider points that are at most 15 positions apart in S_y.

Algorithm Closest-Pair: Closest Pair of Points

```
Input: Set of $n$ points $P$ in 2D space
Output: The closest pair of points
1  Function ClosestPair($P$):
2      if $n \leq 3$ then
3          return BruteForce($P$) // Base case:  use brute-force for small input
4      end
5      Sort $P$ by $x$-coordinate;
6      Split $P$ into left ($P_L$) and right ($P_R$) halves;
7      $(p_1, p_2)$ := ClosestPair($P_L$);
8      $(p_3, p_4)$ := ClosestPair($P_R$);
9      $\delta := \min(d(p_1, p_2), d(p_3, p_4))$;
10     $S$ := points within $\delta$ of the dividing line;
11     Sort $S$ by $y$-coordinate;
12     for each point $p$ in $S$ do
13         for each of the next 15 points $q$ do
14             if $d(p, q) < \delta$ then
15                 Update $\delta$ and closest pair;
16             end
17         end
18     end
19     return closest pair;
```

Consider the following set of points:

$$P = \{(1,2), (3,7), (5,4), (8,9), (9,6), (10,3)\}$$

Now the algorithm can be summarized as:

1. **Sorting:** Arrange by x-coordinate: $[(1,2), (3,7), (5,4), (8,9), (9,6), (10,3)]$
2. **Divide:** Split into two halves:

$$P_L = [(1,2), (3,7), (5,4)]$$
$$P_R = [(8,9), (9,6), (10,3)]$$

3. **Recursive Closest Pair:** - Left half: Closest pair is $(1, 2)$ and $(5, 4)$ with $d = 4.47$. - Right half: Closest pair is $(9, 6)$ and $(10, 3)$ with $d = 3.16$.
4. **Combine Step:** - Points near the vertical dividing line: are $(5, 4), (8, 9), (9, 6)$. - Checking only points within $\delta = 3.16$:
- $(5, 4)$ and $(9, 6)$ have $d = 4.47$ (not better).
- $(8, 9)$ and $(9, 6)$ have $d = 3.16$ (same as right half).
- $(5, 4)$ and $(9, 6)$ have $d = 2.24$ (new minimum).
5. **Final Output:** $(5, 4)$ and $(9, 6)$ with distance 2.24.

The time complexity is as follows. For sorting the points, we spend $O(n \log n)$ time. To recursively solving subproblems, we use $O(n \log n)$ time. For checking points in the strip we spend $O(n)$ time. Thus, the overall time complexity is $O(n \log n)$.

9.5 Counting Inversions in an Array using Divide and Conquer

Given an array $A[1 \dots n]$, an **inversion** is a pair of indices (i, j) such that $i < j$ and $A[i] > A[j]$. The goal is to count the total number of inversions in the array efficiently. A brute-force approach iterates through all pairs and checks if $A[i] > A[j]$, requiring $O(n^2)$ time. Using Merge Sort with an additional counting step, we can solve this problem in $O(n \log n)$ time.

We modify the *Merge Sort* algorithm to count inversions efficiently: (1) Recursively count inversions in the left and right halves. (2) Count the number of inversions across the two halves while merging. (3) Sum up all these counts.

Algorithm CountingInversions: Finding the number of inversions in an array.

```
Input: Array $A$ of size $n$
Output: Total number of inversions
Function CountInversions($A, left, right$):
    if $left \geq right$ then
        return 0 // Base case: single element has no inversions
    end
    $mid := (left + right)/2$;
    $invLeft :=$ CountInversions($A, left, mid$);
    $invRight :=$ CountInversions($A, mid + 1, right$);
    $invMerge :=$ MergeAndCount($A, left, mid, right$);
    return $invLeft + invRight + invMerge$;
Function MergeAndCount($A, left, mid, right$):
    // left as an exercise
```

Consider the array:

$$A = [8, 4, 2, 1]$$

Step-by-step inversion count:

- Left half $[8, 4]$: 1 inversion $(8, 4)$.
- Right half $[2, 1]$: 1 inversion $(2, 1)$.
- Merging left and right halves:

- $(8, 2), (8, 1), (4, 2), (4, 1)$ contribute 4 more inversions.

Total inversions: 6.

It is easy to see that the merge sort recursion takes $O(n \log n)$ time. The Merge step inversion counting takes $O(n)$ time. Thus, the overall time complexity is $O(n \log n)$.

9.6 Planar Convex Hull

Suppose that we are given S, a set of points in a plane. The convex hull of S is the smallest convex polygon that contains all points of S. The convex polygon can be described by the ordered list of points (in the clockwise direction) that defines the boundary of the polygon. Our task is to compute the convex hull of S.

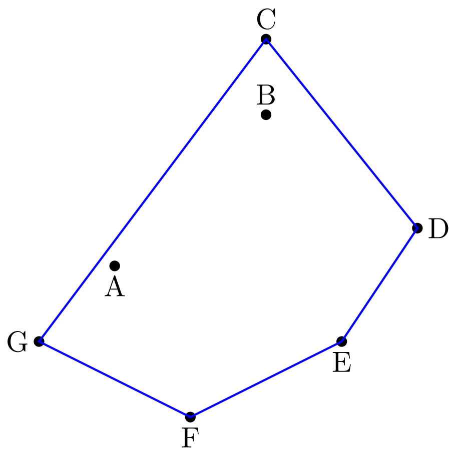


Figure 9.1: An example of Planar Convex Hull

Let points $\{p_1, ..., p_n\}$ be sorted in their x-coordinate. For simplicity, we assume that x coordinates are unique. It is easy to verify that the point with the smallest x-coordinate, p_1 and the largest x-coordinate p_n are always in the convex hull. These two points also divide up the convex hull into two sets: the upper hull and the lower hull. The upper hull consists of all points in the convex hull starting from p_1 and ending at p_n (but not including p_n) in the clockwise direction. The lower hull consists of all points starting from p_n and ending at p_1 (but not including p_1).

To compute the convex hull of S, we divide the set of points into two sets S_1 and S_2 such that S_1 has first $n/2$ points with smaller x coordinate and S_2 has $n/2$ points with the larger x coordinate. Similar to mergesort, we can compute convex hulls of S_1 and S_2. The only task left is to merge the convex hull of S_1 and S_2 to get the convex hull of S. We first get the upper hulls of S_1 and S_2 and merge them to get the upper hull of S. This procedure can be repeated for lower hulls. To merge upper hulls, we need to determine the upper common tangent of $UH(S_1)$ and $UH(S_2)$. It is known that the upper common tangent can be determined sequentially in $O(n)$ time. Hence, we get the following recurrence:

$$T(n) \leq 2T(n/2) + O(n)$$

Solving this recurrence, we get $T(n) = O(n \log n)$.

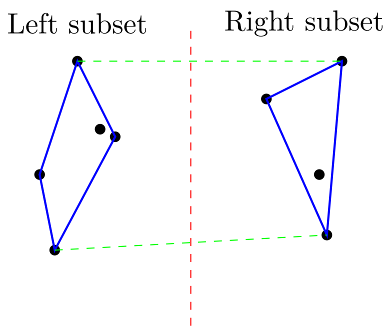


Figure 9.2: Using Divide and Conquer for Planar Convex Hull

9.7 Karatsuba's Multiplication Algorithm

A notable application of divide and conquer approach is Karatsuba's multiplication algorithm, which computes the product of two large integers faster than the classical grade-school method. Introduced by Anatoly Karatsuba in 1960, it reduces the complexity from $O(n^2)$ to $O(n^{\log_2 3}) \approx O(n^{1.585})$, where n is the number of digits.

Consider multiplying two polynomials $A(x) = a_0 + a_1 x$ and $B(x) = b_0 + b_1 x$, where coefficients are single-digit numbers (e.g., base-10 digits). Their product is:

$$C(x) = A(x) \cdot B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + a_1 b_1 x^2.$$

Naive multiplication computes each term $(a_0 b_0, a_0 b_1, a_1 b_0, a_1 b_1)$, requiring 4 multiplications and 1 addition, yielding $O(n^2)$ for degree-n polynomials.

Now, represent two n-digit numbers $X = x_1 \cdot 10^{n/2} + x_0$ and $Y = y_1 \cdot 10^{n/2} + y_0$ (assuming n is even), where x_1, x_0, y_1, y_0 are $n/2$-digit numbers. Their product is:

$$X \cdot Y = (x_1 \cdot 10^{n/2} + x_0)(y_1 \cdot 10^{n/2} + y_0) = x_1 y_1 \cdot 10^n + (x_1 y_0 + x_0 y_1) \cdot 10^{n/2} + x_0 y_0,$$

akin to polynomial multiplication with $x = 10^{n/2}$. Naive computation requires 4 multiplications of $n/2$-digit numbers, suggesting $O(n^2)$ overall. Karatsuba's algorithm optimizes this to 3 multiplications.

Karatsuba's insight reduces the number of multiplications by computing: 1. $p_1 = x_0 y_0$, 2. $p_2 = x_1 y_1$, 3. $p_3 = (x_0 + x_1)(y_0 + y_1)$.

Then:

$$X \cdot Y = p_2 \cdot 10^n + [p_3 - (p_1 + p_2)] \cdot 10^{n/2} + p_1.$$

$p_3 = x_0 y_0 + x_0 y_1 + x_1 y_0 + x_1 y_1 = p_1 + p_2 + (x_1 y_0 + x_0 y_1)$, Thus, $p_3 - (p_1 + p_2) = x_1 y_0 + x_0 y_1$.

This requires only 3 multiplications (p_1, p_2, p_3) instead of 4, plus additional additions/subtractions, which are $O(n)$.

For n-digit numbers:

- Split: $X = x_1 \cdot 10^{n/2} + x_0$, $Y = y_1 \cdot 10^{n/2} + y_0$.
- Recursively compute p_1, p_2, p_3 on $n/2$-digit numbers.
- Combine using the formula above.

The recurrence is:

$$T(n) = 3T(n/2) + O(n),$$

solving to $T(n) = O(n^{\log_2 3})$ via the Master Theorem (Case 1: $a = 3, b = 2, \log_b a \approx 1.585$).

Algorithm Karatsuba: Karatsuba's Multiplication Algorithm

```
Input: Numbers $X$, $Y$, digit length $n$ (power of 2)
Output: Product $X \cdot Y$
if $n = 1$ then
    return $X \cdot Y$ // base case: single-digit multiplication
end
$x_1 := \lfloor X/10^{n/2} \rfloor$, $x_0 := X \mod 10^{n/2}$ // high and low halves of $X$
$y_1 := \lfloor Y/10^{n/2} \rfloor$, $y_0 := Y \mod 10^{n/2}$ // high and low halves of $Y$
$p_1 :=$ Karatsuba$(x_0, y_0, n/2)$ // low product
$p_2 :=$ Karatsuba$(x_1, y_1, n/2)$ // high product
$p_3 :=$ Karatsuba$(x_0 + x_1, y_0 + y_1, n/2)$ // middle term helper
return $p_2 \cdot 10^n + [p_3 - (p_1 + p_2)] \cdot 10^{n/2} + p_1$
```

Consider $X = 1234$, $Y = 5678$, $n = 4$ (pad to power of 2 if needed).

Step 1: Divide

- $10^{n/2} = 10^2 = 100$,
- $x_1 = \lfloor 1234/100 \rfloor = 12$, $x_0 = 1234 \mod 100 = 34$,
- $y_1 = \lfloor 5678/100 \rfloor = 56$, $y_0 = 5678 \mod 100 = 78$.

Step 2: Conquer (Recurse)

- $p_1 = 34 \cdot 78 = 2652$ (base case, $n/2 = 2$),
- $p_2 = 12 \cdot 56 = 672$ (base case),
- $p_3 = (34 + 12) \cdot (78 + 56) = 46 \cdot 134 = 6164$ (base case).

Step 3: Combine

- $p_3 - (p_1 + p_2) = 6164 - (2652 + 672) = 6164 - 3324 = 2840$,
- $X \cdot Y = 672 \cdot 10^4 + 2840 \cdot 10^2 + 2652 = 6720000 + 284000 + 2652 = 7006652$.

Verification: $1234 \cdot 5678 = 7006652$, correct.

Karatsuba's algorithm reduces multiplications from 4 to 3 per level, trading them for additions/subtractions. For $n = 2^k$:
- Levels: $k = \log_2 n$,
- Multiplications: $3^k = 3^{\log_2 n} = n^{\log_2 3}$,
- Additions: $O(n \log n)$.

Compared to FFT ($O(n \log n)$), Karatsuba is slower but avoids complex numbers, making it simpler for integer arithmetic. The example shows its practicality for small numbers, scaling efficiently for larger n.

9.8 Strassen's Matrix Multiplication

Algorithm Strassen: Strassen's Matrix Multiplication

```
   Input: $A, B$: $n \times n$ matrices, $n$ a power of 2
   Output: $C$: $n \times n$ matrix, $C = A \cdot B$
1  if $n = 1$ then
2      $C[1,1] := A[1,1] \cdot B[1,1]$ ;
3      Return $C$ ;
4  end
5  Divide $A$ into $A_{11}, A_{12}, A_{21}, A_{22}$ ;          // Each $n/2 \times n/2$
6  Divide $B$ into $B_{11}, B_{12}, B_{21}, B_{22}$ ;          // Each $n/2 \times n/2$
7  $M_1 :=$ Strassen$(A_{11} + A_{22}, B_{11} + B_{22})$ ;
8  $M_2 :=$ Strassen$(A_{21} + A_{22}, B_{11})$ ;
9  $M_3 :=$ Strassen$(A_{11}, B_{12} - B_{22})$ ;
10 $M_4 :=$ Strassen$(A_{22}, B_{21} - B_{11})$ ;
11 $M_5 :=$ Strassen$(A_{11} + A_{12}, B_{22})$ ;
12 $M_6 :=$ Strassen$(A_{21} - A_{11}, B_{11} + B_{12})$ ;
13 $M_7 :=$ Strassen$(A_{12} - A_{22}, B_{21} + B_{22})$ ;
14 $C_{11} := M_1 + M_4 - M_5 + M_7$ ;
15 $C_{12} := M_3 + M_5$ ;
16 $C_{21} := M_2 + M_4$ ;
17 $C_{22} := M_1 - M_2 + M_3 + M_6$ ;
18 Return $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$ ;
```

Strassen's algorithm is a divide-and-conquer approach to matrix multiplication that reduces the time complexity from the naive $O(n^3)$ to $O(n^{\log_2 7}) \approx O(n^{2.807})$ for multiplying two $n \times n$ matrices. Unlike the standard method, which performs 8 recursive multiplications for 2×2 submatrices, Strassen's method uses 7 multiplications by cleverly combining submatrix operations, trading some multiplications for additional additions.

Given two $n \times n$ matrices A and B, where n is a power of 2 (padded with zeros if necessary), Strassen's algorithm divides each matrix into four $n/2 \times n/2$ submatrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

The product $C = A \cdot B$ is computed as:

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where traditionally:

- $C_{11} = A_{11}B_{11} + A_{12}B_{21}$,
- $C_{12} = A_{11}B_{12} + A_{12}B_{22}$,
- $C_{21} = A_{21}B_{11} + A_{22}B_{21}$,
- $C_{22} = A_{21}B_{12} + A_{22}B_{22}$.

This requires 8 multiplications and 4 additions of $n/2 \times n/2$ matrices. Strassen's insight is to define 7 intermediate products (M_1 to M_7) that allow the computation of C with fewer multiplications:

The recurrence relation for Strassen's algorithm is:

$$T(n) = 7T(n/2) + O(n^2),$$

where 7 is the number of recursive multiplications, and $O(n^2)$ accounts for the additions/subtractions of $n/2 \times n/2$ matrices. Using the Master Theorem ($a = 7$, $b = 2$, $f(n) = O(n^2)$, $\log_b a = \log_2 7 \approx 2.807 > 2$):

$$T(n) = O(n^{\log_2 7}) \approx O(n^{2.807}).$$

This is an improvement over the standard $O(n^3)$.

9.9 LLP Algorithms based on Divide and Conquer

Viewing the problem as finding an appropriate element in a distributive lattice, the divide-and-conquer approach corresponds to viewing the problem as finding two appropriate elements in two distributive lattices. This search is carried out independently (and in parallel). Once these two elements are determined, we use them to find the required element in the original lattice.

Let L be a finite distributive lattice and B be any lattice-linear predicate. We call B splittable if there exists two lattices L_1 and L_2 and two lattice-linear predicates B_1 and B_2 such that given the least solution for B_1 and B_2 in L_1 and L_2, one can determine the least solution for B in L. The effort to determine the least solution for B in L is reduced if we have the solutions for B_1 and B_2. If B_1 and B_2 are also splittable, then we can apply this strategy recursively. Once the lattice is trivial, i.e., it is totally ordered, then finding the least solution is simple. Formally,

Definition 9.1 (Splittable Lattice-Linear Predicate) *A nonempty Lattice-Linear Predicate B is splittable for a finite distributive lattice L, if there exists two nonempty lattices L_1 and L_2 and two lattice-linear predicates B_1 and B_2 such that there exists a map $f : L_1 \times L_2 \to L$, where $f(G_1, G_2)$ is the least element satisfying B in L such that G_1 and G_2 are the least elements in L_1 and L_2 satisfying B_1 and B_2.*

As an example of a predicate that is not splittable, consider the stable marriage problem. Given the distributive lattice of all assignments, the predicate B that the assignment is a stable marriage is not splittable.

The predicate B that is true on all elements is trivially splittable into B_1 and B_2 also as true elements. The lattice L can be split into arbitrary $L_1 \times L_2$. We now give many examples of nontrivial splittable predicates.

Quicksort: Revisited

Let A be any array. Without loss of generality assume that all the entries are unique. Sorting the array is equivalent to finding the permutation such that when the permutation is applied to the array we get a sorted array. A permutation can be represented in many ways. We represent it using the inversion vector. The inversion number of any entry i is the number of entries greater than i that appear to the left of i in the permutation.

Sorting is equivalent to finding the inversion vector which sorts the array. For example, if the array is $[7, 3, 9]$, then the inversion vector is $[0, 1, 0]$ We can order the inversion vectors based on component-wise comparison. Under this order, the set of inversion vectors forms a distributive lattice L. For an array of size 3, there are six elements in the lattice L: $[0, 0, 0]$, $[0, 0, 1]$, $[0, 1, 0]$, $[0, 1, 1]$, $[0, 0, 2]$ and $[0, 1, 2]$. An inversion vector G satisfies B if when G is applied to A, it results in a sorted vector. Formally,

$$B(G) \equiv A \text{ has inversion vector } G$$

We claim that B is splittable. Let *pivot* be an arbitrary element in the array which is not an extremal element. We define L_1 to be the lattice of inversion vectors for entries in an array that are at most *pivot*. Similarly, let L_2 be the lattice of inversion vectors for entries that are strictly bigger than *pivot*. Let G_1 and G_2 be the inversion vectors for entries in L_1 and L_2, respectively. Then, G is given by the inversion vector (G_1, G_2) applied to respective entries in L_1 and L_2. As a concrete example, suppose that $A = [5, 9, 16, 4, 11]$. Let the pivot be 5. There are two entries in A that are less than or equal to *pivot*, and three entries that are bigger than *pivot*. Then, L_1 corresponds to all inversion vectors of length 2 and L_2 corresponds to all inversion vectors of length 3. G_1 equals $[0, 1]$ because the inversion vector $[0, 1]$ sorts the subarray $[5, 4]$ and G_2 equals $[0, 0, 1]$ which sorts the subarray $[9, 16, 11]$. Then, the inversion vector G equals $[0, 1, 0, 0, 1]$ applied to respective entries sorts the array. In our example, we get $[0, 1]$ applied to $[5, 4]$ and $[0, 0, 1]$ applied to $[9, 16, 11]$. Hence, the sorted array is $[4, 5, 9, 11, 16]$. Of course, sorting the array by this method requires the array to be split into two parts: entries less than or equal to pivot and the entries greater than pivot. However, in the worst case our partition may be trivial. For example, when the pivot is the greatest element in the array or the smallest element in the array, one side of the partition is just one less than the original array. We note here that in expectation, the partitioned arrays are not unevenly sized and the algorithm performs well in practice.

Merge sort: Revisited

In the above example, we first partitioned the array such that all elements on the left side of the split are less than the right side of the split. Once the left side and the right side are sorted, there is nothing else to be done. An alternative version of the split would be to split the lattice trivially, but the merging of the solution requires more effort. In our example of the sorted array $A = [5, 9, 16, 4, 11]$, suppose that we split the lattice into two lattices L_1 and L_2 such that L_1 consists of the first two components and L_2 corresponds to the last three components. The vector $S_1 = [5, 9]$ is the sorted version of the left side $[5, 9]$ and the vector $S_2 = [4, 11, 16]$ is the sorted version of the right side $[16, 4, 11]$. Now, our task is to merge the two sorted arrays. The function f in definition of *splittable lattice-linear predicate* corresponds to merging of these two arrays.

To sort an array of size n, we divide it into two subarrays of size $n/2$ and sort each of the parts in parallel. The main trick is to determine how to merge these two sorted arrays. To understand, an application of the LLP algorithm for mergesort, we consider a related problem of counting the number of inversions of each entry in an array A. Given an array A, our goal is to compute another array G such that for any index i in the array,

$$G[i] = \#\{j < i \mid A[j] > A[i]\} \tag{9.1}$$

To keep things simple, we let all entries of A be unique. To exploit divide-and-conquer, we divide the array into two equal parts (or nearly equal, if the array A is of odd size). Let these two parts be called $Left$ and $Right$. For the left part, computing G remains identical irrespective of the total size of the array, i.e., we still have Equation 9.1. However, for the right part, we have that $G[i] = \#\{(j < i) \wedge (j \in Left) \mid A[j] > A[i]\} + \#\{(j < i) \wedge (j \in Right) \mid A[j] > A[i]\}$. The second part of the above equation is identical to Equation 9.1 for the right part of the array, but we need to find the value of the first part of the equation.

Algorithm LLP-MergeSort: MergeSort

```
LLP-MergeSort(A, low, high)
if $low < high$ then
    $mid = \lfloor (lo + high)/2 \rfloor$ ;
    $B$ := LLP-MergeSort(A, low, mid) in parallel with $C$ := LLP-MergeSort(A, mid + 1, high)
    $G := BinSrch(B, C)$;
    return $A$ with inversions $(G, 0)$;
else return $A$;
```

Now, we merge the two sorted subarrays to get the sorted version of the original array. How do we view merging of two sorted array A_l and A_r as an LLP algorithm? Let us model this problem as searching for a particular inversion vector in the lattice of all inversion vectors. We consider the number of elements less than $A[i]$ that appear to the right of $A[i]$. If all entries are unique, then there is only one inversion vector that corresponds to the correct answer. We know that A_r is a zero vector if A_r is sorted. For each entry x in A_l, we need to determine the number of entries in A_l and A_r that are smaller than x. Since A_l is sorted we know that the number of entries smaller than x in A_l is simply $j - 1$. To find the number of entries smaller than x in A_r, we use binary search because A_r is sorted. We call this method BinSrch.

Algorithm LLP-BinSrch: LLP Computing Inversion vector for A_l

```
var
$G$: array$[1 \ldots m]$ of int initially $-1$;
forbidden(j): $(G[j] < 0)$;
    advance $G[j] := (j - 1) + binarySearch(A_l[j], A_r)$;
```

The method *binarySearch* returns the number of elements less than $A_i[j]$ in A_r. Here, the vector lattice has the bottom element as $[-1, -1, \ldots -1]$. We are searching for the least vector G such that $G[j]$ is at least the number of inversions of $A[j]$. Formally,

$$B(G) \equiv \forall j : G[j] \geq inversions(j, A)$$

Let us compute the time complexity. We have the following recurrence

$$T(n) = 2T(n/2) + O(n); T(1) = O(1)$$

Solving this recurrence, we get the sequential time complexity as $T(n) = O(n \log n)$.

Nearest Neighbors in the Euclidean Space

Counting Inversions

Planar Convex Hull

Karatsuba's Algorithm

Let the two numbers be X and Y. We view them as two-dimensional vectors. Our goal is to compute the product (polynomial multiplication) of these two vectors. We are computing the least polynomial that is divisible by both the vectors. Let L be the lattice of three dimensional vector. Our search is for the least three dimensional vector that is divisible by both X and Y. Formally, we are searching for the least vector G such that

$$B(G) \equiv (X divides G) \wedge (Y divides G)$$

Strassen Matrix Multiplication

9.10 Problems

1. Solve the problem of computing the Fast-Fourier-Transform in using the divide and conquer approach.

9.11 Bibliographic Remarks

The general divide-and-conquer strategy and the Master Theorem are thoroughly treated in [CLRS09]. Strassen's matrix multiplication (Section 10), first proposed by [Str69], reduces the complexity of a matrix multiplication of size $n \times n$ from $O(n^3)$ to $O(n^{2.807})$. Karatsuba's multiplication introduced by [KO62] reduces the complexity of multiplying two n digit numbers from $O(n^2)$ to $O(n^{1.585})$. The Closest Pair problem uses a divide-and-conquer approach from [SH75], to obtain the time complexity $O(n \log n)$. Counting Inversions via Mergesort, achieving $O(n \log n)$, is a standard application of the divide-and-conquer approach. The Planar Convex Hull is based on [Gra79], with its $O(n \log n)$ time complexity. For further reading, we refer the readers to [Hoa62], [Str69], [KO62], [SH75], and [Gra79].

Chapter 10

Dynamic Programming

10.1 Introduction

Dynamic programming is applicable to problems where it is easy to set up a recurrence relation such that the solution of the problem can be derived from the solutions of problems with smaller sizes. The problem can be solved using recursion; however, recursion can result in many duplicate computations. By using memoization, we can avoid recomputing previously computed values. We assume that the problem is solved using dynamic programming with such a bottom-up approach in this chapter.

The LLP algorithm views solving a problem as searching for an element in a finite distributive lattice [Bir67, DP90, Gar15] that satisfies a given predicate B. The predicate is required to be lattice-linear. For all the problems considered in this chapter, the longest subsequence problem, the optimal binary search tree problem, and the knapsack problem, this is indeed the case.

There are also some key differences between dynamic programming (the bottom-up approach) and the LLP algorithm. The usual dynamic programming problem seeks a structure that minimizes (or maximizes) some scalar. For example, the longest subsequence problem asks for the subsequence in an array $A[1..n]$ that maximizes the sum. In contrast, the LLP algorithm seeks to minimize or maximize a *vector*. In the longest subsequence problem with the LLP approach, we are interested in the longest subsequence in the array $A[1..i]$ for each $i \leq n$ that ends at the index i. Thus, instead of asking for a scalar, we ask for the vector of size n. We get an array $G[1..n]$, and the solution to the original problem is just the maximum value in the array G. Similarly, the optimal binary search tree problem [Knu71] asks for the construction of an optimal binary search tree on n symbols such that each symbol i has probability p_i of being searched. Our goal is to find the binary search tree that minimizes the expected cost of the search in the tree. The LLP problem seeks the optimal binary search tree for all ranges $i \ldots j$ instead of just one range $1..n$. Finally, the knapsack problem [HS74, IK75] asks for the maximum valued subset of items that can fit in a knapsack so that the profit is maximized and the total weight of the knapsack is at most W. The LLP problem seeks the maximum profit obtained by choosing items from $1..i$ and the total weight from $1..W$. In all these problems, traditionally we are seeking a single structure that optimizes a single scalar; whereas the LLP algorithm asks for a vector. It turns out that in asking for an optimal *vector* instead of an optimal *scalar*, we do not lose much, since the existing solutions also end up finding the optimal solutions for the subproblems. The LLP algorithm returns a vector G such that $G[i]$ is optimal for i.

Yet another difference between dynamic programming and the LLP algorithm is that we can use the LLP algorithm to solve a constrained version of the problem, so long as the constraint itself is lattice-linear. Suppose that we are interested in the longest subsequence such that successive elements differ by at least 2. It can be (easily) shown that this constraint is lattice-linear. Hence, the LLP algorithm is applicable because we are searching for an element that satisfies a conjunction of two lattice-linear predicates. Since the set of lattice-linear predicates is closed under conjunction, the resulting predicate is also lattice-linear, and the LLP algorithm is applicable. Similarly, the predicate that the symbol i is not a parent of the symbol j is lattice-linear, and the constrained optimal binary search tree algorithm returns the optimal tree that satisfies the given constraint. In the Knapsack problem, it is easy to solve the problem with the additional constraint that if the item x is included in the Knapsack, then the item y is also included.

This chapter is organized as follows. Section 10.4 applies the LLP method to the longest subsequence problem. Section 10.5 gives an LLP algorithm for the optimal binary search tree construction problem. Section 10.7 gives an LLP algorithm for the knapsack problem.

10.2 Recursion vs Dynamic Programming

Dynamic programming is applicable to problems in which it is easy to set up a recurrence relation so that the solution of the problem at hand can be derived from the solutions to problems with smaller sizes. The problem can be solved using recursion; however, recursion may result in many duplicate computations.

As a simple example, suppose that our goal is to compute the Fibonacci number F_i such that it satisfies the following conditions:

- $F_0 = 1$
- $F_1 = 1$
- $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$

A recursive algorithm is as follows:

```
F(n):
  if n < 2:
    return 1
  else:
    return F(n-1) + F(n-2)
```

If we use recursion to find $F(4)$, we will call $F(3)$ and $F(2)$. Now, $F(3)$ will call $F(2)$ and $F(1)$. Thus, $F(2)$ is called twice.

Using *memoization* for dynamic programming, we avoid such duplicate computations. Memoization is used to recall previously computed values.

An implementation using memoization is as follows:

```
F(n):
  Table: map of (int, int)
  if (n, v) exists in Table for n:
    return v
  else:
```

```
  if n < 2:
    return 1
  else:
    answer = F(n-1) + F(n-2)
    Store (n, answer) in Table
```

Alternatively, instead of computing $F(n)$ in a top-down manner, one can compute it in a bottom-up fashion. Assume that we keep an array A such that $A[i]$ will eventually store $F(i)$. We start by filling in the array A from the lower to higher indices. The recurrence relation guarantees that if we have two previous entries, we can fill any entry $A[i]$. Thus, we can compute $F(n)$ in $O(n)$ time.

In this example, we had a simple array. For more nontrivial examples, we may have two- or three-dimensional tables.

10.3 Weighted Interval Scheduling

We revisit the problem of Interval Scheduling from Chapter 5. This time we consider a generalization of the problem in which each interval is assigned *weight* denoting its priority. We would like to compute a subset of intervals such that they are non-overlapping and have the maximum weight. In this generalization, we can no longer guarantee that the interval that finishes earliest and starts after the last chosen interval is always selected.

As before, we assume that the intervals are sorted according to their *finish* times. Let $Opt(j)$ denote the optimal value that can be obtained from the intervals $1 \ldots j$. When j is zero, the set of intervals selected is empty and $Opt(0)$ is zero. Let $p(j)$ be the largest interval before the interval j which is disjoint from the interval j. Then we can set up a recurrence relation as follows. The maximum value we can obtain from the intervals $1 \ldots j$ is equal to the maximum of two values obtained by including the interval j or not including the interval j. If we include the interval j, then the value obtained is $w_j + Opt(p(j))$. If we do not include the interval j, then the value obtained is $Opt(j-1)$. Thus, we have

$$Opt(j) = \max\{w_j + Opt(p(j)), Opt(j-1)\}$$

This recurrence relation allows us to obtain the algorithm for weighted interval scheduling as shown in Fig. WeightedIntervalSequential

The sequential algorithm takes $O(n)$ time when p array is available. This time complexity is optimal. Computing p array takes $O(n \log n)$ time.

Consider the following 5 intervals, sorted by finish time:

Interval	s_i	f_i	w_i
1	1	3	4
2	2	5	6
3	4	6	5
4	6	8	3
5	5	9	7

The intervals are shown in Fig. 10.1. We first compute $p(i)$ (the last nonoverlapping interval before i):
- $p(1) = 0$ (none before 1)
- $p(2) = 0$ ($f_1 = 3 > s_2 = 2$)

Algorithm WeightedIntervalSequential: A Sequential Program for the Weighted Interval Scheduling problem

```
// jobs are sorted by their finish times
Input: $s$: array[$0 \dots n-1$] of int; // start times
Input: $f$: array[$0 \dots n-1$] of int ; // finish times
Input: $w$: array[$0 \dots n-1$] of int; //weight of the interval
var $G$: array[$0 \dots n-1$] of 0..1 initially 0;
    $p$: array[$0 \dots n-1$] of int; //previous non-overlapping interval
    $opt$: array[$0 \dots n-1$] of int; //optimal value
// $p[j]$ is the largest interval $i$ such that $f[i] < s[j]$
$p[0] := 0$;
// Compute $p[j]$ as the largest $i$ such that $f[i] < s[j]$;
// Since $f$ is sorted, we can compute all $p[j]$ in $O(\log n)$ time using binary search
$opt[0] := 0$;
for $int\ cur := 1$ to $n-1$ do
    $opt[cur] := opt[cur-1]$;
    if $(w[cur] + opt[p[cur]] \geq opt[cur-1])$ then
        $opt[cur] := w[cur] + opt[p[cur]]$
        $G[cur] := 1$;
    end
    else $G[cur] := 0$;;
end
```

- $p(3) = 1$ ($f_1 = 3 < s_3 = 4,\ f_2 = 5 > s_3$)
- $p(4) = 3$ ($f_3 = 6 = s_4$, compatible)
- $p(5) = 2$ ($f_2 = 5 = s_5$, compatible)

Now, we compute $OPT(i)$:
- $OPT(0) = 0$
- $OPT(1) = \max(OPT(0), w_1 + OPT(p(1))) = \max(0, 4 + 0) = 4$ (select 1)
- $OPT(2) = \max(OPT(1), w_2 + OPT(p(2))) = \max(4, 6 + 0) = 6$ (select 2)
- $OPT(3) = \max(OPT(2), w_3 + OPT(p(3))) = \max(6, 5 + 4) = 9$ (select 1, 3)
- $OPT(4) = \max(OPT(3), w_4 + OPT(p(4))) = \max(9, 3 + 9) = 12$ (select 1, 3, 4)
- $OPT(5) = \max(OPT(4), w_5 + OPT(p(5))) = \max(12, 7 + 6) = 13$ (select 2, 5)

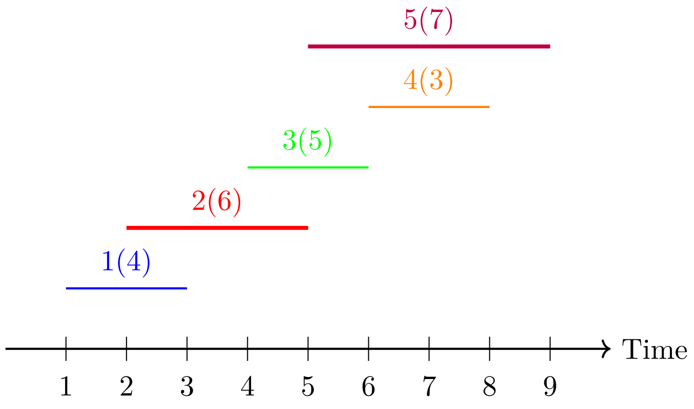


Figure 10.1: Weighted intervals (weights in parentheses) with optimal selection (thick lines).

10.4 Longest Increasing Subsequences

The Longest Increasing Subsequence (LIS) problem involves finding a subsequence of a given array of integers, where the elements are in strictly increasing order and the subsequence is as long as possible. A subsequence is derived by deleting zero or more elements from the original array without changing the order of the remaining elements.

For example, in the array $[10, 9, 2, 5, 3, 7, 101, 18]$, one possible LIS is $[2, 5, 7, 101]$ with length 4. The problem has applications in data analysis, sequence alignment, and more. We present an iterative dynamic programming solution to compute the LIS length efficiently.

Our problem can be stated as follows. Given an array $A = [a_1, a_2, \ldots, a_n]$ of n integers, find the length of the longest subsequence $[a_{i_1}, a_{i_2}, \ldots, a_{i_k}]$ such that:

$$i_1 < i_2 < \cdots < i_k \quad \text{and} \quad a_{i_1} < a_{i_2} < \cdots < a_{i_k}.$$

We use an iterative dynamic programming approach to solve the LIS problem. Define $dp[i]$ as the length of the longest increasing subsequence ending at index i (including a_i).

The idea is to build the dp array iteratively by comparing each element with all the previous elements.

- Initialize $dp[i] = 1$ for all i (each element is an LIS of length 1 itself).
- For each i, check all $j < i$. If $a[j] < a[i]$, update $dp[i]$ as $\max(dp[i], dp[j] + 1)$.
- The length of the LIS is the maximum value in the dp array.

Algorithm LongestIncreasingSubsequence: Longest Increasing Subsequence (Iterative)

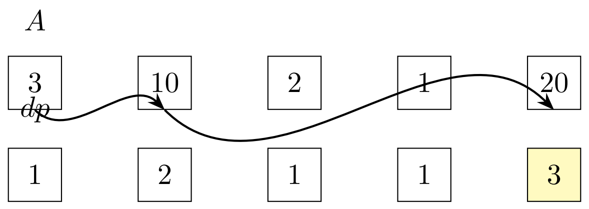
```
Input: Array $A$ of $n$ integers
Output: Length of the LIS
var
    $dp$: array $[0 \dots n-1]$: $\forall i : dp[i] = 1$;
for $i = 1$ to $n - 1$ do
    for $j = 0$ to $i - 1$ do
        if $A[j] < A[i]$ then $dp[i] = \max(dp[i], dp[j] + 1)$ ;
    end
end
return $\max(dp[0], dp[1], \dots, dp[n-1])$;
```

$dp[i]$ represents the longest increasing subsequence ending in $A[i]$, considering all prior elements. By checking all $j < i$, we ensure that every possible subsequence ending at i is evaluated. The maximum $dp[i]$ gives the overall length of the LIS, as it captures the longest chain possible.

Consider the array $A = [3, 10, 2, 1, 20]$. We compute the LIS length step-by-step and visualize the process.

- $A[0] = 3$, $dp[0] = 1$ (base case).
- $A[1] = 10$, $3 < 10$, $dp[1] = \max(1, 1 + 1) = 2$.
- $A[2] = 2$, $3, 10 > 2$, $dp[2] = 1$.
- $A[3] = 1$, $3, 10, 2 > 1$, $dp[3] = 1$.
- $A[4] = 20$, $3 < 20$, $10 < 20$, $2 < 20$, $1 < 20$, $dp[4] = \max(1, 1 + 1, 2 + 1, 1 + 1, 1 + 1) = 3$.

Final $dp = [1, 2, 1, 1, 3]$, LIS length $= \max(dp) = 3$. An LIS is $[3, 10, 20]$.



The highlighted $dp[4] = 3$ indicates the LIS length. Arrows show one possible LIS: $[3, 10, 20]$.

The algorithm has two nested loops:
Outer loop: i from 1 to $n - 1$, $O(n)$ iterations.
Inner loop: j from 0 to $i - 1$, up to $O(n)$ iterations per i.

The total comparisons are equal to $0 + 1 + 2 + \cdots + (n - 1) = \frac{(n-1)n}{2}$, which is $O(n^2)$.

Thus, the overall time complexity is $O(n^2)$. The space complexity is $O(n)$ for the dp array.

Longest Increasing Subsequence Using LLP Method

We are given an integer array as input. For simplicity, we assume that all entries are distinct. Our goal is to find for each index i the length of the longest increasing sequence that ends at i. For example, suppose that the array A is `{35 38 27 45 32}`. Then, the desired output is `{1 2 1 3 2}`. The corresponding longest increasing subsequences are: `(35), (35, 38), (27), (35, 38, 45), (27, 32).`

We can define a graph H with indices as vertices. For this example, we have five vertices numbered v_1 to v_5. We draw an edge from v_i to v_j if i is less than j and $A[i]$ is also less than $A[j]$. This graph is clearly acyclic, as an edge can only go from a lower index to a higher index. We use $pre(j)$ to be the set of indices that have an incoming edge to j. The length of the longest increasing subsequence ending at index j is identical to the length of the longest path ending at j.

To solve the problem using LLP, we model it as a search for the smallest vector G that satisfies the constraint $B \equiv \forall j : G[j] \geq 1 \wedge \forall j : G[j] \geq \max\{G[i] + 1 \mid i \in pre(j)\}$.

The underlying lattice we consider is that of all vectors of natural numbers less than or equal to the maximum element in the lattice. A vector in this lattice is *feasible* if it satisfies B. We first show that the constraint B is lattice-linear.

Lemma 10.1 *The constraint* $B \equiv (\forall j : G[j] \geq 1) \wedge (\forall j : G[j] \geq \max\{G[i]+1 \mid i \in pre(j)\})$ *is lattice-linear.*

Proof: Since the predicate B is a conjunction of two predicates, it suffices to show that each of them is lattice-linear. The first conjunct is lattice-linear because the constant function 1 is monotone. The second conjunct can be viewed as a conjunction over all j. For a fixed j, the predicate $G[j] \geq \max\{G[i] + 1 \mid i \in pre(j)\}$ is lattice-linear.

■

Our goal is to find the smallest vector in the lattice that satisfies B. To ensure that no $G[j]$ is updated more than once, we introduce a Boolean $fixed$ for each index so that we update $G[j]$ only when it is not fixed and all its predecessors are fixed. We now have the Algorithm LLP-Longest-Increasing-Subsequence.

Algorithm LLP-Longest-Increasing-Subsequence: Finding the Longest Increasing Subsequence.

```
$P_j$: Code for thread $j$
input: $A$:array[1 ... $n$] of int;
var $G$: array[1 ... $n$] of int;
    $fixed$: array[1 ... $n$] of boolean;
init: $G[j] = 1$; $fixed[j] := false$;
    $pre(j) := \{i \in 1..j-1 | A[i] < A[j]\}$;
forbidden: $\neg fixed[j] \wedge (\forall i \in pre(j) : fixed[i])$;
    advance: $G[j] := \max\{G[i] + 1 \mid i \in pre(j)\}$;
        $fixed[j] := true$;
```

Let us now analyze the complexity of the algorithm. The sequential complexity is simple because we can maintain the list of all vertices that are forbidden because all its predecessors are fixed. Once we have processed a vertex, we never process it again. This is similar to a topological sort sequential algorithm. In this case, we examine a vertex exactly once only after all its predecessors are fixed. The time complexity of this algorithm is $O(n^2)$.

We now add lattice-linear constraints to the problem. Instead of the longest increasing subsequence, we may be interested in the longest increasing subsequence that satisfies an additional predicate.

Lemma 10.2 *All the following predicates are lattice linear.*

1. *For any j, $G[j]$ is greater than or equal to the longest increasing subsequence of odd integers ending at j.*

2. *$G[j]$ is greater than or equal to the longest increasing subsequence such that j^{th} element in the subsequence exceeds $(j-1)^{th}$ element by at least k.*

Proof:

1. Since the lattice-linear predicates are closed under conjunction, it is sufficient to focus on a fixed j. If $G[j]$ is less than the length of the longest increasing subsequence of odd integers ending at j, then the index j is forbidden. Unless j is increased, the predicate can never become true.

2. We view this predicate as redrawing the directed graph H such that we draw an edge from v_i to v_j if i is less than j and $A[i]+k$ is less than or equal to $A[j]$.

■

10.5 Optimal Binary Search Tree

The Optimal Binary Search Tree problem involves constructing a binary search tree (BST) for n symbols that minimizes the expected search cost, given the probability of each symbol's occurrence. Each symbol i (ranging from 0 to $n-1$) has a probability $p[i]$ of being searched, and these probabilities sum to 1, i.e., $\sum_{i=0}^{n-1} p[i] = 1$. The objective is to arrange the symbols in a BST to minimize the total cost, defined as the sum of each symbol's probability multiplied by its depth, where the root has depth 0. For instance, given three symbols A, B, and C with probabilities $p = [0.4, 0.3, 0.3]$, an optimal BST might place A as the root to reduce the average search time. This problem finds applications in areas like database indexing and compiler design, where efficient search structures are crucial. We present an iterative dynamic programming solution to compute the minimum cost.

Formally, given n symbols indexed from 0 to $n-1$, each associated with probability $p[i]$, and defining the sum of probabilities from i to j as $s(i,j) = \sum_{k=i}^{j} p[k]$, the task is to construct a BST that minimizes the expected search cost, expressed as $\text{Cost} = \sum_{i=0}^{n-1} p[i] \cdot \text{depth}(i)$, where depth$(i)$ represents the depth of the symbol i in the tree.

To solve this, an iterative dynamic programming approach is employed. Two key arrays are defined: $dp[i][j]$ represents the minimum cost of a BST for symbols from i to j (inclusive), and $s(i,j)$ denotes the sum of probabilities from i to j, which serves as the cost increment at each level. The solution builds the dp table iteratively over increasing lengths of subarrays. When considering a single symbol ($j = i$), the cost is simply $dp[i][i] = p[i]$, reflecting the contribution of a leaf node. For larger ranges, each symbol r from i to j is tested as the root, combining the left subtree $[i, r-1]$ and right subtree $[r+1, j]$. The total cost for a root r is computed as $dp[i][r-1] + dp[r+1][j] + s(i,j)$, where $s(i,j)$ accounts for the increased depth of all symbols in the subtree. To optimize, $s(i,j)$ is precomputed iteratively.

Here is the pseudocode algorithm:

The correctness of this approach comes from the fact that $dp[i][j]$ computes the minimum cost by exhaustively trying each possible root r and combining the optimal costs of the resulting subproblems.

Algorithm Optimal-Binary-Search-Tree: Optimal Binary Search Tree (Iterative)

```
   Input: Array $p[0 \ldots n-1]$ of probabilities
   Output: Minimum cost of Optimal BST
1  Initialize $dp[0 \ldots n-1][0 \ldots n-1]$ and $s[0 \ldots n-1][0 \ldots n-1]$ with 0;
2  for $i = 0$ to $n-1$ do
3      $dp[i][i] = p[i]$;
4      $s[i][i] = p[i]$;
5  end
6  for $l = 1$ to $n-1$ do
7      for $i = 0$ to $n-1-l$ do
8          $j = i + l$;
9          $s[i][j] = s[i][j-1] + p[j]$;
10         $dp[i][j] = \infty$;
11         for $r = i$ to $j$ do
12             $left\_cost = 0$;
13             $right\_cost = 0$;
14             if $r > i$ then $left\_cost = dp[i][r-1]$ ;
15             if $r < j$ then $right\_cost = dp[r+1][j]$ ;
16             $cost = s[i][j] + left\_cost + right\_cost$;
17             $dp[i][j] = \min(dp[i][j], cost)$;
18         end
19     end
20 end
21 return $dp[0][n-1]$;
```

The term $s(i, j)$ ensures that the cost reflects the increase in depth for all symbols in the subtree. By filling the table bottom-up, from single symbols to the full range, the iterative method guarantees that $dp[0][n-1]$ yields the optimal cost.

Consider an example with $n = 4$ symbols A, B, C, and D, with probabilities $p = [0.25, 0.2, 0.3, 0.25]$. The computation proceeds as follows, and key steps are highlighted using itemization for clarity:

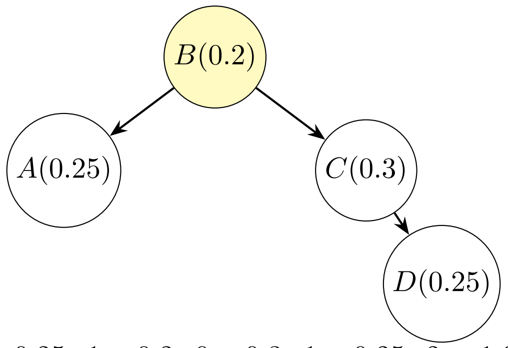
- For length $l = 0$, base cases are set: $dp[0][0] = 0.25$, $dp[1][1] = 0.2$, $dp[2][2] = 0.3$, $dp[3][3] = 0.25$, with corresponding $s[i][i] = p[i]$.

- For $l = 1$, consider $i = 0$, $j = 1$: $s[0][1] = 0.25 + 0.2 = 0.45$. Testing $r = 0$ gives $0.45 + 0 + 0.2 = 0.65$, $r = 1$ gives $0.45 + 0.25 + 0 = 0.7$, so $dp[0][1] = 0.65$ (root A). Similarly, $dp[1][2] = 0.65$ (root C), $dp[2][3] = 0.65$ (root C).

- For $l = 2$, consider $i = 0$, $j = 2$: $s[0][2] = 0.45 + 0.3 = 0.75$. Testing $r = 0$ gives $0.75 + 0 + 0.65 = 1.4$, $r = 1$ gives $0.75 + 0.25 + 0.3 = 1.3$, $r = 2$ gives $0.75 + 0.65 + 0 = 1.4$, so $dp[0][2] = 1.3$ (root B).

- For $l = 3$, $i = 0$, $j = 3$: $s[0][3] = 0.75 + 0.25 = 1.0$. Testing $r = 0$ gives $1.0 + 0 + 1.15 = 2.15$ (using $dp[1][3] = 1.15$), $r = 1$ gives $1.0 + 0.25 + 0.65 = 1.9$, $r = 2$ gives $1.0 + 0.65 + 0.25 = 1.9$, $r = 3$ gives $1.0 + 1.3 + 0 = 2.3$, so $dp[0][3] = 1.9$ (root B or C).

The minimum cost is $dp[0][3] = 1.9$, with B as a possible optimal root. The resulting optimal BST can be visualized as follows:



Cost $= 0.25 \cdot 1 + 0.2 \cdot 0 + 0.3 \cdot 1 + 0.25 \cdot 2 = 1.9$

In this tree, B is the root at depth 0, A and C are at depth 1, and D is at depth 2, resulting in a cost of $0.25 \cdot 1 + 0.2 \cdot 0 + 0.3 \cdot 1 + 0.25 \cdot 2 = 1.9$, matching $dp[0][3]$.

To analyze the time complexity, consider the three nested loops in the algorithm. The outer loop iterates over lengths l from 1 to $n - 1$, which takes $O(n)$ time. For each l, the middle loop iterates over starting indices i from 0 to $n - 1 - l$, also $O(n)$ per length. Within these, the inner loop tests each root r from i to j, which can be up to $O(n)$ iterations. This results in a total of $O(n) \cdot O(n) \cdot O(n) = O(n^3)$ operations. Precomputing $s[i][j]$ requires $O(n^2)$ time, as each entry builds on the previous, but this is dominated by the main computation. The overall time complexity is thus $O(n^3)$, with a space complexity of $O(n^2)$ for the dp and s tables.

In conclusion, this iterative dynamic programming solution constructs an OBST with the minimum expected search cost in $O(n^3)$ time.

Optimal Binary Search Tree with LLP

Suppose that we have a fixed set of n symbols called *keys* with some associated information called *values*. Our goal is to build a dictionary based on binary search tree out of these symbols. The dictionary supports a single operation search that returns the value associated with the given key. We are also given the frequency of each symbol as an argument for the search query. The cost of any search for a given key is given by the length of the path from the root of the binary search tree to the node containing that key. Given any binary search tree, we can compute the total cost of the tree for all searches. We would like to build the binary search tree with the least cost.

Let the frequency of the key i being searched be p_i. We assume that the keys are sorted in the increasing order of the symbols. Our algorithm is based on building progressively bigger binary search trees. The main idea is as follows. Suppose that symbol k is the root of an optimal binary search tree for symbols in the range $[i..j]$. The root k divides the range into three parts: the range of indices strictly less than k, the index k, and the range of indices strictly greater than k. The left or right range may be empty. Then, the left subtree and the right subtree must themselves be optimal for their respective ranges. Let $G[i,j]$ denote the least cost of any binary search tree built from symbols in the range $i..j$. We use the symbol $s(i,j)$ as the sum of all frequencies from the symbol i to j, i.e.,

$$s(i,j) = \sum_{k=i}^{j} p_k$$

For convenience, we let $s(i,j)$ equal to 0 whenever $i > j$, i.e., the range is empty.

We now define a lattice linear constraint on $G[i,j]$. Let $i \leq k \leq j$. Consider the cost of the optimal tree so that the symbol k is at the root. The cost has three components: the cost of the left subtree if any, the cost of the search ending at this node itself, and the cost of the search in the right subtree. The cost of the left subtree is

$$G[i,k-1] + s(i,k-1)$$

whenever $i < k$. The cost of the node itself is $s(k,k)$. The cost of the right subtree is

$$G[k+1,j] + s(k+1,j)$$

Combining these expressions, we get

$$G[i,j] = \min_{i \leq k \leq j} (G[i,k-1] + s(i,j) + G[k+1,j])$$

This is also the least value of $G[i,j]$ such that

$$G[i,j] \geq \min_{i \leq k \leq j} (G[i,k-1] + s(i,j) + G[k+1,j])$$

We now show that the predicate above is lattice-linear.

Lemma 10.3 *The constraint* $B \equiv \forall i,j : G[i,j] \geq \min_{i \leq k \leq j}(G[i,k-1]+s(i,j)+G[k+1,j])$ *is lattice-linear.*

Proof: Suppose that B is false, i.e., $\exists i,j : G[i,j] < \min_{i \leq k \leq j}(G[i,k-1]+s(i,j)+G[k+1,j])$. This means that there exists i,j,k with $i \leq k \leq j$ such that $G[i,j] < (G[i,k-1]+s(i,j)+G[k+1,j])$. This means that the index (i,j) is forbidden and unless $G[i,j]$ increases, the predicate B can never become true, regardless of how other components of G are increased.

■

We now have our LLP-based algorithm for the optimal binary search tree as an Algorithm LLP-OptimalBinarySearchTree. The program has a single variable G. It is initialized so that $G[i,i]$ equals $p[i]$ and $G[i,j]$ equals zero whenever i is not equal to j. The algorithm advances $G[i,j]$ whenever it is smaller than $\min_{i\le k\le j} G[i,k-1]+s(i,j)+G[k+1,j]$. In Algorithm LLP-OptimalBinarySearchTree, we have used the **always** clause as a macro that uses $s(i,j)$ as a short form for $\sum_{k=i}^{j} p[k]$.

Algorithm LLP-OptimalBinarySearchTree: Finding An Optimal Binary Search Tree

```
1 $P_{i,j}$: Code for thread $(i,j)$
2 input: $p$:array of real;// frequency of each symbol
3 init: $G[i,j] = 0\ \forall i \ne j$;
4     $G[i,i] = p[i]$;
5 always: $s(i,j) = \sum_{k=i}^{j} p[k]$
6 ensure:
7     $G[i,j] \ge \min_{i\le k\le j} G[i,k-1] + s(i,j) + G[k+1,j]$
8 priority: $(j-i)$
```

Although the above algorithm will give us correct answers, it is not efficient as it may update $G[i,j]$ before $G[i,k]$ and $G[k,j]$ for $i \le k \le j$ have stabilized. However, the following scheduling strategy ensures that we update $G[i,j]$ at most once. We check whether $G[i,j]$ is forbidden in the order of $j-i$. Hence, initially, all $G[i,j]$ such that $j = i+1$ are updated. This is followed by all $G[i,j]$ such that $j = i+2$, and so on. We capture this scheduling strategy with the **priority** statement. We choose $G[i,j]$ to update so that $(j-i)$ have minimal values. Of course, our goal is to compute $G[1,n]$. With the above strategy of updating $G[i,j]$, we get that $G[i,j]$ is updated at most once. Since there are $O(n^2)$ possible values for $G[i,j]$ and each takes $O(n)$ work to update, we get the work complexity of $O(n^3)$.

We now consider the constrained versions of the problem.

Lemma 10.4 *All the following predicates are lattice linear.*

1. *The key x is not a parent for any key.*

2. *The difference in the sizes of the left subtree and the right subtree is at most* 1.

Proof:

1. This requirement changes the ensure predicate to $G[i,j] \ge \min_{i\le k\le j, k\ne x} G[i,k-1]+s(i,j)+G[k+1,j]$. The right-hand side of the constraint continues to be monotonic, and therefore it is lattice linear.

2. This requirement changes the ensure predicate to $G[i,j] \ge \min_{i\le k\le j, |k-1-i, j-k-1|\le 1} G[i,k-1] + s(i,j) + G[k+1,j]$. This change simply restricts the values of k, and the right-hand side continues to be monotonic.

■

Chain Matrix Multiplication

A problem very similar to the Optimal Binary Search tree is that of constructing an optimal way of multiplying a chain of matrices. Since matrix multiplication is associative, the product of matrices $(M_1 * M_2) * M_3$ is equal to $M_1 * (M_2 * M_3)$. However, depending on the dimensions of the matrices, the computational effort may be different. For example, considering the dimensions (30 times 10), (10 times 30), (30 times 2) result in a matrix of size 30 times 2. If we multiply the first two matrices using the standard simple matrix multiplication, we get 30 times 10 times 30 = 9000 operations to get the 30 times 30 matrix. Multiplying M_3 requires 1800 additional operations. If we multiply M_2 and M_3 first, then we need 600 operations first. Multiplying M_1 adds $30 \times 10 \times 2 = 600$ additional operations. We let the dimension of matrix M_i be $m_{i-1} \times m_i$. Note that this keeps the matrix product well defined because the dimension of the matrix M_{i+1} would be $m_i \times m_{i+1}$ and the product $M_i \times M_{i+1}$ is well defined. We can view any evaluation of a chain as a binary tree where the intermediate notes are the multiplication operation and the leaves are the matrices themselves. Suppose that our goal is to compute the optimal binary tree for multiplying matrices in the range $M_i \dots M_j$. Taking ideas from the previous section, we let $G[i, j]$ denote the optimal cost of computing the product of matrices in the range $M_i \dots M_j$. Suppose that this product is broken into products of $M_i \dots M_k$ and $M_{k+1} \dots M_j$ and then the multiplication of these two matrices. We can compute the cost of this tree as,

$$G[i, k] + G[k + 1, j] + m_{i-1}m_k m_j$$

Then, we have the following predicate on G.

$$G[i, j] \geq \min_{i \leq k < j} (G[i, k] + m_{i-1}m_k m_j + G[k + 1, j]$$

The reader will notice the similarity with the optimal binary search tree problem and this problem, and the same algorithm can be adapted to solve this problem.

10.6 Segmented Least Squares Problem

The Segmented Least Squares problem involves fitting a set of n points in the x-y plane with a sequence of straight-line segments. The goal is to minimize the total sum of squared errors between the points and their corresponding line segments, plus a penalty C for each segment used. This balances fitting accuracy with the model simplicity. We present a dynamic programming solution, followed by an example with illustrations.

Given n points $p_1 = (x_1, y_1), p_2 = (x_2, y_2), \dots, p_n = (x_n, y_n)$, sorted by increasing x-coordinates ($x_1 < x_2 < \cdots < x_n$), we aim to:

- Partition the points into segments.
- For each segment, compute the least squares line that minimizes the sum of squared errors.
- Minimize the total cost: sum of squared errors in all segments plus C times the number of segments.

For points $p_i, p_{i+1}, \dots, p_j$, the least squares line minimizes:

$$e_{ij} = \sum_{k=i}^{j} (y_k - (ax_k + b))^2,$$

where a and b are the slope and intercept of the line, computed as:

$$a = \frac{n \sum (x_k y_k) - \sum x_k \sum y_k}{n \sum x_k^2 - (\sum x_k)^2}, \quad b = \frac{\sum y_k - a \sum x_k}{n},$$

for $k = i$ to j, and $n = j - i + 1$.

The total cost for a segmentation is:

$$\text{Cost} = \sum_{\text{segments}} e_{ij} + C \cdot (\text{number of segments}).$$

Define $OPT(j)$ as the minimum cost of segmenting points $p_1, p_2, \ldots, p_j$. The recurrence relation is:

$$OPT(j) = \min_{0 \le i < j} \left[OPT(i) + e_{i+1,j} + C \right],$$

where:

- $OPT(0) = 0$ (base case: no points),
- $e_{i+1,j}$ is the error of fitting a single line to points p_{i+1} to p_j,
- C is the penalty for adding a new segment starting at p_{i+1}.

To compute $OPT(j)$ for all $j = 1$ to n:

1. Precompute e_{ij} for all i, j pairs ($O(n^3)$ time with $O(n^2)$ space).
2. For each j, evaluate the minimum over all $i < j$ ($O(n)$ per j).
3. Store the optimal i for each j to reconstruct the segments.

Total time complexity: $O(n^3)$, space complexity: $O(n^2)$.

Algorithm Segmented-Least-Squares: Segmented Least Squares

```
Input: Points $P = \{(x_1, y_1), \dots, (x_n, y_n)\}$, penalty $C$
Output: Optimal cost $OPT(n)$, segment boundaries
// Precompute error $e_{ij}$ for all segments;
for $i = 1$ to $n$ do
    for $j = i$ to $n$ do
        Compute $e_{ij}$ using least squares fit for $p_i$ to $p_j$;
    end
end
// Initialize DP array;
$OPT[0] = 0$;
for $j = 1$ to $n$ do
    $OPT[j] = \infty$;
    for $i = 0$ to $j - 1$ do
        $cost = OPT[i] + e_{i+1,j} + C$;
        if $cost < OPT[j]$ then
            $OPT[j] = cost$;
            $last[j] = i$ // Store the last break point;
        end
    end
end
// Reconstruct segments;
$segments = []$;
$j = n$;
while $j > 0$ do
    $i = last[j]$;
    Add segment from $p_{i+1}$ to $p_j$ to $segments$;
    $j = i$;
end
return $OPT[n], segments$
```

Consider $n = 6$ points: $(1, 1), (2, 2), (3, 4), (4, 3), (5, 2), (6, 1)$, with penalty $C = 1$.
Compute e_{ij} for key segments (simplified for brevity):

- $e_{1,2}$: Line for $(1, 1), (2, 2)$, $y = x$, error $= 0$.
- $e_{1,3}$: Line for $(1, 1), (2, 2), (3, 4)$, $y = 1.5x - 0.5$, error $= 0.5$.
- $e_{3,4}$: Line for $(3, 4), (4, 3)$, $y = -x + 7$, error $= 0$.
- $e_{4,6}$: Line for $(4, 3), (5, 2), (6, 1)$, $y = -x + 7$, error $= 0$.
- $e_{1,6}$: Line for all points, $y = 2.2 - 0.2x$, error $= 7.6$.

- $OPT(0) = 0$,
- $OPT(1) = e_{1,1} + C = 0 + 1 = 1$,

- $OPT(2) = \min(OPT(0) + e_{1,2} + C, OPT(1) + e_{2,2} + C) = \min(0 + 0 + 1, 1 + 0 + 1) = 1$ (segment $1 - 2$),
- $OPT(3) = \min(OPT(0) + e_{1,3} + C, OPT(2) + e_{3,3} + C) = \min(0 + 0.5 + 1, 1 + 0 + 1) = 1.5$ (segment $1 - 3$),
- $OPT(4) = \min(OPT(0) + e_{1,4} + C, OPT(2) + e_{3,4} + C, OPT(3) + e_{4,4} + C) = \min(2.67 + 1, 1 + 0 + 1, 1.5 + 0 + 1) = 2$ (segments $1 - 2, 3 - 4$),
- $OPT(5) = \min(\ldots) = 3$ (segments $1 - 2, 3 - 4, 5 - 5$),
- $OPT(6) = \min(OPT(0) + e_{1,6} + C, OPT(2) + e_{3,6} + C, OPT(3) + e_{4,6} + C) = \min(7.6 + 1, 1 + 1 + 1, 1.5 + 0 + 1) = 2.5$ (segments $1 - 3, 4 - 6$).

Optimal solution: Segments $1 - 3$ ($y = 1.5x - 0.5$) and $4 - 6$ ($y = -x + 7$), cost $= 0.5 + 0 + 2C = 2.5$.

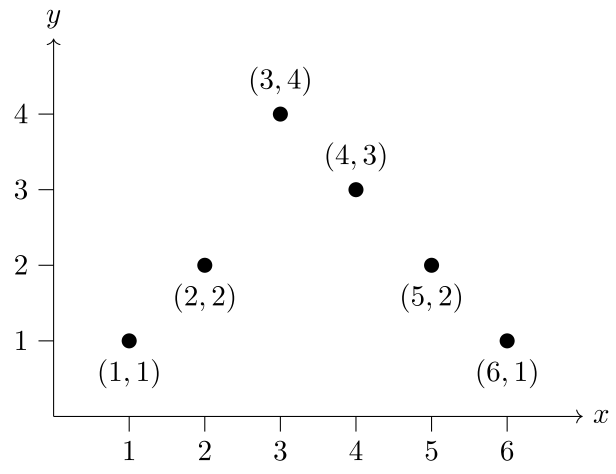


Figure 10.2: Input points $(1, 1), (2, 2), (3, 4), (4, 3), (5, 2), (6, 1)$ in the x-y plane.

10.7 Knapsack Problem

We are given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. We are also given a knapsack that has a capacity of W. Our goal is to determine the subset of items that can be carried in the knapsack and that maximizes the total value. The standard dynamic programming solution is based on the memoization of the following dynamic programming formulation [Vaz01, DPW10]. Let $G[i, w]$ be the maximum value that can be obtained by picking items from $1..i$ with the capacity constraint of w. Then, $G[i, w] = max(G[i - 1, w - w_i] + v_i, G[i - 1, w])$. The first argument of the max function corresponds to the case when the item i is included in the optimal set from $1..i$, and the second argument corresponds to the case when the item i is not included and hence the entire capacity can be used for the items from $1..i - 1$. If $w_i > w$, then the item i cannot be in the knapsack and can be skipped. The base cases are simple. The value of $G[0, w]$ and $G[i, 0]$ is zero for all w and i. Our goal is to find $G[n, W]$. By filling in the two-dimensional array G for all values of $0 \leq i \leq n$ and $0 \leq w \leq W$, we get an algorithm with time complexity $O(nW)$.

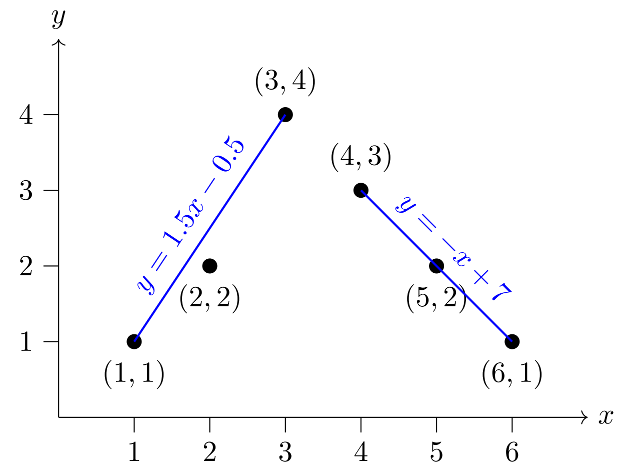


Figure 10.3: Segmented least squares solution: segments $1 - 3$ ($y = 1.5x - 0.5$) and $4 - 6$ ($y = -x + 7$).

The following algorithm computes $G[n, W]$ and tracks the selected items:

Algorithm Knapsack Algorithm: 0/1 Knapsack Dynamic Programming

```
Input: Weights $w[1..n]$, Values $v[1..n]$, Capacity $W$
Output: Maximum value and selected items
for $w = 0$ to $W$ do  $G[0, w] := 0$ ;
for $i = 0$ to $n$ do  $G[i, 0] := 0$ ;
for $i = 1$ to $n$ do
    for $w = 1$ to $W$ do
        if $w_i > w$ then  $G[i, w] := G[i-1, w]$ ;
        else  $G[i, w] := \max(G[i-1, w], G[i-1, w-w_i] + v_i)$ ;
    end
end
$selected := \emptyset$;
$i := n$;
$w := W$;
while $(i > 0) \wedge (w > 0)$ do
    if $G[i, w] \neq G[i-1, w]$ then
        $selected := selected \cup \{i\}$;
        $w := w - w_i$;
    end
    $i := i - 1$;
end
return $G[n, W], selected$;
```

Consider $n = 4$ items with weights $w = [2, 3, 4, 5]$, values $v = [3, 4, 5, 6]$, and capacity $W = 8$. The dynamic programming table $G[i, w]$ is computed as follows (key steps shown):

- $G[0, w] = 0$ for $w = 0$ to 8.

- $i = 1, w_1 = 2, v_1 = 3$:
  - $w = 2$: $G[1,2] = \max(0,3) = 3$
  - $w = 8$: $G[1,8] = 3$
- $i = 2, w_2 = 3, v_2 = 4$:
  - $w = 5$: $G[2,5] = \max(3,7) = 7$
  - $w = 8$: $G[2,8] = 7$
- $i = 3, w_3 = 4, v_3 = 5$:
  - $w = 8$: $G[3,8] = \max(7,9) = 9$
- $i = 4, w_4 = 5, v_4 = 6$:
  - $w = 8$: $G[4,8] = \max(9,9) = 9$

Final value: $G[4,8] = 9$. Backtracking: - $G[4,8] = G[3,8]$, item 4 not included. - $G[3,8] > G[2,8]$, item 3 included ($w = 8 - 4 = 4$). - $G[2,4] > G[1,4]$, item 2 included ($w = 4 - 3 = 1$). - $G[1,1] = 0$, item 1 not included.

Selected items: $\{2,3\}$, weight $= 3 + 4 = 7 \leq 8$, value $= 4 + 5 = 9$.

Knapsack Using LLP Formulation

We can model this problem using lattice-linear predicates as follows. We model the feasibility as $G[i,w] \geq \max(G[i-1, w-w_i] + v_i, G[i-1,w])$ for all $i, w > 0$ and $w_i \leq w$. Also, $G[i,w] = 0$ if $i = 0$ or $w = 0$. Our goal is to find the minimum vector G that satisfies the feasibility.

Lemma 10.5 *The constraint $B \equiv \forall i, w : G[i,w] \geq \max(G[i-1, w-w_i] + v_i, G[i-1,w])$ for $w_i \leq w$ is lattice-linear.*

Proof: If the predicate B is false, there exist i and w such that $G[i,w] < \max(G[i-1, w-w_i] + v_i, G[i-1,w])$. The value $G[i,w]$ is forbidden; unless $G[i,w]$ is increased, the predicate can never become true.

■

Algorithm LLP-Knapsack: Finding An Optimal Solution to the Knapsack Problem

```
1 $P_{i,j}$: Code for thread $(i,j)$
2 input: $w, v$:array[1..$n$] of int;// weight and value of each item
3 var: $G$:array[$0 \ldots n$, $0 \ldots W$] of int;
4 init: $G[i,j] = 0$ $if$ $(i = 0) \vee (j = 0)$;
5 ensure:
6 $G[i,j] \geq \max\{G[i-1, j-w_i] + v_i, G[i-1,j]\}$    if  $j \geq w_i$
7            $\geq G[i-1,j]$, otherwise.
```

Algorithm LLP-Knapsack updates the value of $G[i,j]$ based only on the values of $G[i-1,.]$. Furthermore, $G[i,j]$ is always at least $G[i-1,j]$. Based on this observation, we can simplify the algorithm as follows. We consider the problem of adding only one item to the knapsack given the constraint that the total weight does not exceed W. We maintain the list of all optimal configurations for each weight less than W.

Algorithm LLP-IncrKnapsack2: Finding An Optimal Solution to the Incremental Knapsack Problem

```
1 $P_j$: Code for thread $j$
2 input: $w, v$: int;// weight and value of the next item
3     $C$: array$[0 \ldots W]$ of int;
4 var $G$:array$[0 \ldots W]$ of int;
5 init $\forall j : G[j] = C[j]$;
6 ensure:
7     $G[j] \geq C[j-w] + v$   if  $j \geq w$
```

The incremental algorithm can be implemented in $O(1)$ parallel time using $O(W)$ processors as shown in Fig. LLP-IncrKnapsack2. Each processor j can check whether $G[j]$ needs to be advanced.

We can now invoke the incremental Knapsack algorithm as Algorithm Knapsack2.

Algorithm Knapsack2: Finding An Optimal Solution to the Knapsack Problem

```
1 $P_j$: Code for thread $j$
2 input: $w, v$:array$[1..n]$ of int;// weight and value of each item
3 var $G$:array$[0 \ldots W]$ of int;
4 init: $\forall j : G[j] = 0$;
5 for $i := 1$ to $n$ do
6     $G := IncrKnapsack2(w[i], v[i], G)$;
7 end
```

We now add some lattice-linear constraints to the knapsack problem. In many applications, some items may be related, and the constraint $x_a \Rightarrow x_b$ means that if the item x_a is included in the knapsack, then the item x_b must also be included. Thus, the item x_a has a profit of zero if x_b is not included. The item x_b has utility even without x_a but not vice versa. Without loss of generality, we assume that all weights are strictly positive and that index $b < a$. In the following lemma, we use an auxiliary variable $S[i, j]$ that keeps the set of items included in $G[i, j]$ and not just the profit from those items.

Lemma 10.6 *First, assume that $(i \neq a)$. Let $B(i, w) \equiv G[i, w] \geq \max(G[i-1, w-w_i] + v_i, G[i-1, w])$ for $(w_a \leq w)$ and $G[i, w] \geq G[i-1, w]$, otherwise. This predicate corresponds to any item i different from a. The value with a bag of capacity w is always greater than or equal to the choice of picking the item or not picking the item.*

Let $B(a, w) \equiv G[a, w] \geq \max(G[a-1, w-w_a] + v_a, G[a-1, w])$ if $b \in S[a-1, w-w_a] \wedge (w_a \leq w)$ and $G[a, w] \geq G[a-1, w]$, otherwise.

Then, $B(i, w)$ is lattice-linear for all i and w.

Proof: Suppose that $B(i, w)$ is false for some i and w. Unless $G[i, w]$ increases, it can never become true.

■

10.8 Problems

1. We are given a directed acyclic graph (V, E) with n nodes and m edges. Our goal is to assign a number, *label* to each vertex from $1..n$ such that for all edges $(i, j) \in E$, $label[i] < label[j]$. Define B as $\forall(i, j) \in E : label[j] \geq label[i] + 1$. Show that B is lattice-linear predicate.

2. Modify the algorithm for Optimal Binary Search Tree to return not only the optimal cost but also the tree itself.

3. **(Longest Path in Directed Acyclic Graphs)** We are given a directed acyclic graph (V, E) with n nodes and m edges such that each edge has a positive real cost. We are also given a distinguished vertex v_0. Give an LLP based algorithm to find the longest path from v_0 to all vertices.

4. We are given a polygon and we are required to triangulate it optimally. We are given a weight function that takes a triangle on the vertices v_i, v_j and v_k and returns a real-valued function $w(v_i, v_j, v_k)$. Our goal is to triangulate the given polynomial and triangulate it to minimize the sum of weights of all triangles. *(Hint: Devise a recurrence relation similar to the matrix chain product problem.)*

10.9 Bibliographic Remarks

This chapter 14 explores the paradigm of solving optimization problems by breaking them into overlapping subproblems, with applications to the Longest Increasing Subsequence (LIS), Optimal Binary Search Tree (OBST), Knapsack, Weighted Interval Scheduling, Segmented Least Squares, and Chain Matrix Multiplication. The chapter introduces dynamic programming (DP) as a method to establish recurrence relations, avoiding redundant computations through memoization or bottom-up computation, as originally formalized by Bellman [Bel52]. Cormen et al. [CLRS01] provide a comprehensive treatment of sequential DP algorithms for these problems.

The chapter contrasts traditional DP with the Lattice-Linear Predicate (LLP) algorithm, which models problems as searches for extremal vectors in a finite distributive lattice. This approach, detailed in Garg [Gar22], enables parallel computation with minimal synchronization overhead and supports lattice-linear constraints.

The OBST problem, introduced by Knuth [Knu71], minimizes expected search cost in a binary search tree, with the LLP approach achieving $O(n \log n)$ parallel time. The Knapsack problem, formulated in Horowitz and Sahni [HS74] and Ibarra and Kim [IK75], is solved in $O(nW)$ time via DP, with LLP reducing parallel time to $O(n)$ using W processors. Vazirani [Vaz01] and Williamson [DPW10] provide modern DP formulations for Knapsack, including approximation techniques.

Weighted Interval Scheduling and Segmented Least Squares, covered in Kleinberg and Tardos [KT06], extend greedy and DP techniques to optimization problems with weighted objectives and error minimization, respectively. The Chain Matrix Multiplication problem, structurally similar to OBST, is addressed using a recurrence akin to Knuth's formulation, as noted in Aho, Hopcroft, and Ullman [AHU74]. Papadimitriou and Steiglitz [PS82] offer additional context for combinatorial optimization problems like Knapsack and matrix multiplication.

Chapter 11

Max Flow

11.1 Introduction

The Ford-Fulkerson algorithm is a greedy algorithm that computes the maximum flow in a flow network. It was published in 1956 by L. R. Ford, Jr. and D. R. Fulkerson. The maximum flow problem is one of the most fundamental problems in network theory with diverse applications, ranging from traffic systems and communication networks to supply chains and project scheduling. Understanding how to efficiently compute the maximum flow allows optimizing resources and improving system performance.

This chapter provides a comprehensive overview of the Ford-Fulkerson algorithm, including its theoretical foundation, correctness proof, and complexity analysis. Additionally, we explore related concepts such as the Max-Flow Min-Cut Theorem and discuss strongly polynomial algorithms in the context of network flows.

Definition 11.1 (Flow Network) *A flow network is a directed graph $G = (V, E)$ with a source vertex $s \in V$, a sink vertex $t \in V$, and a capacity function $c : E \to \mathbb{R}^+$ that assigns a non-negative capacity $c(u, v)$ to each edge $(u, v) \in E$.*

Definition 11.2 (Flow) *A flow in a flow network G is a function $f : E \to \mathbb{R}^+$ that satisfies:*

1. *Capacity constraint: $\forall(u, v) \in E, 0 \leq f(u, v) \leq c(u, v)$*
2. *Flow conservation: $\forall v \in V \setminus \{s, t\}, \sum_{u:(u,v)\in E} f(u, v) = \sum_{w:(v,w)\in E} f(v, w)$*

Definition 11.3 (Value of Flow) *The value of a flow f is defined as the total flow out of the source:*

$$|f| = \sum_{v:(s,v)\in E} f(s, v)$$

Definition 11.4 (Residual Network) *Given a flow network $G = (V, E)$ and a flow f, the residual network $G_f = (V, E_f)$ consists of edges with remaining capacity, defined as:*

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

where $c_f(u, v)$ is the residual capacity defined as:

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Definition 11.5 (Augmenting Path) *An augmenting path is a simple path from the source s to the sink t in the residual network G_f.*

11.2 The Ford-Fulkerson Algorithm

The algorithm works by finding augmenting paths in the residual network and increasing the flow along these paths until no more augmenting paths can be found.

Algorithm Ford-Fulkerson Algorithm: Ford-Fulkerson Algorithm

```
Input: Graph $G$, source $s$, sink $t$
Output: Maximum flow $f$
1 Initialize flow $f(u,v) = 0$ for all edges $(u,v)$ in $G$;
2 while there exists an augmenting path $p$ from $s$ to $t$ in the residual network $G_f$ do
3     $c_f(p) := \min\{c_f(u,v) : (u,v) \in p\}$ be the residual capacity of path $p$;
4     for each edge $(u,v)$ in $p$ do
5         if $(u,v) \in E$ then $f(u,v) = f(u,v) + c_f(p)$ // Increase flow ;
6         else $f(v,u) = f(v,u) - c_f(p)$ // Decrease flow in the reverse edge ;
7     end
8 end
9 return $f$
```

Theorem 11.6 *The Ford-Fulkerson algorithm correctly computes the maximum flow in any flow network with finite capacities.*

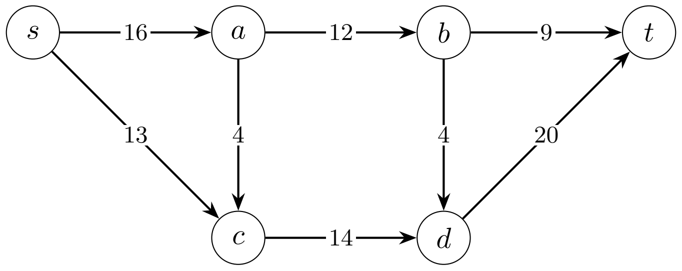
Proof: The algorithm repeatedly increases the flow along augmenting paths found in the residual graph. Each augmentation strictly increases the total flow value by at least the minimum residual capacity of the augmenting path. Since the capacities are finite and nonnegative, the total flow is bounded above.

The algorithm terminates when there are no more augmenting paths in the residual graph. By the Max-Flow Min-Cut Theorem, a flow is maximum if and only if there is no augmenting path in the residual graph. Therefore, when the algorithm terminates, the current flow is maximal.

■

Let us consider a simple flow network and demonstrate the Ford-Fulkerson algorithm step by step.

Consider a flow network with six vertices $\{s, a, b, c, d, t\}$ and the following capacities:

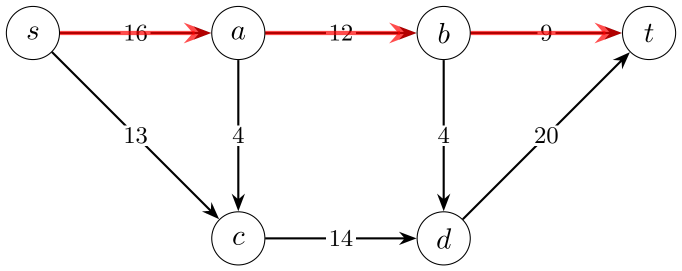


Initially, all flow values are 0. The edge labels show capacity.

First Iteration

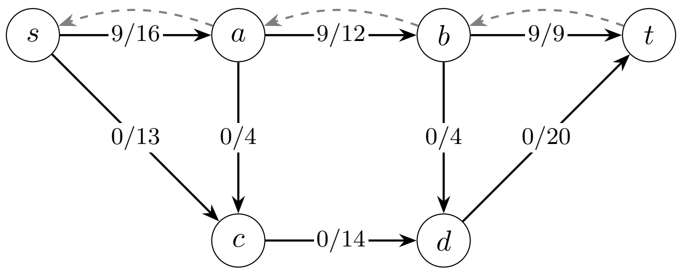
Finding the First Augmenting Path

We look for a path from s to t in the residual network. Let's choose the path $s \to a \to b \to t$.



The minimum residual capacity on this path is $\min\{16, 12, 9\} = 9$. We augment the flow by 9 units along this path.

Updated Flow After First Iteration

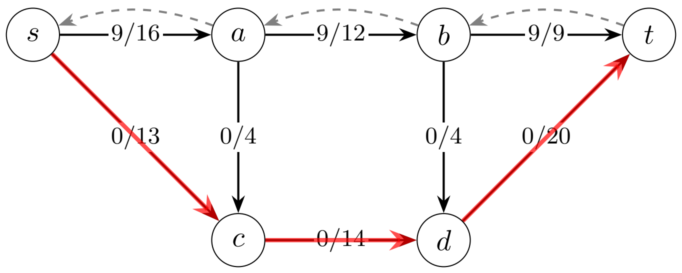


The flow value is now 9. The edge labels show flow/capacity. The residual network now includes reverse edges where the flow was pushed.

Second Iteration

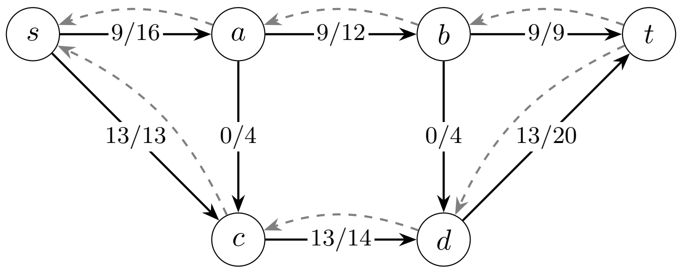
Finding the Second Augmenting Path

We find another path in the residual network: $s \to c \to d \to t$.



The minimum residual capacity on this path is $\min\{13, 14, 20\} = 13$. We augment the flow by 13 units.

Updated Flow After Second Iteration

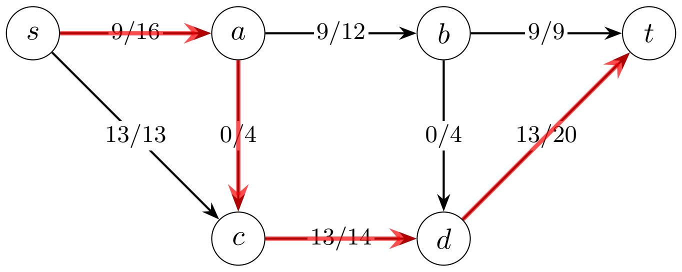


The flow value is now $9 + 13 = 22$.

Third Iteration

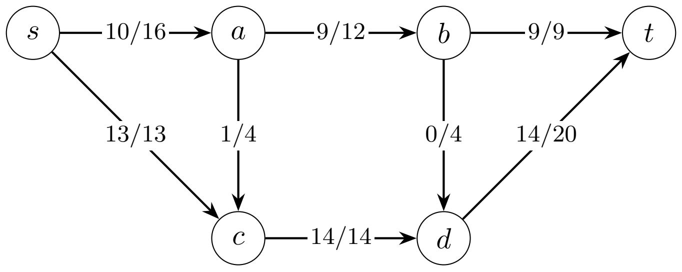
Finding the Third Augmenting Path

We find another path: $s \to a \to c \to d \to t$.



The minimum residual capacity on this path is $\min\{7, 4, 1, 7\} = 1$. We augment the flow by 1 unit.

Updated Flow After Third Iteration

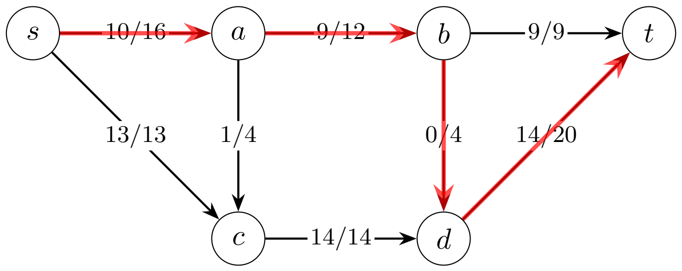


The flow value is now 10 + 13 = 23.

Fourth Iteration

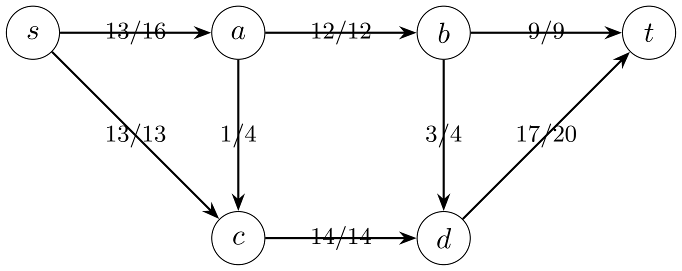
Finding the Fourth Augmenting Path

We find one more path: $s \to a \to b \to d \to t$.



The minimum residual capacity on this path is $\min\{6, 3, 4, 6\} = 3$. We augment the flow by 3 units.

Final Flow



The flow value is now 13 + 13 = 26. There are no more augmenting paths in the residual network, so the algorithm terminates. The maximum flow is 26.

11.3 Min-Cut Interpretation

Theorem 11.7 *In any flow network, the maximum value of a flow is equal to the minimum capacity of an s-t cut.*

Proof: Let f be a flow with maximum value and let (S, T) be any cut of the network, where $s \in S$ and $t \in T$. The value of the flow $|f|$ cannot exceed the capacity of the cut $c(S, T)$, since no more than $c(S, T)$ units of flow can cross from S to T.

When the Ford-Fulkerson algorithm terminates, there are no augmenting paths in the residual graph. Define S as the set of vertices reachable from s in the residual graph, and $T = V \setminus S$. Since no augmenting paths remain, all edges from S to T in the original graph are saturated, and all edges from T to S carry no flow.

Therefore, the value of the flow equals the capacity of the cut (S, T), proving that the maximum flow equals the minimum cut capacity.

■

According to the Max-Flow Min-Cut theorem, the value of the maximum flow equals the capacity of the minimum cut. In our example, we can identify the minimum cut by finding the set of vertices reachable from s in the final residual network.

Let us identify the cut. The set of vertices reachable from s in the final residual network is $\{s, a, c\}$. The remaining vertices $\{b, d, t\}$ form the other side of the cut.

The cut edges are:

- (a, b) with capacity 12
- (c, d) with capacity 14

The total capacity is $12 + 14 = 26$, which is exactly equal to the maximum flow, confirming the Max-Flow Min-Cut theorem.

Let us determine the running time of the Ford-Fulkerson algorithm. Suppose that all capacities in the graph are integral. Let f^* be the maximum flow in the graph. Every iteration of Ford-Fulkerson's algorithm finds an augmenting path in the residual network of at least one unit. This step takes $O(|E|)$ time. Hence, the total running time is $O(f^*|E|)$. This algorithm is pseudo-polynomial because it depends on the unary representation of the capacity C of the edges. It is desirable to get an algorithm with the time complexity that is dependent only on the number of vertices and the edges of the graph and not the capacity. For large capacities, this can lead to inefficiencies, motivating strongly polynomial algorithms like the Edmond-Karp and Dinitz algorithm.

11.4 Edmond-Karp Algorithm

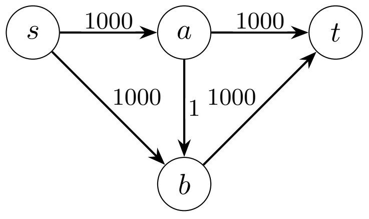
The Edmond-Karp modification improves the Ford-Fulkerson algorithm by selecting augmenting paths with the fewest possible edges, using a breadth-first search (BFS) to find the shortest paths in the residual graph. This ensures that each augmentation is performed efficiently.

This approach guarantees a strongly polynomial runtime of $O(|E|^2|V|)$, independent of the numerical values of capacities. The algorithm works in phases, where each phase builds a level graph and finds blocking flows, reducing the number of necessary augmentations. Overall, Edmonds-Karp's algorithm does at most $(|E||V|)$ augmentations and each augmentation takes at $O(|E|)$ time (via BFS) giving us the time complexity of $O(|E|^2|V|$.

By focusing on the shortest paths and limiting the number of augmentations, the Edmond-Karp algorithm provides significant performance improvements, especially for large-scale networks.

Consider a network with vertices s, a, b, t and edges: $s \to a$ with capacity 1000, $s \to b$ with capacity 1000, $a \to b$ with capacity 1, $a \to t$ with capacity 1000, $b \to t$ with capacity 1000.

In the basic Ford-Fulkerson algorithm, choosing poor augmenting paths may lead to many iterations due to the small capacity edge $a \to b$. In the example with vertices s, a, b, t, Ford-Fulkerson may choose paths like $s \to a \to b \to t$ repeatedly. Due to the bottleneck capacity of 1 on the edge $a \to b$, only 1 unit of flow is added per iteration. This leads to many redundant iterations before utilizing the larger capacity paths effectively. However, with Edmonds-Karp's method using BFS, the shortest paths are found, and the blocking flow quickly saturates the large-capacity edges in just a few phases, significantly reducing the total number of iterations.



Algorithm Edmonds-Karp Algorithm: Edmonds-Karp Algorithm

```
Input: Graph $G = (V, E)$, capacities $c$, source $s$, sink $t$
Output: Maximum flow $f$
1  Initialize $f(u, v) \leftarrow 0$ for all $(u, v) \in E$;
2  while there exists an augmenting path in $G_f$ do
3      Use BFS to find shortest path $p$ from $s$ to $t$ in $G_f$;
4      Compute $c_f(p) = \min_{(u,v)\in p} c_f(u, v)$;
5      for each edge $(u, v) \in p$ do
6          $f(u, v) \leftarrow f(u, v) + c_f(p)$;
7          $f(v, u) \leftarrow f(v, u) - c_f(p)$;
8      end
9  end
10 return $f$;
```

11.5 Lattice of Mincuts

In this section, we first show that the property of a cut (S, T) being a mincut is lattice-linear. A set $S \subseteq V$ is a mincut if:

- $s \in S$, $t \notin S$ (i.e., S is a cut).
- $\text{val}(S) = \sum_{(u,v)\in E: u\in S, v\in V\setminus S} c(u, v) \leq \text{val}(S')$ for all cuts S'.

We can now show:

Lemma 11.8 *The predicate $B(S) = (s \in S$ and $t \notin S)$ and (val(S) is minimum) is lattice-linear with efficient advancement.*

Proof: We use submodularity of the cut function. For min cuts S_1, S_2, the submodularity inequality is:

$$\text{val}(S_1) + \text{val}(S_2) \geq \text{val}(S_1 \cup S_2) + \text{val}(S_1 \cap S_2).$$

If $\text{val}(S_1) = \text{val}(S_2) = \mu$ (the min cut capacity), then $\text{val}(S_1 \cup S_2) + \text{val}(S_1 \cap S_2) \leq 2\mu$. Since $\text{val}(S) \geq \mu$ for any cut, each term must equal μ. Thus, $S_1 \cap S_2$ and $S_1 \cup S_2$ are min cuts (noting $s \in S_1 \cap S_2$, $t \notin S_1 \cup S_2$).

For the advancement property, suppose S is not a min cut (e.g., $\text{val}(S) > \mu$). If S equals $V - \{t\}$, then we know that there is no $S' \supseteq S$ such that S' is a mincut and thus all indices are forbidden. Otherwise, there is at least some vertex v such that there exists an edge (u, v), $u \in S$, $v \notin S$, with residual capacity $c(u, v) > f(u, v)$. If v equals t, we know that there is no $S' \supseteq S$ such that S' is a mincut and thus all indices are forbidden. Otherwise, unless we advance by including v in S, the cut can not be a mincut.

■

11.6 An Algorithm to Find Mincuts Satisfying a Lattice-Linear Predicate

Since finding a mincut satisfying B is NP-complete in general, we focus on special cases of B. We first assume that the given predicate B is lattice-linear. This means that the set of cuts that satisfy B form an inf-semilattice and we have efficient advancement property. Since the set of mincuts form a sublattice, we get that the set of mincuts that satisfy B also forms an inf-semilattice.

We now have a simple algorithm to find a mincut that satisfies B.

Algorithm LLP-MincutWithB: To find the minimum vector that satisfies B

```
vector function getLeastMincutSatisfyingPred( $B$: predicate)
var $G$: vector of reals initially $\forall i : G[i] = 0$;
$G := LeastMincut$
while $\neg B(G)$ do
    while $\exists j : forbidden(G, j, B)$ do
            if $(G[j] = 1)$ then return null;
            else $G[j] := 1$;
    endwhile;
    if there does not exist LeastMincut $\geq G$
        return null; //no such cut
    else $G :=$ LeastMincut $\geq G$
endwhile;
return $G$; // the optimal solution
```

The algorithm alternates between finding the mincut after G and finding G that satisfies B. If it cannot find a mincut or a cut satisfying B, it returns null. Assuming that the complexity of computing $\exists j : forbidden(G, j, B)$ is $O(m)$, the above algorithm takes $O(MF(n, m) + mn)$ given the join-irreducible elements of the mincut lattice. The outer while loop executes at most n times because at least one bit in the vector G becomes 1 after every iteration.

We now show that for many lattice-linear predicates, the complexity of the above algorithm is $O(MF(n, m) + kn)$ where k is a parameter dependent upon the predicate B.

To concretize our discussion, consider the predicate $B(S)$ that holds whenever either both vertices u and v belong to S or neither belong to S. We are interested in finding S such that S is a mincut and satisfies $B(S)$.

We note here that the predicate when evaluated over an increasing sets of S, can transition at most twice. Either it is initially false and then turns true. This happens when S initially has u or v, and then later it gets both. Alternatively, it may be initially true, then turn false and then finally turn true. This happens when initially, neither u nor v is in S. Then, one of them gets added to S and finally both of them get added.

The predicate B that can transition at most k times is termed as *k-Transition predicate.* Formally,

Definition 11.9 *A predicate $B : J(P) \to \{true, false\}$ on the ideals of a poset P is a* k-Transition Predicate *if, for any chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \in J(P)$, the sequence $B(I_1), B(I_2), \ldots, B(I_n)$ has at most k transitions, where a transition occurs at index i if $B(I_i) \neq B(I_{i+1})$.*

For example, the *stable* predicates [CL85] are 1-Transition predicates. Similarly, predicates that are true on a single mincut [GS24] are also 1-Transition predicates. We will assume that the algorithm for detecting whether B is true on a cut S is $O(m)$.

Thus, if the lattice-linear predicate is a k-transition predicate, then the time complexity is bounded by $O(MF(n,m) + km)$.

11.7 Problems

1. Consider the flow network below, with capacities on edges. Use the Ford-Fulkerson algorithm to compute the maximum flow from s to t. Show one possible sequence of augmenting paths and the final flow value.

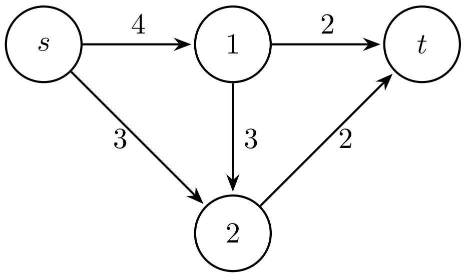


Figure 11.1: Flow network with capacities.

2. Let $G = (V, E)$ be a directed graph and let s and t be two nodes in V. Describe an efficient algorithm to find all the edge-disjoint paths from s to t in G. (Edge-disjoint means that none of the paths have any edges in common; they are allowed to have vertices in common.)

3. Suppose you are given a flow network $G = (V, E)$ with source s, sink t, integer capacities, and a maximum flow, f of G along with the residual graph G_f. There are n vertices and m edges in the flow network.

   (a) We increase the capacity of a single edge $(u, v) \in E$ by one. Give a $O(m + n)$ time algorithm to update the maximum flow.

   (b) We decrease the capacity of a single edge $(u, v) \in E$ by one. Give a $O(m + n)$ time algorithm to update the maximum flow.

11.8 Bibliographic Remarks

The maximum flow problem and its solutions have been extensively studied in combinatorial optimization and network theory. The original Ford-Fulkerson algorithm was introduced in 1956 [JF56], with the foundational Max-Flow Min-Cut Theorem proving its correctness. The Dinitz algorithm [Din70], and the algorithm by Karp and Edmonds [EK72], offered significant improvements with strongly polynomial time guarantees.

More advanced algorithms such as Goldberg and Tarjan's push-relabel method [GT88] provide even better performance in practice and theory, particularly for dense networks. For further reading and comprehensive treatments of flow algorithms, we refer to standard textbooks such as Ahuja, Magnanti, and Orlin [AMO93], Schrijver [Sch03], and Korte and Vygen [KV18].

Chapter 12

Bipartite Matching

12.1 Introduction

In this chapter, we discuss algorithms for matching problems in bipartite graphs. This chapter is organized as follows. Section 12.2 describes the classical sequential augmenting path algorithm. The algorithm allows one to increase the size of any matching by finding an augmenting path in the bipartite graph. Section 12.3 describes an algorithm to find the chain cover of a poset given a matching algorithm.

12.2 A Bipartite Matching Algorithm

Given an undirected graph (V, E), a *matching* M is a subset of edges E such that no two edges are incident on the same vertex. As an example, consider a bipartite graph (L, R, E) such that the vertex set L denotes a set of men, R denotes a set of women and E denotes a symmetric "likes" relation, Then, a matching M, is simply a set of edges such that no man or a woman is adjacent to two edges in M. If the number of men and the number of women are equal to n and every man is matched, then we call that matching a *perfect matching*. The perfect matching may not always exist. The following famous Theorem gives a necessary and sufficient condition for a perfect matching to exist in a bipartite graph. It is based on the notion of *overdemanded* subset. We say that $L' \subseteq L$ is overdemanded if there exists $R' \subseteq R$ such that the set of neighbors of R' in the graph is L' and the size of R' is strictly bigger than L'. Intuitively, L' is overdemanded because R' can only match with vertices in L' but the size of L' is smaller than the size of R'. We can now state the Theorem.

Theorem 12.1 (Hall's Theorem) *The necessary and sufficient condition for a perfect matching to exist in a bipartite graph (L, R, E) where $|L| = |R|$, is that L does not have any overdemanded subset L'.*

Observe that even though Hall's theorem gives us insight about the perfect matching, it cannot be directly applied to efficiently check for perfect matching because there are 2^n possible subsets of L.

We now give an efficient algorithm for a more general problem called the *maximum cardinality matching problem*. This problem requires us to find a matching of the largest size. When the number of men and the number of women are equal to n, then a perfect matching exists iff the size of the maximum cardinality matching is n. The algorithm is based on the idea of increasing the size of a matching by the idea of an

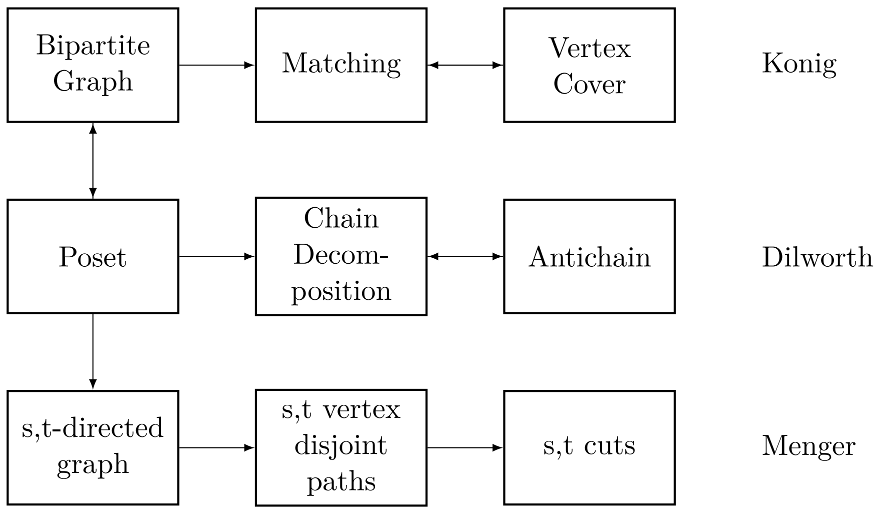


Figure 12.1: Various Structures and Transformations between them

augmenting path. Suppose that the algorithm has a matching M_t in iteration t. The algorithm is then guaranteed to find a matching M_{t+1} of size one bigger than M_t whenever a matching with a larger size exists. If we can make this idea work, then we can start with M_0 as the empty matching and then keep applying the idea to reach a maximum cardinality matching.

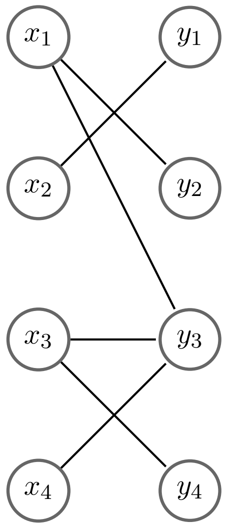


Figure 12.2: A Bipartite Graph

To understand the idea of an augmenting path for a matching M, we define a vertex as *exposed* if it is not incident on any edge in M. Now suppose that there exists a path from an exposed vertex to another exposed vertex that alternates between unmatched and matched edges. Such a path must start with an unmatched edge and end with an unmatched edge (by the definition of exposed vertices). Furthermore, such a path has an odd number of edges, with the unmatched edges exactly one more than the matched edges. Then, by simply switching the matched with unmatched edges along that path, we can increase the size of matching by one.

Algorithm Seq-AugmentingPath: Finding a maximum cardinality matching

```
1 $M := \{\}$;
2 while there exists an alternating path $P$ in $(L, R, E)$ for $M$ do
3     $M :=$ matching $M$ with edges switched in the path $P$
4 end
5 return $M$;
```

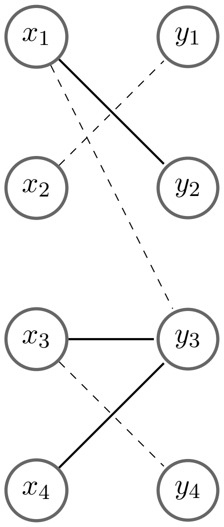


Figure 12.3: A Matching M shown with dashed edges in the Bipartite Graph

How do we find an alternating path? One can start with any exposed vertex and do a breadth-first-search such that the BFS tree alternates between unmatched and matched edges. Such a search will result in either finding an alternating path or an overdemanded set. In our example of the matching M in Fig. 12.3, if we start from x_4, we get the alternating path x_4, y_3, x_1, y_2. If there are $2n$ vertices and m edges, every iteration of the *while* loop takes $O(m)$ time for breadth-first-search and there can be at most n iterations, giving us the time complexity of $O(nm)$. By augmenting along multiple paths in every iteration, the complexity can be reduced to $O(\sqrt{n}m)$ [HK71].

We now give a sequential algorithm for a slight variant of the matching problem in a bipartite graph (L, R, E). Let the vertices of L be numbered $1..n$. Let L_i denote the set of vertices $\{v_1, v_2, \ldots, v_i\}$, for all $1 \leq i \leq n$. The output of our algorithm is a vector G such that $\sum_i G[i]$ is the size of the maximum matching in the graph (L_i, R, E_i) where E_i are the edges restricted to the set $L_i \cup R$. For simplicity, we set L_0 to $\emptyset$. We let S be the matched vertices in R. It is sufficient to describe the sequential algorithm when a new vertex v_i is added to the left side. We simply look for an alternating path from v_i to any unmatched vertex in R. If we find any such path, then the newly matched vertex in R is added to S. If there is no path, then L_i is a constricted set, and the set of vertices in R matched to L_{i-1} is identical to the set of vertices matched to L_i. In either case, we continue the procedure to the next vertex in L, if any.

Algorithm BipartiteMatching: A Sequential Algorithm for Bipartite Matching

```
input: $(L, R, E)$: bipartite graph
var: $G$: array[1..$n$] of int ;
init: $\forall j : G[j] := 0$;
$S$: array[1..$n$] of 0..$n$;
init: $\forall j : S[j] := 0$;
for $j := 1$ to $n$ do
    if there exists an alternating path from $v_j$ to any vertex $k$ in $R$ with $(S[k] = 0)$ then
        Assign $S[]$ in the alternating path;
        $G[j] := 1$;
    end
end
return $G$
```

Algorithm BipartiteMatching searches the element in the boolean lattice B_n such that $G[i]$ equals 1 if the size of the bipartite matching for (L_i, R, E_i) is one more than for (L_{i-1}, R, E_{i-1}), and 0 otherwise. The time complexity of the algorithm is $O(nm)$ because the *for* loop has n iterations and each iteration takes $O(m)$ to search for a path. The matching itself is stored in all nonzero S entries.

A key advantage of Algorithm BipartiteMatching is that it is a *deterministic* algorithm. Given a labeling of indices in L, it produces a single fixed G.

12.3 Chain Partition of a Poset

In this section, we show that maximum matching in a bipartite graph also allows us to partition a poset into the minimum number of chains.

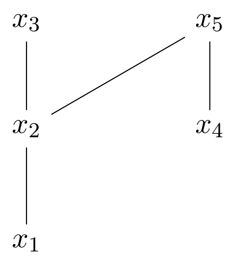


Figure 12.4: A poset

Assume that we have an algorithm for bipartite matching. How do we use it for chain decomposition? This relationship between chain decomposition and bipartite matching is based on the idea of a strict split of a poset.

Definition 12.2 (Strict Split of a Poset) *Given a poset P, the* **strict split** *of the poset P, denoted by $S(P)$, is a bipartite graph (L, R, E) where $L = \{x^-|x \in P\}$, $R = \{x^+|x \in P\}$, and $E = \{(x^-, y^+)|x < y \text{ in } P\}$.*

We state the following theorem due to Fulkerson [Ful56].

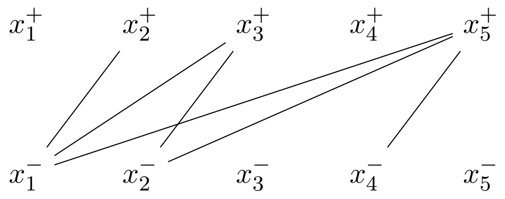


Figure 12.5: A Strict Split of the Poset in Fig. 12.4

Theorem 12.3 *[Ful56] Any matching in $S(P)$ can be reduced to a chain cover of P.*

Proof: We start with the trivial chain partition of P with each element in P being a chain. Thus, initially, we have n chains for n elements. We now consider each edge (x^-, y^+) in the matching. Each such edge implies that $x < y$ in P and we combine the chains corresponding to x and y. Since there can be at most one edge incident on x^- in a matching, x is the top element of its chain and similarly y is the lowest element of the chain. Therefore, these chains can be combined into one chain. If there are e edges in the matching, after this procedure we have a chain cover with $n - e$ chains.

■

Thus, any algorithm for bipartite matching can be used to determine a chain cover.

12.4 Problems

1. Show that the minimum size of a vertex cover is equal to the maximum size of a matching in a bipartite graph.
2. Give a linear program formulation for the maximum matching problem and show that its dual corresponds to the minimum vertex cover problem.
3. Prove Hall's Theorem.
4. Edmond's matrix is defined for a bipartite graph (U, V, E) with $|U| = |V| = n$ as follows. Let the vertices in U be $u_1, u_2, \ldots u_n$ and the vertices in V be $v_1, v_2, \ldots, v_n$. We set $A[i, j] = x_{i,j}$ if $(u_i, v_j) \in E$ and 0 otherwise. Show that Edmond's matrix of a balanced bipartite graph has a perfect matching iff the polynomial corresponding to the determinant of Edmond's matrix is not identically zero.
5. We are given an unweighted graph with two distinguished vertices s and t. We are required to find the total number of paths from s to t such that these paths do not share any edge. Give an algorithm to find this number, and justify its correctness. Suppose that the graph has n vertices and m edges, give the time complexity of your algorithm.
6. Modify Algorithm BipartiteMatching to output the vertex cover in addition to bipartite matching.

12.5 Bibliographic Remarks

The sequential augmenting path algorithm, detailed in Section 12.2, is a cornerstone of matching theory, achieving a time complexity of $O(nm)$ for a graph with n vertices and m edges. This algorithm is enhanced by the seminal work of Hopcroft and Karp [HK71], who introduced a method to augment multiple paths per iteration, reducing the complexity to $O(\sqrt{n}m)$. Their approach remains a standard for efficient bipartite matching. Hall's Theorem, presented in the chapter, is a fundamental result characterizing the existence of perfect matching, with roots in early combinatorial optimization. For a deeper exploration of matching theory, including Hall's Theorem and its extensions, Lovász and Plummer [LP86] offer a classic and authoritative reference. Section 12.3 connects bipartite matching to poset chain partitions through strict split construction, leveraging Fulkerson's theorem [Ful56]. Fulkerson's result elegantly reduces matchings in the bipartite graph $S(P)$ to chain covers in the poset P.

Chapter 13

Intractability

13.1 Introduction

This chapter discusses NP-completeness, a foundational concept in computational complexity that identifies the hardest problems in NP. We define P and NP, explore polynomial-time reductions, and prove NP-completeness for a broad set of problems: 3-SAT, Clique, Vertex Cover, Hamiltonian Path, Independent Set, and TSP. We also discuss methods to deal with the NP-complete problem.

The classes P and NP classify computational problems by their solvability and verifiability. P includes problems with polynomial-time deterministic algorithms, while NP includes those verifiable in nondeterministic polynomial time. The unresolved question $P = NP$ drives much of complexity theory.

The problems are specified as requiring an *output* for the given *input*. We will use n for the size of the input. Our interest would be in determining the time required to compute the output. The output we require would be *binary*. For example, our input may be a Boolean expression B on Boolean variables. We would require the output to be 1 if the Boolean expression is satisfiable and 0 otherwise. At first, it may seem that in practical applications, the user may be interested in finding a satisfying assignment rather than simply that B is satisfiable. However, if someone gave us a method f to efficiently check whether B is satisfiable, then we can use f to determine a satisfying assignment. If B is satisfiable, then we can easily compute another Boolean expression B' in which the Boolean variable x_1 is set to true. We now ask whether B' is satisfiable. If B' is satisfiable, we continue with the other variables. If B' is not satisfiable, then we know that x_1 must be false. We compute a Boolean expression B'' from B by setting x_1 to false. Thus, in n applications of f, we would have found a satisfying assignment. This idea is applicable to all problems considered in this chapter, and we will restrict ourselves to such problems.

13.2 Class P

The complexity class **P** is the set of all decision problems $L \subseteq \{0,1\}^*$ for which there exists a deterministic Turing machine M and a polynomial function $p : \mathbb{N} \to \mathbb{N}$ such that:

1. M halts on every input $x \in \{0,1\}^*$ in at most $p(|x|)$ steps, where $|x|$ denotes the length of x,

2. M accepts x if and only if $x \in L$.

Formally, $\mathbf{P} = \{L \mid$ there exists a deterministic Turing machine M and a polynomial p such that M decides
where n is the input size.

- $\mathbf{P}$: The complexity class of decision problems solvable in polynomial time by a deterministic Turing machine. The boldface emphasizes it as a formal class name.
- L: A language, representing a decision problem, which is a set of strings over the binary alphabet $\{0, 1\}$.
- M: A deterministic Turing machine, a theoretical model of computation with a single, fixed sequence of steps for any input. For our purposes, we can use any practical programming language such as Java or C++.
- p: A polynomial function $p : \mathbb{N} \to \mathbb{N}$, mapping natural numbers (input sizes) to natural numbers (time bounds).

Informally, a problem L is in P if we can write a program that determines if the given instance x is in L in time that is polynomial in the size of x. Most of the problems that we have seen in this book so far belong to $\mathbf{P}$.

13.3 Polytime Reductions

We define a relation between various problems as follows. We say that a problem π_1 is at least as hard as another problem π_2 if by using solutions to the problem π_2 we can solve the problem π_1 in time polynomial in the size of the problem. Formally, π_2 is *polynomially reducible* to π_1 if arbitrary instances of π_2 can be solved using a function that is a polynomial in the size of input and makes at most polynomial calls to a function that solves π_1. We denote this relation as $\pi_2 \leq_P \pi_1$. It is easy to verify that $\leq_P$ is a reflexive and transitive relation.

A consequence of the above definition is that if there exists an efficient algorithm to solve π_1, then we can use that solution to solve π_2. Another consequence, important to this chapter, is that if π_2 is *hard* to solve, then π_1 is also hard to solve.

Independent Set $\leq_P$ Vertex Cover

Let us show that the problem of vertex cover is harder than the problem of independent set in an undirected graph. Let $G = (V, E)$ be an undirected graph with the vertex set V and the edge set E.

The *Independent Set* problem asks: Given a graph G and an integer k, does there exist a subset $S \subseteq V$ of at least k vertices such that no two vertices in S are adjacent (i.e., $\forall u, v \in S, \{u, v\} \notin E$)? Such a set S is called an *independent set.*

The *Vertex Cover* problem asks: Given a graph G and an integer k, does there exist a subset $C \subseteq V$ of at most k vertices such that every edge in E is incident to at least one vertex in C (i.e., $\forall \{u, v\} \in E, u \in C$ or $v \in C$)? Such a set C is called a *vertex cover.*

We argue that the Vertex Cover problem is at least as hard as the Independent Set problem by providing a polynomial-time reduction from Independent Set to Vertex Cover. Given an instance $(G = (V, E), k)$ of Independent Set, construct an instance of Vertex Cover as follows:

- Use the same graph $G = (V, E)$.

- Set the parameter $k' = |V| - k$.

The reduction is polynomial-time since it only involves computing $|V|$ and subtracting k.

We claim that G has an independent set of size at least k if and only if G has a vertex cover of size at most $k' = |V| - k$.

- ($\Rightarrow$): Suppose $S \subseteq V$ is an independent set with $|S| \geq k$. Define $C = V \setminus S$. Since S is independent, no edge $\{u, v\} \in E$ has both $u, v \in S$. Thus, for every edge $\{u, v\} \in E$, at least one of u or v is in $C = V \setminus S$. Hence, C is a vertex cover. Moreover, $|C| = |V| - |S| \leq |V| - k = k'$.
- ($\Leftarrow$): Suppose $C \subseteq V$ is a vertex cover with $|C| \leq k' = |V| - k$. Define $S = V \setminus C$. For any edge $\{u, v\} \in E$, since C is a vertex cover, at least one of u or v is in C, so at most one is in S. But if both $u, v \in S$, then neither is in C, contradicting that C covers $\{u, v\}$. Thus, no edge connects vertices in S, so S is an independent set. Moreover, $|S| = |V| - |C| \geq |V| - k' = |V| - (|V| - k) = k$.

Since Independent Set is NP-complete, and we have reduced Independent Set to Vertex Cover in polynomial time, Vertex Cover is at least as hard as Independent Set (in the sense that solving Vertex Cover efficiently would imply solving Independent Set efficiently). Formally, Independent Set $\leq_p$ Vertex Cover.

Vertex Cover $\leq_P$Independent Set

Conversely, we argue that the Independent Set problem is at least as hard as the Vertex Cover problem by providing a polynomial-time reduction from Vertex Cover to Independent Set.

Given an instance $(G = (V, E), k)$ of Vertex Cover, construct an instance of Independent Set as follows:

- Use the same graph $G = (V, E)$.
- Set the parameter $k' = |V| - k$.

The reduction is polynomial-time, identical to the previous reduction.

Since we have reduced Vertex Cover to Independent Set in polynomial time, Independent Set is at least as hard as Vertex Cover. Formally, Vertex Cover $\leq_p$ Independent Set.

The bidirectional reductions show that Independent Set and Vertex Cover are computationally equivalent in terms of polynomial-time solvability: Independent Set $\leq_p$ Vertex Cover and Vertex Cover $\leq_p$ Independent Set. The reductions exploit the complementary relationship: S is an independent set if and only if $V \setminus S$ is a vertex cover.

Set Cover Problem: Vertex Cover $\leq_P$ Set Cover

We now show that the problem of *Set Cover* is harder than the vertex cover. The *Set Cover problem* is defined as follows: Given a universe U of n elements $\{e_1, e_2, \ldots, e_n\}$, a collection $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ of subsets of U such that $\bigcup_{i=1}^{m} S_i = U$, and an integer k, does there exist a subcollection $\mathcal{C} \subseteq \mathcal{S}$ of at most k subsets such that $\bigcup_{S_i \in \mathcal{C}} S_i = U$? In other words, we seek a cover of U using at most k subsets from $\mathcal{S}$.

It is easy to see that the vertex cover is a special case of set cover. Let $\{e_1, e_2, \ldots, e_n\}$ be the edges in an undirected graph. For each vertex v_i, we define a set S_i as all the edges that are incident on v_i. Then, choosing k sets to cover the universe is identical to choosing k vertices that cover all the edges. Hence, we conclude that Vertex Cover $\leq_P$ Set Cover.

Satisfiability and 3-SAT

So far, our problems were graph-theoretic. Let us now investigate problems that relate to satisfiability of boolean expression. A Boolean expression is said to be in Conjunctive Normal Form (CNF) if it is a conjunction of *clauses* where each clause is a disjunction of *literals*. A literal is any boolean variable or its negation. We call the problem of checking satisfiability of a boolean expression in CNF as the problem SAT. We define 3-SAT as the special case of SAT where each clause has exactly three literals. We now investigate the complexity of 3-SAT problem compared to the independent set problem.

SAT $\leq_P$ 3-SAT

It is clear that SAT is a harder problem than 3-SAT because 3-SAT is a special case of the SAT problem. Somewhat surprisingly, we now show that 3-SAT is as hard as SAT problem. Suppose that we are given an instance of the SAT problem. We convert it into 3-SAT problem as follows. Every clause with a single literal can be changed to four clauses with Given a CNF formula $\phi = C_1 \wedge \cdots \wedge C_m$, where $C_i = l_{i1} \vee \cdots \vee l_{ik_i}$, we construct a 3-CNF formula ψ such that ϕ is satisfiable if and only if ψ is satisfiable.

For each clause C_i:

- **Case 1:** $k_i = 1$ (e.g., (l_1)): Introduce dummy variables y_i, z_i. Replace with:
$$(l_1 \vee y_i \vee z_i) \wedge (l_1 \vee y_i \vee \neg z_i) \wedge (l_1 \vee \neg y_i \vee z_i) \wedge (l_1 \vee \neg y_i \vee \neg z_i).$$
- **Case 2:** $k_i = 2$ (e.g., $(l_1 \vee l_2)$): Introduce y_i. Replace with:
$$(l_1 \vee l_2 \vee y_i) \wedge (l_1 \vee l_2 \vee \neg y_i).$$
- **Case 3:** $k_i = 3$: Keep unchanged: $(l_1 \vee l_2 \vee l_3)$.
- **Case 4:** $k_i > 3$ (e.g., $(l_1 \vee \cdots \vee l_k)$): Introduce $y_{i,1}, \ldots, y_{i,k-2}$. Replace with:
$$(l_1 \vee l_2 \vee y_{i,1}) \wedge (\neg y_{i,1} \vee l_3 \vee y_{i,2}) \wedge \cdots \wedge (\neg y_{i,k-3} \vee l_{k-1} \vee l_k).$$

It is easy to verify that ϕ is true iff ψ is true. It is also easily verified that this reduction can be done in polynomial time.

3-SAT $\leq_P$ Independent set

We claim that 3-SAT $\leq_P$ Independent set. Suppose that someone gives us an instance of 3-SAT problem defined as follows: Given a Boolean formula ϕ in conjunctive normal form (CNF), where each clause contains exactly three literals, does there exist an assignment of truth values to the variables that satisfies ϕ? We construct a reduction from a 3-SAT instance ϕ with n variables and m clauses to an Independent Set instance $(G = (V, E), k)$ such that ϕ is satisfiable if and only if G has an independent set of size at least k. The graph G encodes the clauses and variable assignments, ensuring that selecting a valid independent set corresponds to a satisfying truth assignment. Given a 3-SAT formula ϕ with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$, where each clause $C_j = (l_{j1} \vee l_{j2} \vee l_{j3})$ and l_{jk} is a literal (x_i or $\neg x_i$), construct the graph $G = (V, E)$ and integer k as follows:

For each clause C_j ($j = 1, 2, \ldots, m$), create three vertices v_{j1}, v_{j2}, v_{j3}, one for each literal l_{j1}, l_{j2}, l_{j3}. These form a *clause gadget* for C_j, represented as a triangle (complete subgraph K_3). Thus, $V = \{v_{j1}, v_{j2}, v_{j3} \mid j = 1, 2, \ldots, m\}$. We construct edges as follows.

- **Within clause gadgets**: For each clause C_j, add edges to form a triangle: $\{v_{j1}, v_{j2}\}$, $\{v_{j2}, v_{j3}\}$, $\{v_{j3}, v_{j1}\}$. This ensures at most one vertex per clause is selected in an independent set.

- **Between clause gadgets**: For any two vertices v_{jk} and $v_{j'k'}$ ($j \neq j'$), add an edge $\{v_{jk}, v_{j'k'}\}$ if the literals l_{jk} and $l_{j'k'}$ are *inconsistent*. Two literals are inconsistent if one is x_i and the other is $\neg x_i$ for the same variable x_i.

Set $k = m$, the number of clauses. We claim that ϕ is satisfiable if and only if G has an independent set of size at least $k = m$.

- ($\Rightarrow$) : Suppose ϕ is satisfiable with a truth assignment $\tau : \{x_1, \ldots, x_n\} \to \{\text{true}, \text{false}\}$. For each clause $C_j = (l_{j1} \vee l_{j2} \vee l_{j3})$, at least one literal, say l_{jk}, is true under τ. Select vertex v_{jk} in G for clause C_j. Construct $S = \{v_{jk} \mid l_{jk} \text{ is true in } C_j, j = 1, \ldots, m\}$, choosing one vertex per clause. Thus, $|S| = m$. It is easily verified that S is independent:

- ($\Leftarrow$): Suppose G has an independent set S with $|S| \geq m$. Since each clause gadget is a triangle, at most one vertex per clause can be in S (vertices v_{j1}, v_{j2}, v_{j3} are pairwise adjacent). With m clauses and $|S| \geq m$, S must include exactly one vertex from each clause's triangle, so $|S| = m$. For each C_j, let $v_{jk} \in S$. Assign truth values to make l_{jk} true:

  - If $l_{jk} = x_i$, set $x_i = \text{true}$.
  - If $l_{jk} = \neg x_i$, set $x_i = \text{false}$.

  This assignment is consistent because if $x_i = \text{true}$ and $x_i = \text{false}$ were assigned (via $v_{jk} = x_i$ and $v_{j'k'} = \neg x_i$ in S), then $\{v_{jk}, v_{j'k'}\} \in E$, contradicting that S is independent. Thus, the assignment satisfies each clause C_j (since l_{jk} is true), making ϕ satisfiable.

13.4 NP-Complete Problems

In this section, we define the class of problems in NP and NP-Complete. As we saw earlier, the class P includes problems that are solvable in $O(n^k)$ time with input size n for some fixed k. The class NP includes problems, where a *yes* certificate of the problem is verifiable in $O(n^k)$. What is a *yes* certificate? It is an input for all the *yes* instances of the problem that is at most polynomial in size of n and certifies that the given instance is indeed an *yes* instance. For the satisfiability problem, the assignment to the Boolean variables gives us a *yes* certificate. The assignment itself is of size n when the boolean expression is on n variables. Furthermore, by substituting the variables using the assignment in the certificate, we can verify in polynomial time that the boolean expression is indeed true. As another example, consider the subset sum problem. We are given a set, S, of integers, another integer t, and we are asked if there is a subset S' such that the subset S' sums to t. Then, the *yes* certificate is simply the set S'. The following polynomial algorithm can be used to check that the certificate is indeed a *yes* certificate.

Algorithm SubsetSumVerify: SubsetSumVerify(S, t)

```
Input  : Set $S$ of integers, target $t$
Output: True if a subset sums to $t$, False otherwise
Input  : Certificate: Subset $S' \subseteq S$
int $sum := 0$ ;
forall $x$ in $S'$ do
    $sum := sum + x$ ;
end
if $sum = t$ then return true ;
else return false;
```

Observe that we ask for certification for only *yes* instances of the problem.

We now claim that

$$P \subseteq NP.$$

Given any problem π in P, there exists a polynomial time algorithm to solve π. Hence, certification is trivial. It could be just one bit specifying whether π is a *yes* instance or not. Since the problem itself is in polynomial class, we can simply run the algorithm as a checker to verify that the problem is a *yes* instance.

This naturally leads to the problem whether the containment is strict, i.e., is there any problem that is in NP but not in P. In other words, is there a problem for which a solution can be checked efficiently but computing the solution is inefficient. This is the most fundamental problem of the theory of computation.

A problem is *NP-complete* if it is in NP and every NP problem reduces to it in polynomial time. Reductions transform instances while preserving answers, establishing relative hardness.

Theorem 13.1 (Cook-Levin Theorem) *SAT is NP-complete.*

Proof: *Sketch*: SAT is in NP (verify assignment in $O(n)$). Any Nondeterministic computation can be encoded as a satisfiable Boolean formula in polynomial time, reducing all NP problems to SAT.

■

The following figure illustrates the relationship between P, NP-complete, and NP. P is a subset of NP, as polynomial-time solvable problems are verifiable in polynomial time. NP-complete problems are the hardest in NP, with every NP problem reducing to them in polynomial time. If P $\neq$ NP, NP-complete problems are disjoint from P.

3-SAT $\leq_P$ Clique

As yet another reduction, we show that the Clique problem is at least as hard as 3-SAT problem. Given $G = (V, E)$ and integer k, does G have a k-clique?

We construct a polynomial-time reduction from 3-SAT to Clique, showing that a 3-SAT formula ϕ is satisfiable if and only if the constructed graph G has a clique of size at least k. Given a 3-SAT formula ϕ with n variables $x_1, x_2, \ldots, x_n$ and m clauses $C_1, C_2, \ldots, C_m$, where each clause $C_j = (l_{j1} \vee l_{j2} \vee l_{j3})$ and l_{jk} is a literal (x_i or $\neg x_i$), construct an instance $(G = (V, E), k)$ of Clique as follows:

1. **Vertices**: For each literal l_{jk} in clause C_j (i.e., the k-th literal in clause j, where $k = 1, 2, 3$), create a vertex v_{jk}. Thus, $V = \{v_{jk} \mid j = 1, \ldots, m, k = 1, 2, 3\}$, with $|V| = 3m$.

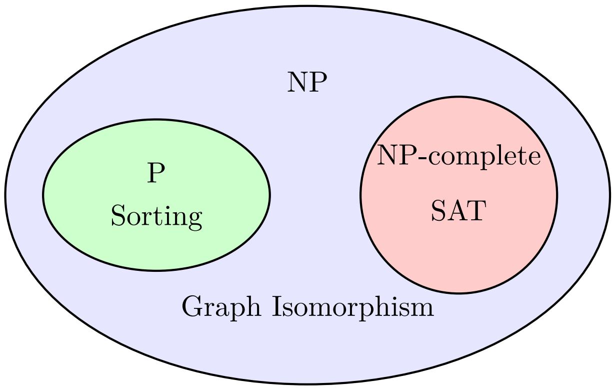


Figure 13.1: Relationship between P, NP-complete, and NP. P is a subset of NP, NP-complete problems are a subset of NP (disjoint from P if P $\neq$ NP), and NP contains problems with polynomial-time verifiable yes instances.

2. **Edges**: Add an edge $\{v_{jk}, v_{j'k'}\}$ between vertices v_{jk} and $v_{j'k'}$ (where $j \neq j'$) if:

   - The literals l_{jk} and $l_{j'k'}$ are *consistent*, meaning they do not contradict each other (e.g., $l_{jk} = x_i$ and $l_{j'k'} = \neg x_i$ is inconsistent, so no edge is added).

   No edges are added between vertices within the same clause ($j = j'$), as they represent literals in the same clause.

3. **Parameter**: Set $k = m$, the number of clauses.

We claim that ϕ is satisfiable if and only if G has a clique of size at least $k = m$.

- ($\Rightarrow$): Suppose ϕ is satisfiable with a truth assignment $\tau : \{x_1, \ldots, x_n\} \to \{\text{true}, \text{false}\}$. For each clause C_j, at least one literal l_{jk} is true under τ. Select one such vertex v_{jk} for each clause C_j, forming $S = \{v_{jk} \mid l_{jk} \text{ is true}, j = 1, \ldots, m\}$. Since we pick one vertex per clause, $|S| = m$. Check if S is a clique:

  – For $v_{jk}, v_{j'k'} \in S$ $(j \neq j')$, an edge exists if l_{jk} and $l_{j'k'}$ are consistent. Since τ is a valid assignment, l_{jk} and $l_{j'k'}$ are both true, so they cannot be inconsistent (e.g., x_i and $\neg x_i$). Thus, $\{v_{jk}, v_{j'k'}\} \in E$.

  – No vertices from the same clause are in S, so intra-clause edges are irrelevant.

  Hence, S is a clique of size $m = k$.

- ($\Leftarrow$): Suppose G has a clique S with $|S| \geq k = m$. Since vertices from the same clause are not connected (no edges within C_j), S can include at most one vertex per clause. With $|S| \geq m$ and m clauses, S must include exactly one vertex per clause, so $|S| = m$. For each clause C_j, let $v_{jk} \in S$, corresponding to literal l_{jk}. Construct a truth assignment τ:

  – If $l_{jk} = x_i$, set $x_i = \text{true}$.

  – If $l_{jk} = \neg x_i$, set $x_i = \text{false}$.

The assignment is consistent because if $x_i = \text{true}$ (from $v_{jk} = x_i$) and $x_i = \text{false}$ (from $v_{j'k'} = \neg x_i$), then l_{jk} and $l_{j'k'}$ are inconsistent, so $\{v_{jk}, v_{j'k'}\} \notin E$, contradicting that S is a clique. Thus, τ is consistent, and since each clause C_j has a true literal l_{jk}, ϕ is satisfiable.

3D-Matching $\leq_P$ Subset Sum

As yet another reduction, we show that the subset sum problem is harder than the 3D-Matching problem. The *Subset Sum problem* asks whether, given a set of positive integers $S = \{a_1, \ldots, a_n\}$ and a target t, there exists a subset $S' \subseteq S$ summing to t. We prove Subset Sum is NP-complete by showing it is in NP and NP-hard via a polynomial-time reduction from *3D-Matching* (3DM), using only $3n$ digits in base $m+1$ to ensure carry-free addition. It is easy to verrify that the Subset Sum problem is in NP. We reduce 3DM to Subset Sum. Given a 3DM instance with sets $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$, $Z = \{z_1, \ldots, z_n\}$, and triples $T = \{t_1, \ldots, t_m\} \subseteq X \times Y \times Z$, we construct a Subset Sum instance (S, t) with $3n$ digits in base $m + 1$, such that a perfect matching of size n exists if and only if a subset sums to t. Our construction is as follows.

- **Elements**: $3n$ elements: $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$.
- **Base**: Use base $b = m + 1$, where $m = |T|$. Each element appears in at most m triples, so sums in each digit position are at most $n \leq m < b$, ensuring no carry.
- **Numbers**: Create a number a_j for each triple $t_j = (x_{j1}, y_{j2}, z_{j3})$, with $3n$ digits corresponding to elements. In base b:
$$a_j = b^{3n-j_1} + b^{2n-j_2} + b^{n-j_3},$$
where digits for x_{j1}, y_{j2}, z_{j3} (positions $3n - j_1 + 1$, $2n - j_2 + 1$, $n - j_3 + 1$) are 1, others 0.
- **Target**:
$$t = \sum_{k=1}^{n} (b^{3n-k} + b^{2n-k} + b^{n-k}).$$
In base b, t has 1 in each of the $3n$ digit positions.
- **Set** S: $S = \{a_1, \ldots, a_m\}$.

If 3DM has a matching: Let $M \subseteq T$ be a matching of n triples, covering each element exactly once. Then, we select $\{a_j \mid t_j \in M\}$. Each element appears in one triple, so each digit position sums to 1. The sum $\sum_{j \in M}(b^{3n-j_1} + b^{2n-j_2} + b^{n-j_3}) = t$, as each digit is 1 exactly once. Conversely, suppose that the Subset Sum has a solution. Let $S' \subseteq S$ sum to t. t has 1 in each of $3n$ digits. Since each a_j has three 1s, and sums are carry-free, exactly n numbers are chosen ($3n$ ones require n triples). Each element digit is 1, so each element is covered exactly once. The n triples form a perfect matching.

We now list other problems which have been shown to be NP-complete. Table 13.2 summarizes six NP-complete decision problems: 3-Colorability, 3D-Matching, Knapsack, Hamiltonian Path, Hamiltonian Cycle, and Traveling Salesman. For each problem, we provide its definition and identify a known NP-complete problem that can be polynomially reduced to it to prove NP-completeness. Each problem is in NP (solutions are verifiable in polynomial time), and NP-hardness is established via the specified reduction.

1. **3-Colorability**:

Problem	**Definition**	**Reduction From**
3-Colorability	Graph G 3-colorable?	SAT
3D-Matching	Perfect matching in $X \times Y \times Z$?	3-SAT
Knapsack	Items fit W, value $\geq V$?	Subset Sum
Hamiltonian Path	Path visits all vertices once?	3-SAT
Hamiltonian Cycle	Cycle visits all vertices once?	Hamiltonian Path
Traveling Salesman	Tour with weight $\leq B$?	Hamiltonian Cycle

Figure 13.2: NP-Complete Problems and Reductions

- *Definition*: Given a graph $G = (V, E)$, can the vertices be colored with three colors such that no adjacent vertices share the same color?
- *Reduction From*: SAT. Construct a graph with gadgets for variables (true/false colors) and clauses (ensuring at least one true literal), where a 3-coloring encodes a satisfying assignment.

2. **3D-Matching**:
   - *Definition*: Given sets X, Y, Z, each of size n, and triples $T \subseteq X \times Y \times Z$, is there a subset of n triples covering each element exactly once?
   - *Reduction From*: 3-SAT. Create variable gadgets (true/false triples) and clause gadgets (triples for true literals), where a matching corresponds to a satisfying assignment.
3. **Knapsack**:
   - *Definition*: Given items with weights w_i, values v_i, capacity W, and target V, is there a subset with weight $\leq W$, value $\geq V$?
   - *Reduction From*: Subset Sum. Set $v_i = w_i$, $W = t$, $V = t$, making Knapsack equivalent to Subset Sum.
4. **Hamiltonian Path**:
   - *Definition*: Given a graph $G = (V, E)$, is there a path visiting each vertex exactly once?
   - *Reduction From*: 3-SAT. Build a graph with variable paths (true/false) and clause nodes, where a Hamiltonian path encodes a satisfying assignment.
5. **Hamiltonian Cycle**:
   - *Definition*: Given a graph $G = (V, E)$, is there a cycle visiting each vertex exactly once?
   - *Reduction From*: Hamiltonian Path. Add a vertex connected to all others, where a Hamiltonian cycle implies a Hamiltonian path in the original graph.
6. **Traveling Salesman**:
   - *Definition*: Given a complete graph with edge weights and bound B, is there a tour of weight $\leq B$?
   - *Reduction From*: Hamiltonian Cycle. Set weights: 1 for original edges, 2 otherwise, $B = n$. A tour of weight n is a Hamiltonian cycle.

13.5 co-NP Class of Problems

In computational complexity theory, the class co-NP plays a crucial role alongside NP in understanding the structure of decision problems. The complexity class co-NP consists of all decision problems whose *complements* are in NP. Formally, a decision problem $L \subseteq \{0,1\}^*$ is in co-NP if its complement $\overline{L} = \{0,1\}^* \setminus L$ is in NP. Equivalently, a problem L is in co-NP if there exists a polynomial-time verifiable certificate for *no* instances (i.e., instances $x \notin L$). This means there is a polynomial-time Turing machine M and a polynomial p such that for every input $x \notin L$, there exists a certificate c of length at most $p(|x|)$ where $M(x,c) = 1$, and for every $x \in L$, no such certificate exists.

In contrast, a problem L is in NP if there exists a polynomial-time verifiable certificate for *yes* instances ($x \in L$). Thus, co-NP is the class of problems where the evidence for rejection is efficiently verifiable, while NP focuses on evidence for acceptance.

The relationship between co-NP and NP is symmetric due to their complementary definitions:

- If $L \in$ NP, then $\overline{L} \in$ co-NP.
- If $L \in$ co-NP, then $\overline{L} \in$ NP.

This symmetry implies that NP and co-NP are *dual* classes. Notably:

- P $\subseteq$ NP$\cap$co-NP, since if a problem is in P, both its yes and no instances can be decided in polynomial time, providing trivial certificates.
- NP $\cap$ co-NP contains problems like Integer Factorization (deciding if a number n has a factor d with $1 < d < n$), where both existence (NP) and non-existence (co-NP) of factors are verifiable with certificates (a factor or a primality certificate).

A central question is whether NP = co-NP. If NP = co-NP, then for every NP problem, its complement would also have efficiently verifiable yes certificates, implying a symmetry in verification. However, it is widely believed that NP $\neq$ co-NP, as their equality would collapse the polynomial hierarchy and have profound implications for problems like SAT and UNSAT.

The following figure illustrates the relationship between NP, co-NP, and P. P is explicitly drawn as a distinct region within the intersection of NP and co-NP, representing problems solvable in polynomial time. NP and co-NP are dual classes, where a problem is in co-NP if its complement is in NP. Their intersection, NP $\cap$ co-NP, contains P and problems like Integer Factorization, not known to be in P.

We present several examples of problems in co-NP, focusing on their complements in NP and their verification mechanisms.

- The *Tautology problem* asks: Given a Boolean formula ϕ in disjunctive normal form (DNF), is ϕ true for all possible truth assignments (i.e., is ϕ a tautology)? The complement is: Is ϕ not a tautology? This is in NP because a no instance (non-tautology) has a certificate: a truth assignment making ϕ false, verifiable in polynomial time by evaluating ϕ. Thus, Tautology is in co-NP.
- The *Unsatisfiability (UNSAT) problem* asks: Given a Boolean formula ϕ in conjunctive normal form (CNF), is ϕ unsatisfiable (i.e., no truth assignment makes ϕ true)? The complement is SAT (Satisfiability), which is in NP: a yes instance of SAT has a certificate (a satisfying assignment) verifiable in polynomial time. Thus, UNSAT is in co-NP.

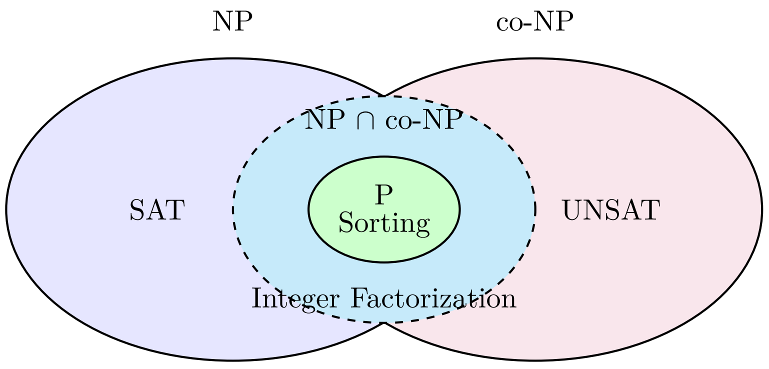


Figure 13.3: Relationship between NP, co-NP, and P. P is drawn within NP ∩ co-NP, which also includes problems like Integer Factorization. It is unknown if NP = co-NP.

- The *Non-Isomorphism problem* for graphs asks: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, are G_1 and G_2 not isomorphic (i.e., there exists no bijection $f : V_1 \to V_2$ preserving edges)? The complement, Graph Isomorphism, is in NP: a yes instance has a certificate (a bijection f) verifiable in polynomial time by checking edge preservation. Thus, Non-Isomorphism is in co-NP. It is not known if Non-Isomorphism is co-NP-complete.

Several open problems in complexity theory involve co-NP, reflecting its central role in understanding computational limits. We highlight key questions:

- Does NP = co-NP? The most significant open problem is whether NP = co-NP. If NP = co-NP, then for every NP-complete problem like SAT, its complement (UNSAT) would also be in NP, implying efficient certificates for unsatisfiability. This would collapse the polynomial hierarchy, as the hierarchy relies on the distinction between NP and co-NP. Most researchers believe NP $\neq$ co-NP due to the intuitive asymmetry: verifying satisfiability (NP) seems easier than verifying unsatisfiability (co-NP).

- Which problems lie in NP ∩ co-NP? Known examples include Primality and Integer Factorization, but identifying others (e.g., Graph Isomorphism) is open. It is conjectured that NP ∩ co-NP $\neq$ NP and NP ∩ co-NP $\neq$ co-NP, but the boundaries are unclear.

13.6 Approximation Algorithms

Approximation algorithms offer polynomial-time solutions for NP-complete optimization problems.

Example: Vertex Cover Approximation

```
Algorithm ApproxVertexCover: ApproxVertexCover(G)
Input  : Graph $G = (V, E)$
Output: Vertex cover $C$
$C := \emptyset$ ;
$E' := E$ ;
while $E' \neq \emptyset$ do
    Pick any edge $(u, v) \in E'$ ;
    $C := C \cup \{u, v\}$ ;
    Remove all edges incident to $u$ or $v$ from $E'$ ;
end
return $C$
```

Time: $O(m)$.
Ratio: 2-approximation.
It is easy to see that each edge requires at least one vertex in OPT. The algorithm picks two per iteration, so $|C| \leq 2 \cdot |OPT|$.

13.7 Problems

1. You are given a graph $G = (V, E)$ with non-negative lengths $w_e \geq 0$ for all edges $e \in E$. You are also given a source node s and destination node t and an integer K. Show that deciding whether there is an s-t path of length exactly K is NP-complete.

2. Consider the *Job Scheduling with Deadlines* problem: Given n jobs with processing times p_i, deadlines d_i, and profits r_i, and a single processor, select a subset of jobs to maximize total profit such that each selected job completes by its deadline (job i takes p_i time units, must finish by d_i, and jobs are non-preemptive). Prove this problem is NP-complete.

13.8 Bibliographic Remarks

The Cook-Levin Theorem (Theorem 13.1), independently established by Cook [Coo71] and Levin [Lev73], marks SAT as the first NP-complete problem, showing that every NP problem can be reduced to SAT in polynomial time. This result laid the groundwork for identifying other NP-complete problems, as detailed in the chapter's reductions (e.g., SAT $\leq_P$ 3-SAT, 3-SAT $\leq_P$ Independent Set, 3-SAT $\leq_P$ Clique). Karp [Kar72] significantly expanded this landscape by proving NP-completeness for 21 fundamental problems, including Clique, Vertex Cover, and Hamiltonian Path, many of which are covered in the chapter's Figure 13.2. For a comprehensive treatment of NP-completeness and its reductions, Garey and Johnson [GJ79] remains a definitive reference.

Polynomial-time reductions, as illustrated in Sections 13.2 and 13.3, are central to establishing the hardness of problems like Independent Set, Vertex Cover, and 3-SAT. These reductions exploit structural relationships, such as the complementary nature of Independent Set and Vertex Cover, to demonstrate computational equivalence. The chapter's treatment of co-NP (Section 13.4) introduces the dual class of problems with efficiently verifiable no certificates, with examples like UNSAT and Tautology. Papadimitriou [Pap94] offers an authoritative resource on complexity classes, including NP, co-NP, and their intersections.

Chapter 14

The Housing Allocation Problem

14.1 Introduction

The housing market problem proposed by Shapley and Scarf [SS74] is a matching problem with one-sided preferences. There are n agents and n houses. Each agent a_i initially owns a house h_i for $i \in \{1, n\}$ and has a completely ranked list of houses. There are variations of this problem when the agents do not own any house initially. In this paper, we focus on the version with the initial endowment of houses for agents. The list of agent preferences is given by $pref[i][k]$ which specifies the k^{th} preference of the agent i. Thus, $pref[i][1] = j$ means that a_i prefers h_j as his top choice. The goal is to come up with an optimal house allocation such that each agent has a house, and no subset of agents can improve the satisfaction of agents in this subset by exchanging houses within the subset. It can be shown that there is a unique such matching called the *core* for any housing market. The standard algorithm for this problem is Gale's Top Trading Cycle Algorithm (TTC), which takes $O(n^2)$ time. This algorithm is optimal in terms of time complexity, since the input size is $O(n^2)$.

Section 14.2 gives Gale's Top Trading Cycle Algorithm. Section 14.3 applies the LLP method to the Housing market problem.

14.2 Gale's Top Trading Cycle Algorithm

Consider the housing market instance shown in Fig. 14.1. There are four agents a_1, a_2, a_3 and a_4. Initially, the agent a_i holds the house h_i. The preferences of the agents are shown in Fig. 14.1.

The Top Trading Cycle (TTC) algorithm attributed to Gale by Shapley and Scarf [SS74] works in stages. At each stage, it has the following steps:

Step 1. We construct the *top choice* directed graph $G_t = (A, E)$ on the set of agents A as follows. We add a directed edge from agent $a_i \in A$ to agent $a_j \in A$ if a_j holds the current top house of a_i. Fig. 14.2 shows the directed graph in the first stage.

Step 2. Since each node has exactly one outgoing edge in G_t, there is at least one cycle in the graph (possibly a self-loop). All cycles are node disjoint. We find all the cycles in the top trading graph and implement the trade indicated by the cycles, that is, each agent that is in any cycle gets its current top house.

$a_1 : h_2, h_3, h_1, h_4$	$a_1 : h_1$	$a_1 : h_2$
$a_2 : h_1, h_4, h_2, h_3$	$a_2 : h_2$	$a_2 : h_1$
$a_3 : h_1, h_2, h_4, h_3$	$a_3 : h_3$	$a_3 : h_4$
$a_4 : h_2, h_1, h_3, h_4$	$a_4 : h_4$	$a_4 : h_3$
Agents' Preferences	Initial Allocation	Matching returned by the TTC algorithm

Figure 14.1: Housing Market and the Matching returned by the Top Trading Cycle Algorithm

Step 3. Remove all agents who get their current top houses, and remove all houses which are assigned to some agent from the preference list of remaining agents.

The above steps are repeated until each agent is assigned a house. At each stage, at least one agent is assigned a final house. Thus, this algorithm takes $O(n)$ stages in the worst case and requires $O(n^2)$ computational steps.

$$\begin{tikzcd}
a_1 \arrow[r, bend right] & a_2 \arrow[l, bend right] \\
a_3 \arrow[u] & a_4 \arrow[u]
\end{tikzcd}$$

Figure 14.2: The top choice graph at the first stage.

14.3 An LLP Algorithm for the Housing Market Problem

We model the housing market problem as that of the predicate detection in a computation. There are n agents and n houses. Each agent proposes to houses in the decreasing order of preferences. These proposals are considered events executed by n processes representing the agents. Thus, we have n events per process. Each event is labeled as (i, h, k), which corresponds to the agent i proposing to the house h as his choice number k.

The global state corresponds to the number of proposals made by each of the agents. Let $G[i]$ be the number of proposals made by the agent i. We will assume that in the initial state, every agent has made his first proposal. Thus, the initial global state $G = [1, 1, .., 1]$. We extend the notation of indexing to subsets $J \subseteq [n]$ such that $G[J]$ corresponds to the subvector given by the indices in J.

We now model the possibility of reallocation of houses based on any global state. Recall that $pref[i][k]$ specifies the k^{th} preference of the agent a_i. Let $wish(G, i)$ denote the house proposed by a_i in the global state G, that is,

$$wish(G, i) = pref[i][G[i]]$$

A global state G satisfies *matching* if every agent proposes a different house, that is,

$$matching(G) \equiv \forall i, j : i \neq j : wish(G, i) \neq wish(G, j).$$

We generalize *matching* to refer to a subset of agents rather than the entire set.

Definition 14.1 (submatching) *Let $J \subseteq [n]$. Then submatching(G, J) iff wish(G, J) is a permutation of the indices in J.*

Intuitively, if *submatching*(G, J) holds, then all agents in J can exchange houses within the subset J. For any G, it is easy to show that

Lemma 14.2 *For all G, there always exists a nonempty J such that submatching(G, J).*

Proof: Given any G, we can create a directed graph as follows. The set of vertices is agents, and there is an edge from i to j if $wish(G, i) = j$. There is exactly one outgoing edge from any vertex in $[n]$ to $[n]$ in this graph. This implies that there is at least one cycle in this graph (possibly a self-loop). The indices of agents in the cycle give us such a subset J.

■

We now show that

Lemma 14.3 *submatching(G, J_1) and submatching(G, J_2) implies submatching$(G, J_1 \cup J_2)$.*

Proof: Any index $i \in J_1 \cup J_2$ is mapped to J_1 if $i \in J_1$ and J_2, otherwise.

■

Hence, there exists the biggest submatching in G. Note that *matching*(G) is equivalent to *submatching*$(G, [n]$

Definition 14.4 (Feasible Global State) *A global state G is feasible for the housing market problem iff it is a matching and for all global states $F < G$, there does not exist any submatching which is better in F than in G. Note that if there exists a submatching J that is better in F than in G, then agents in J can improve their allocation by just exchanging houses within the subset J. Formally, let*

$$B_{housing}(G) \equiv matching(G) \wedge (\forall F < G : \forall J \subseteq [n] : submatching(F, J) \Rightarrow F[J] = G[J]).$$

We show that $B_{housing}(G)$ is a lattice-linear predicate. This result will let us use the lattice-linear predicate detection algorithm for the housing market problem.

Theorem 14.5 *The predicate $B_{housing}(G)$ is lattice-linear.*

Proof: Suppose that $\neg B_{housing}(G)$. This implies that either G is not a matching or it is a matching, but there exists a smaller global state F that has a submatching better than G.

First, consider the case when G is not a matching. Let J be the largest set such that *submatching*(G, J). Consider any index $i \notin J$ such that $wish(G, i) \in J$. We claim $forbidden(G, i, B_{housing})$. Let H be any global state greater than G such that $G[i] = H[i]$. We consider two cases.

Case 1: $H[J] > G[J]$.
Then, from the second conjunct of $B_{housing}$, we know that $\neg B_{housing}(H)$ because *submatching*(G, J) and $H[J] \neq G[J]$.

Case 2: $H[J] = G[J]$.

Since $wish(H,i) = wish(G,i)$, $wish(G,i) \in J$, and $G[J] = H[J]$, we get that H is not a matching because the house given by $wish(G,i)$ is also in the wish list of some agent in J.

Now consider the case where G is a matching, but $\neg B_{housing}(G)$. This implies,

$$\exists F < G : \exists J \subseteq [n] : submatching(F,J) \wedge F[J] < G[J]).$$

However, the same F will also result in guaranteeing $\neg B_{housing}(H)$ for any $H \geq G$.

■

It is also easy to see from the proof that if an index is part of a submatching, then it will never become forbidden.

This theorem gives us the algorithm shown in Fig. LLP-Housing-Market-Algorithm. Let G be the initial global state. Let $S(G)$ be the biggest submatching in G. All agents such that they are not in $S(G)$ and wish a house which is part of $S(G)$ are forbidden and can move on to their next proposal. The algorithm terminates when no agent is forbidden. This algorithm is a parallel version of the top trading cycle (TTC) mechanism attributed to Gale in [SS74].

Algorithm LLP-Housing-Market-Algorithm: An LLP algorithm to find the optimal housing market

```
var
    $G$: array[1..$n$] of int initially 1;// every agent starts with the top choice
always
    $S(G)$ = largest $J$ such that $submatching(G,J)$
forbidden: $(j \notin S(G)) \wedge (wish(G,j) \in S(G))$
    advance: $G[j] := G[j] + 1$;
```

We now show that

Theorem 14.6 *There exists at least one feasible global state G such that $B_{housing}(G)$.*

Proof: Every agent has his own house on the list of preferences. If he ever makes a proposal to his own house, he forms a submatching. That particular event is never forbidden because it is part of a submatching. Hence, the lattice-linear predicate detection algorithm will never mark that event as forbidden. Since such an event exists for all processes, we are guaranteed to never go beyond this global state.

■

The above proof also shows that agents can never be worse off participating in the algorithm. Each agent will either get his own house back or get a house that he prefers to his own house.

14.4 Problems

1. Consider a housing market with four agents, $\{A_1, A_2, A_3, A_4\}$, who own houses $\{h_1, h_2, h_3, h_4\}$, where A_i owns h_i. Preferences are:

   Apply the TTC algorithm to find the final allocation.

Agent	Preference List
A_1	$h_2 \succ h_4 \succ h_1 \succ h_3$
A_2	$h_1 \succ h_3 \succ h_2 \succ h_4$
A_3	$h_3 \succ h_1 \succ h_4 \succ h_2$
A_4	$h_3 \succ h_2 \succ h_4 \succ h_1$

Table 14.1: Preferences for the agents.

2. Is the outcome of the TTC algorithm unique for a given set of truthful preferences and initial endowments? Justify your answer.
3. Give an algorithm to determine whether the given matching is the core of the housing market.
4. Prove that the allocation returned by the TTC algorithm is Pareto optimal.
5. Prove or disprove: The Top Trading Cycle algorithm is strategyproof, i.e., no agent can benefit by misreporting their preferences.

14.5 Bibliographic Remarks

The housing market problem has been studied by many researchers [SS74, HZ79, Zho90, AS98, AS99, RP77, Rot82, Dav13]. Possible applications of the housing market problem include assigning virtual machines to servers in cloud computers, allocating graduates to trainee positions, professors to offices, and students to roommates. This problem has also been studied by Zheng and Garg [ZG19] where it is shown that the problem of verifying that a matching is a core is in NC, but the problem of computing the core is CC-hard. The class CC (Comparator Circuits) is the complexity class containing decision problems that can be solved by comparator circuits of polynomial size. The paper [ZG19] also provides a *distributed* message passing algorithm to find the core with $O(n^2)$ messages. A parallel algorithm is described in [Gar21].

Chapter 15

The Assignment Problem

15.1 Introduction

In this chapter, we consider the assignment problem in which we seek an assignment of items to bidders such that the total payoff is maximized. We view items and bidders as nodes in a bipartite graph. The weight $v_{b,i}$ of an edge between the bidder b and the item i is the utility of assigning the item i to the bidder b. The assignment problem then corresponds to finding the maximum weight bipartite matching.

The assignment problem has a rich history. The most famous algorithm for this problem is due to an American scientist Kuhn, who called it the Hungarian algorithm because it is inspired by the earlier works of two Hungarian mathematicians, K"onig and Egervary. Another American mathematician, Munkres, observed that the algorithm is strongly polynomial, and the algorithm is also known as the Kuhn-Munkres algorithm. It was later discovered that the problem had previously been solved by Jacobi in the 19th century and was published in Latin in the year 1890.

15.2 Problem Formulation

Let I be a set of indivisible n items and U, a set of n bidders. Each item $i \in I$ receives a valuation $v_{b,i}$ from each bidder $b \in U$. The valuation of any item i is an integer between 0 and $T[i]$. Each item i has a price $G[i]$ which is also an integer between 0 and $T[i]$.

We will assume that the number of items is equal to the number of bidders. If one of the sets is smaller, say there are fewer items than bidders, then we can add virtual items such that all bidders give a valuation of zero to those items. We also note that we are considering the maximum weight bipartite matching. By subtracting the valuation from the maximum valuation for each edge and calling it the cost of that edge, we can transform this problem to minimum-cost bipartite matching.

It is important to note that it is the relative valuation of the items that is important, not the absolute values. Suppose that there are three items and a bidder provides valuation of $(10, 8, 4)$ for these items, then the assigned matching will not change if his valuation is $(6, 4, 0)$ instead. The weight of the matching found for the valuation $(6, 4, 0)$ would be exactly less by 4 because in each assignment a bidder is assigned exactly one item. Hence, one can assume that there is at least one item for each bidder that is valued as 0. Similarly and dually, suppose that an item is valued by three bidders as $(10, 8, 4)$. Then, the assignment will not change if this item is valued as $(6, 4, 0)$ instead. If we view V as a square matrix such that $V[b, i]$ equals $v_{b,i}$, then we can assume that there is a zero in every row by subtracting the minimum value in each row from every entry in the row. Similarly, we can assume that there is at least one zero in every column.

15.3 Market Clearing Price

In this section, we apply the LLP technique to the problem of finding a market clearing price. The set of feasible price vectors is given by

$$\{G \mid 0 \leq G[i] \leq T, 1 \leq i \leq n\}$$

This set of price vectors forms a distributive lattice under the component-wise comparison of the vectors. The minimum is given by the zero vector and the maximum is given by the vector T.

Given a price vector G, we define the bipartite graph $(I, U, E(G))$ as follows. One side of the bipartite graph is the set of items I. The other side of the graph is the set of bidders U. We now add edges between items and the bidders as follows.

$$(j, b) \in E(G) \equiv \forall i : (v_{b,j} - G[j]) \geq (v_{b,i} - G[i]).$$

Informally, there is an edge between item i and bidder b if the payoff for the bidder (the bid minus the price) is maximized with that item. Given any set $U' \subseteq U$, let $N(U', G)$ denote all the items that are adjacent to the vertices in U' in the graph $(I, U, E(G))$. A price vector G is a *market clearing price* if the bipartite graph $(I, U, E(G))$ has a perfect matching.

We construct a computation graph $(E, \rightarrow)$ for this problem as follows (see Fig. 15.1). There are n processes corresponding to n items in the computation such that each process has C events. In Fig. 15.1, we have three items and three bidders, such that both prices and valuations are integers between 0 and 4. If the process i has executed k events, then $G_i = k$. The global state G can be viewed as the price vector p where $G[i] = p[i]$. A price vector G is a *market clearing price*, denoted by $B_{clearingPrice}(G)$ if the bipartite graph $(I, U, E(G))$ has a perfect matching. We now claim that

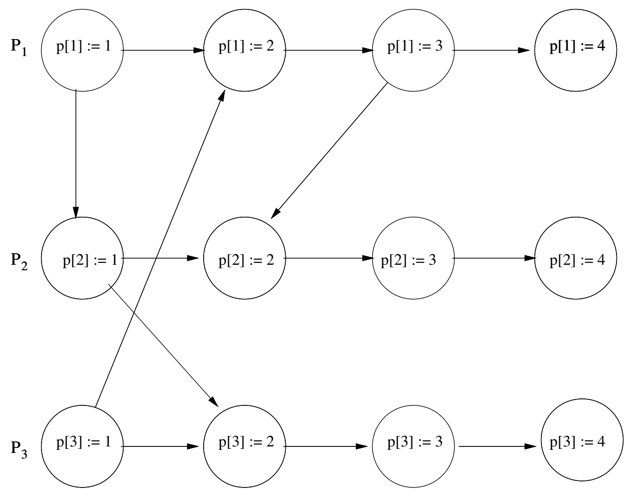


Figure 15.1: The computation graph for a market with three items and three bidders. The valuation of and price for any item is a number between 0 and 4.

Lemma 15.1 *The predicate $B_{clearingPrice}(G)$ is a lattice-linear predicate on the lattice of price vectors.*

Proof: Suppose that the price vectors G and H satisfy $B_{clearingPrice}$. Let $K = min(G, H)$. We show that K also satisfies $B_{clearingPrice}$. Suppose that $(I, U, E(K))$ does not have a perfect matching. This implies that there exists a minimal overdemanded set of items in K. Let an item i be one of the minimal overdemanded items in K. Without loss of generality, assume that $K[i]$ equals $G[i]$ (the argument for the case where $K[i]$ equals $H[i]$ is identical). The item i is not an overdemanded item in G. But this is a contradiction because for all other items i', the price of the item i' has either stayed the same or increased in going from K to G.

■

Algorithm ConstrainedMarketClearingPrice: Algorithm ConstrainedMarketClearingPrice to find the minimum cost assignment vector

```
P_j: Code for thread j
shared var G: array[1..n] of 0..maxint;
Market Clearing Prices: Demange-Gale-Sotomayor algorithm
input: $v[b,i]$: int for all $b, i$
init: $G[j] := 0$;
always:
    $E = \{(k,b) \mid \forall i : (v[b,k] - G[k]) \geq (v[b,i] - G[i])\}$;
    $demand(U') = \{k \mid \exists b \in U' : (k,b) \in E\}$;
    $overDemanded(J) \equiv \exists U' \subseteq U : (demand(U') = J) \wedge (|J| < |U'|)$
forbidden(j): $(\exists minimal\ J : OverDemanded(J) \wedge (j \in J)$
    advance: $G[j] := G[j] + 1$;
```

In Fig. ConstrainedMarketClearingPrice, we have used $\alpha(G, j)$ as simply one unit of price. For any item j that is part of a minimal over-demanded set of items, we can increase its price by the minimum amount to ensure that some bidder b can switch to her second most preferred item. We now give an implementation LLP-Assignment based on this idea.

15.4 Constrained Market Clearing Price Problem

We now generalize the problem of finding a market clearing price to that of finding a constrained market clearing price. For example, constraints on the clearing prices of the form $(G[j] \geq k) \Rightarrow (G[i] \geq k')$ for $1 \leq k, k' \leq C$ are lattice-linear. The constraint says that if item j is priced at least k, then item i must be priced at least k'. The constraint $G[i] \geq G[j]$ is also lattice-linear. Observe that the constraint $G[i] \geq G[j]$ is equivalent to

$$(G[j] \geq 1 \Rightarrow G[i] \geq 1) \wedge (G[j] \geq 2 \Rightarrow G[i] \geq 2) \ldots (G[j] \geq C \Rightarrow G[i] \geq C)$$

Similarly, the constraint $(G[i] = G[j])$ can be modeled as $(G[i] \geq G[j]) \wedge (G[j] \geq G[i])$. Also, constraint of the form $G[j] \geq f(G)$ for monotone f is lattice-linear. Given any set of valuations, and a boolean predicate B that is a conjunction of lattice-linear constraints, a price vector G is a *constrained market clearing price*, denoted by $constrainedClearing(G)$ iff $clearing(G) \wedge B(G)$. Since $B(G)$ is lattice-linear, it

Algorithm LLP-Assignment: Finding the minimum clearing price vector

```
var
    $G$: real initially $\forall i : G[i] = 0$;
while (true) do
    $E := \{(i, b) \mid \forall j : (v_{b,i} - G[i]) \geq (v_{b,j} - G[j])\}$;
    if there exists a perfect matching $M$ in $E$ then return $M$;
    else
        $J$ := minimal OverDemanded set in the bipartite graph with edges $E(G)$;
    end
    forall $j \in J$ do
        $\delta_j = \min\{(v_{b,j} - G[j]) - \max_{i \neq j}(v_{b,i} - G[i]) \mid (j, b) \in E(G)\}$
        $G[j] := G[j] + \delta_j$;
    end
end
```

is sufficient to give an algorithm for *clearing*(G). It follows that the set of constrained market clearing price vectors is closed under meets. By applying the lattice-linear predicate detection, we get an algorithm to compute the least constrained market clearing price shown in Fig. ConstrainedMarketClearingPrice. We get a generalization of Demange, Gale and Sotomayor's exact auction mechanism [DGS86] to incorporate lattice-linear constraints on the market clearing price.

We construct a computation graph $(E, \rightarrow)$ for this problem as follows (see Fig. 15.2). For any constraint, $(G[j] \geq k) \Rightarrow (G[i] \geq k')$, we put an edge from the event k' in P_i to the event k in P_j. By putting such edges, we get a computation graph and any price vector that does not satisfy constraints is not a consistent global state.

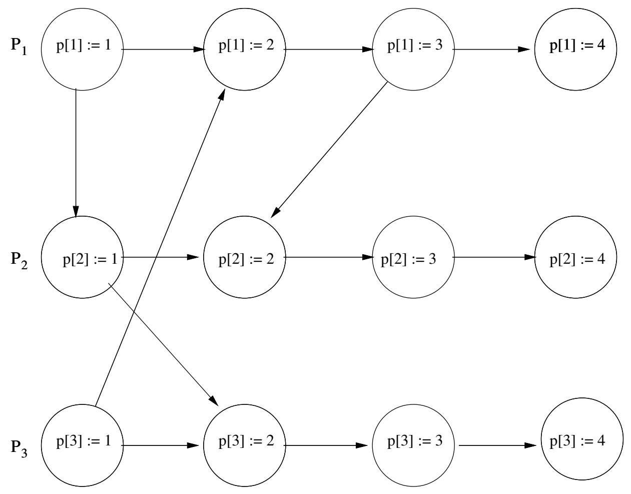


Figure 15.2: The computation graph for a market with three items and three bidders. The valuation of and price for any item is a number between 0 and 4. The computation graph models the constraint $B \equiv (p[2] \geq 2 \Rightarrow p[1] \geq 3) \wedge (p[1] \geq 2 \Rightarrow p[3] \geq 1)$

For example, suppose that the valuation for three items in Fig. 15.2 is as follows. The bidder 1 values them as $[4, 1, 0]$, the bidder 2 values them as $[4, 1, 2]$ and the bidder 3 values them as $[4, 2, 0]$. In the initial global state (with no events on any process), the price vector is $[0, 0, 0]$. This is not a market clearing price since the item 1 is overdemanded. Hence, we advance on P_1 and the new price vector is $[1, 0, 0]$. The item 1 continues to be overdemanded, and we advance on P_1 again to get the price vector $[2, 0, 0]$. This is a market clearing price; however, it does not satisfy the constraint $(p[1] \geq 2 \Rightarrow p[3] \geq 1)$. Hence, we must advance on P_3 to get the price vector $[2, 0, 1]$. This price vector satisfies the constraints, but is not a clearing price. We must advance on the overdemanded item 1, to obtain the price vector $[3, 0, 1]$, which is both clearing and satisfies the constraints. The price vector $[3, 0, 1]$ is the least price vector that is clearing and satisfies constraints. At this price vector, item 1 can be assigned to the bidder 1, item 2 to bidder 3 and item 3 to bidder 2.

15.5 Problems

1. Give the time complexity of implementing LLP-Assignment.
2. Show that the set of market clearing price vectors is also closed under the join operation.
3. Give the linear programming formulation and its dual for solving the assignment problem.
4. Show that the market clearing vector is closed under the component-wise maximum operation.
5. Suppose that your friend gives you an assignment and claims that the assignment is optimal. It is easy to check that the assignment is proper, i.e., every item is assigned to exactly one bidder. But, how can she quickly convince you of the optimality?

15.6 Bibliographic Remarks

Kuhn's Hungarian method for solving the assignment problem is from [Mun57]. Demange-Gale-Sotomayor present the auction-based algorithm [DGS86] for market clearing prices. The notion of a constrained market clearing price is from [Gar20].

Chapter 16

Horn and 2-SAT Satisfiability

16.1 Introduction

It is well-known that checking the satisfiability of a Boolean expression is an NP-complete problem. In this chapter, we consider two important classes of boolean expressions for which satisfiability can be solved efficiently, Horn formulas and 2-SAT formulas.

Throughout this chapter, we assume that the predicate is given in the *conjunctive normal form* (CNF). A Boolean formula B in conjunctive normal form is simply a conjunction of *clauses*, where each clause is a pure disjunction of *literals*. A literal is a variable in its simple form or in a complemented form. For example, suppose that we have n Boolean variables $\{x_1, x_2, \ldots x_n\}$. Then, the formula $(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2)$ is in CNF with two clauses. The first clause has three literals $\{\neg x_1, \neg x_2, x_3\}$, and the second clause has two literals $\{x_1, x_2\}$.

16.2 Horn Satisfiability

A clause is a *Horn clause* if it has at most one positive literal. An example of a Horn clause is $(\neg x_1 \vee \neg x_2 \vee x_3)$. Note that this formula is also equivalent to $(x_1 \wedge x_2) \Rightarrow x_3$. A Horn clause written in this form is also called a *Horn implication* or a *definite clause*. Notice that it has only positive literals on the left-hand side and a single positive literal on the right-hand side. The left-hand side may be empty. Thus, x_3 is also a Horn implication equivalent to $\mathit{true} \Rightarrow x_3$.

Another example of a Horn clause is $(\neg x_1 \vee \neg x_2 \vee \neg x_3)$. This clause does not have any positive literal. It is a *pure negative* clause.

A Horn formula is just a conjunction of Horn clauses. The problem of HornSat is to determine if the given Horn formula is satisfiable.

Consider the following Horn formula B with three variables x_1, x_2, and x_3:

$$(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_1) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$$

This Horn formula B is satisfiable. Consider the assignment: x_1 = false, x_2 = true, and x_3 = false. The first and fourth clauses are true because x_1 is false, and the second and the third clauses are true because x_3 is false. Another assignment that satisfies B is x_1 = false, x_2 = false, and x_3 = false. In fact, this is the *least* satisfying assignment in the Boolean lattice of three variables.

Consider the following set of Horn clauses with two variables x_1 and x_2:

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_1) \wedge (\neg x_1 \vee \neg x_2) \wedge x_1$$

This set of Horn clauses is unsatisfiable because there is no assignment of truth values to x_1 and x_2 that can satisfy all the clauses simultaneously.

16.3 LLP Algorithm for HornSat

We assume that the Horn formula uses n variables $x_1, \ldots x_n$. These n boolean variables define a boolean lattice L of size 2^n that consists of boolean vectors of size n. We first show that a Horn formula is a lattice-linear predicate with respect to L.

Lemma 16.1 *Any Horn formula is a lattice-linear predicate.*

Proof: Let $B = B_1 \wedge B_2 \wedge B_m$ be the Horn formula, where each B_i is a Horn clause. Since lattice linear predicates are closed under conjunction, it is sufficient to show that every Horn clause B_i is lattice linear. We consider two types of Horn clauses.

First, suppose that B_i is a Horn implication, i.e., $B_i \equiv (x_{i_1} \wedge x_{i_2} \wedge x_{i_{k-1}} \Rightarrow x_{i_k})$. Consider any boolean vector G in which B_i is false. This implies that x_{i_1}, $x_{i_2}, \ldots x_{i_{k-1}}$ are true in G but x_{i_k} is false. Consider any H such that $H \geq G$ and $H[i_k] = G[i_k]$. Since $H \geq G$, we have that $(x_{i_1} \wedge x_{i_2} \wedge x_{i_{k-1}})$ holds in H. Furthermore, since $H[i_k]$ is equal to $G[i_k]$, x_{i_k} is false in H. Hence, B_i is also false in H.

Now suppose that B_i is a pure negative clause, i.e., $B_i \equiv (\neg x_{i_1} \vee \neg x_{i_2} \vee \neg x_{i_k})$. Suppose that B_i is false in G. This implies that all the literals are true. Consider any boolean vector $H \geq G$. H must also have all these literals true and therefore B_i is false in H (and any index is trivially forbidden).

■

The proof of Lemma 16.1 also shows us the appropriate advancement function. For Horn implications, we must advance on the right-hand side literal. For pure negative clauses, all Boolean vectors greater than G do not satisfy B. Hence, the algorithm can simply return that "no satisfying vector exists."

We can now apply the LLP algorithm to find a satisfying assignment for a Horn formula.

Algorithm HornSAT-LLP: An Algorithm to find the least assignment that satisfies B

```
1 var $G$: vector of boolean initially $\forall i : G[i] = false$;
2 $P_j$: code for thread $j$
3 forbidden: $\exists h$ : all antecedents of $B_h$ are true, they imply $x_j$, and $x_j$ is false in $G$
4     advance: $G[j] := true$;
5 forbidden: $\exists h$ : a negative clause $B_h$ is false, and $x_j$ is a part of $B_h$
6     return null; // "no satisfying assignment exists"
7 return $G$; // the least assignment that satisfies $B$
```

We leave an efficient implementation of the algorithm HornSAT-LLP as an exercise.

Observe that we also get the following consequence of Lemma 16.1 from its lattice-linearity.

Corollary 16.2 *If G and H satisfy the Horn formula, then $G \sqcap H$ also satisfies that formula.*

A direct proof of this fact is left as an exercise. Although the set of assignments satisfying a Horn formula are closed under meets, they are not closed under joins. For example, consider the Horn formula $B \equiv (\neg x_1 \vee \neg x_2)$. The bit vectors $G = [1, 0]$ and $H = [0, 1]$ satisfy B but their join $[1, 1]$ does not satisfy B.

16.4 Arithmetization of Horn Clauses

In this section, we give another proof that Horn Clauses are lattice linear by considering arithmetic expressions instead of boolean expressions. We use the natural assignment of boolean variables x_i to integer binary variables y_i. If x_i is false then y_i is assigned value 0, otherwise it is assigned 1. Given any clause, we translate it into an arithmetic predicate as follows. Every positive literal x_i in any clause is changed to y_i and every negative literal $\neg x_i$ is changed to $(1-y_i)$. The clause is true iff the sum of all values so replaced in any clause is at least 1. For example, the clause $(\neg x_1 \vee \neg x_2 \vee x_3)$ is equivalent to $(1-y_1)+(1-y_2)+y_3 \geq 1$. If all y_i take values in the set $\{0, 1\}$, then it is easy to verify that the clause is true for $x's$ iff the corresponding arithmetic predicate is true for $y's$. Let us see how implication Horn clauses get translated. An implication Horn clause $(x_{i_1} \wedge x_{i_2} \wedge x_{i_{k-1}} \Rightarrow x_{i_k})$ gets translated to $(1 - y_{i_1}) + (1 - y_{i_2}) + \ldots (1 - y_{i_{k-1}}) + y_{i_k} \geq 1$ which is equivalent to $y_{i_k} \geq y_{i_1} + y_{i_2} + \ldots + y_{i_{k-1}} - (k - 1)$. The right-hand side is a monotone function of y; hence the predicate is lattice-linear. A pure negative clause $(\neg x_{i_1} \vee \neg x_{i_2} \ldots \vee \neg x_{i_k})$ gets translated to $(1 - y_{i_1}) + (1 - y_{i_2}) \ldots + (1 - y_{i_k}) \geq 1$. This predicate is equivalent to $k - 1 \geq y_{i_1} + y_{i_2} + \ldots + y_{i_k}$ which, once it becomes false, it stays false. Hence, predicates for negative clauses are also lattice-linear.

16.5 2-SAT

A conjunctive normal form formula is a 2-SAT formula iff every clause has at most two literals. For example, the formula $B_1 \equiv x_1 \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_3)$ is a 2-SAT formula. If we consider the lattice of boolean vectors, a 2-SAT formula may not be closed under meet. For example, consider the 2-SAT formula $(x_1 \vee x_2)$. The boolean vectors 10 and 01 satisfy the formula, but their meet 00 does not. Hence, there is no *minimum* satisfying assignment and simple LLP algorithm is not applicable.

We now outline an approach to detect if there is any element in the lattice that satisfies B is a 2-SAT formula. The key idea for the 2SAT algorithm is to convert the predicate to an implication graph. We assume that every clause has two literals. A unit clause x_i or $\neg x_i$ forces the value of the variable. It can be replaced in B resulting in a 2SAT formula with fewer variables. Now, we construct an implication graph $I(B)$ for B as follows. The graph has $2n$ vertices, one for each literal. Every clause $(l_i \vee l_j)$ in B is true iff $(\neg l_i \Rightarrow l_j) \wedge (\neg l_j \Rightarrow l_i)$. Therefore, we add two directed edges in the graph — from $\neg l_i$ to l_j and from $\neg l_j$ to l_i.

Consider the following 2-SAT formula with three variables x_1, x_2, and x_3:

$$(\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$$

The implication graph on B is shown in Fig. 16.1.

We leave the following as an exercise.

Exercise 16.1 *1. There exists a satisfying assignment to 2-SAT iff there is no variable x_i such that $\neg x_i$ is reachable from x_i and x_i is reachable from $\neg x_i$ in the graph I.*

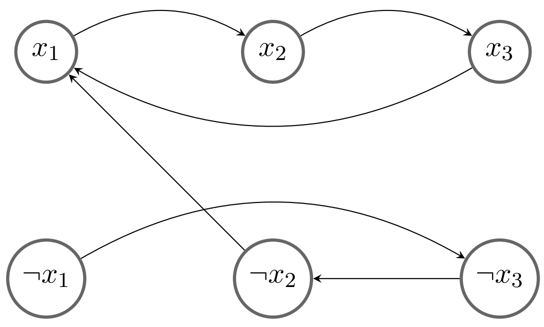


Figure 16.1: Implication Graph of the predicate $B \equiv (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_3) \wedge (\neg x_2 \vee x_3)$

2. *Suppose B is satisfiable. Then, all satisfying assignments set x_j to true iff there exists a path from $\neg x_j$ to x_j in the implication graph.*

3. *Suppose B is satisfiable. Then, all satisfying assignments set x_j to false iff there exists a path from x_j to $\neg x_j$ in the implication graph.*

Algorithm 2SAT: An Algorithm to find a feasible assignment that satisfies B

```
vector function getFeasible($B$: predicate)
var $G$: array$[1 \ldots n]$ of boolean initially $\forall i : G[i] = false$;
    $fixed$: array$[1 \ldots n]$ of boolean initially $\forall i : fixed[i] = false$;
$I(B)$ := implication graph of $B$;
if there exists a path from $\neg x_j$ to $x_j$ and from $x_j$ to $\neg x_j$ in $I(B)$ then
    return null //"no satisfying solution"
end
while $(\exists j : \neg fixed[j])$ do
    if there exists a path from $\neg x_j$ to $x_j$ then $G[j] := true$;;
    $fixed[j] := true$;
    $unitPropagation(B, j)$;
end
return $G$; // a feasible solution
```

Algorithm 2SAT works as follows. The array G keeps the assignment of all binary variables. The array $fixed$ maintains for each i, whether the bit x_i has been set to true or false. The **while** loop in the figure sets these variables as follows. It picks any variable x_j that is not fixed. If there is a path from $\neg x_j$ to x_j, then we set x_j to true; otherwise, it keeps the value of x_j false. We also set the bit $fixed[j]$ to true. Once x_j is set, other variables can also be forced. We call the method *unitPropagation* which sets the other variables that are set because x_j is set. We continue the **while** loop until all variables are set.

In the algorithm, we need to check if there exists a path from x_j to $\neg x_j$ and vice versa. One way to accomplish this is to find strongly connected components (SCCs) in the graph. An SCC is a subgraph in which every vertex is reachable from every other vertex in the same SCC. (One can use Tarjan's algorithm or Kosaraju's algorithm to find SCCs.) The expression is satisfiable if and only if for every variable x_j

and $\neg x_j$ are not in the same SCC. If the expression is satisfiable, an assignment can be constructed by considering the SCCs in reverse topological order. The sequential time complexity of the 2SAT problem is linear in the number of clauses.

Let us now check the time complexity for a parallel algorithm to determine if the given 2SAT expression is satisfiable. The problem reduces to checking if there exists a path from x_j to $\neg x_j$ and vice versa using reachability. Since the reachability problem is in NC, we conclude that checking the satisfiability of a 2SAT formula is in NC.

16.6 Problems

1. Is the following Horn SAT formula satisfiable?
   $(\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_1) \wedge (\neg x_3 \vee \neg x_1) \wedge (\neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2)$

2. Give an efficient implementation of the algorithm in Fig. HornSAT-LLP. In particular, give efficient data structures to represent Horn implications and pure negative clauses. Your algorithm should be linear in the length of the Horn formula.

3. A formula is a Renamable Horn formula if by flipping the polarity of certain variables, the formula can be transformed to a Horn formula. For example, $(\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge (x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee \neg x_4) \wedge (x_1 \vee \neg x_3)$ is not a Horn formula. However, by renaming x_1 as $\neg x_1$ and x_3 as $\neg x_3$, we get a Horn formula. Give an algorithm to determine whether the given formula is a Renamable Horn formula. (Hint: Can you come up with a 2SAT formula that is true iff the given formula is renamable Horn formula).

4. Prove Corollary 16.2 without invoking lattice linearity of Horn Clauses.

5. A dual Horn formula is one in which all clauses have at most one negative literal. Give an algorithm to check satisfiability of dual Horn formulas.

6. Consider a directed graph with n nodes including two distinguished nodes s and t. Show that there exists a 2-SAT formula B on n variables such that the formula is satisfiable iff t is reachable from s.

7. Suppose a 2-SAT formula B is satisfiable. Then, all satisfying assignments of B must have x_j as true iff there exists a path from $\neg x_j$ to x_j in the implication graph.

8. (a) Show that there exists a satisfying assignment to a 2-SAT formula B iff there is no variable x_i such that $\neg x_i$ is reachable from x_i and x_i is reachable from $\neg x_i$ in the implication graph I constructed from B. (b) Show that every literal in any strongly connected component of the implication graph must be assigned the same value, *true* or *false*.

16.7 Bibliographic Remarks

Horn formulas, a subclass of propositional formulas in which each clause has at most one positive literal, are central to efficient satisfiability algorithms. [DG84] present a linear-time algorithm to test the satisfiability of the Horn formula, using unit propagation. The 2SAT problem, determining the satisfiability of a CNF formula with at most two literals per clause, is solvable in linear time. [APT79] introduce a linear-time algorithm using implication graphs and strongly connected components.

Chapter 17

Stable Marriage: Revisited

17.1 Introduction

In this chapter, we revisit the problem of stable marriage that has many applications. In chapter 2 we discussed the upward traversal to find the stable matching. Section 17.2 describes an algorithm to find the stable marriage using downward traversal. Section 17.4 gives a generalization of the problem to the hospital-resident problem in which we match a hospital to residents where a hospital may have more than one seat. Finally, Section 17.5 considers the problem of stable matching in the presence of ties.

17.2 Algorithm β Downward Traversal

We now give the dual of the Algorithm α which performs the downward traversal in the proposal vector lattice and returns the greatest stable marriage less than or equal to I. In the standard literature, one does not consider the dual of the GS algorithm to find the woman-optimal stable marriage. By simply switching the roles of men and women from the man-optimal GS, we get the woman-optimal GS algorithm. We cannot employ this strategy because we assume that the number of men is less than or equal to the number of women. Switching men and women violates this assumption.

Algorithm LLP-β-StableMarriage: A Downward-Traversal Algorithm for Stable Matching

```
1 input: $mpref[i,k]$: int for all $i,k$; $rank[k][i]$: int for all $k,i$; initial proposal vector $I$: array[1..n]
  of $0..m$
2 init: $\forall i : G[i] := min(m, I(i))$;
3 if $(\exists i : G[i] < L[i])$ return null; // no stable matching exists
4 always: $z = mpref[j][G[j]]$;
5 forbidden: $(\exists i : \exists q : wrank[q][i] < wrank[q][$
6     advance: $G[j] := G[j] + 1$;
```

In addition, the downward traversal of the proposal lattice gives different insights into the algorithm for finding a stable matching even when the number of men equals the number of women.

We first give a necessary condition for a stable marriage to exist that is less than or equal to I.

Lemma 17.1 *If $numw(I)$ (the number of unique women who have proposals in any vector less than or equal to I) is less than m, then there cannot be any stable proposal vector less than or equal to I.*

Proof: The claim follows because any proposal vector less than or equal to I cannot be a man-saturating matching if the number of unique women is less than m.

■

If $numw(I) \geq m$, there may or may not be a proposal vector that corresponds to a stable marriage depending upon the women's preferences.

While traversing the proposal lattice in the downward direction, we use the predicate $rForbidden(G, p)$ (short for reverse-Forbidden) which uses the notion of *best blocking pair*.

Definition 17.2 (Best blocking pair) *A blocking pair (p, q) is the* best blocking pair *in G for q if for all blocking pairs (p', q) in G, the woman q prefers p to p'.*

Definition 17.3 (rForbidden) *A man p is* rForbidden *in G if there exists a woman q such that (p, q) is the best blocking pair in G for q.*

We first show that

Lemma 17.4 *Let G be any proposal vector such that $numw(G) \geq m$. There exists a man i such that $rForbidden(G, i)$ iff G is not a stable marriage.*

Proof: First, suppose that there exists i such that $rForbidden(G, i)$. Then, there exists a woman q such that man i and woman q form a blocking pair for G. Hence, G is not a stable marriage.

Conversely, assume that G is not a stable marriage. If G is not a man-saturating matching, then there exists at least one woman q who has not received any proposal in G but has received it earlier because $num(W) \geq m$. Of all such proposals to q let the most favorable proposal be from man i. Then, $rForbidden(G, i)$ holds. If G is a man-saturating matching, but not stable, then there exists at least one blocking pair. Therefore, there exists at least one best blocking pair.

■

Analogous to upward traversal using forbidden predicate, we get that

Lemma 17.5 *Assume $numw(G) \geq m$. If $rForbidden(G, i)$ holds, then there is no stable proposal vector H such that $(H \leq G)$ and $(G[i] = H[i])$.*

Proof: If $rForbidden(G, i)$ holds, there exists a woman q such that q prefers i to all men who have proposed to q until G. First, suppose that H does not have any proposal to q. Then H cannot be stable because q is single and man i prefers q to $\rho(H, i)$. Now suppose that H has a proposal to q. Since $H \leq G$, we know that any proposal to q in H is less preferable to that by man i. Hence, (i, q) continues to be a blocking pair in H.

■

Fig. **??** shows a high-level description of a downward traversal of the proposal lattice. At line (1), we ensure that $G[i]$ is at most m because, due to Lemma 2.2, we know that there cannot be any stable marriage in which any component exceeds m. At line (2), we first ensure that G is at least equal to L, the proposal vector corresponding to the man-optimal stable marriage. Otherwise, there cannot be a stable marriage vector less than or equal to G. At line (3) we pick i such that $rForbidden(G, i)$ holds. This means that there exists a woman q such that (i, q) is a best blocking pair. At line (4), we satisfy the pair (i, q) by decreasing $G[i]$ to $\rho(G, i) = q$. This step corresponds to a downward traversal in the proposal lattice. At line (6), when we exit the while loop, we know that G must be a stable marriage on account of Lemma 17.4. This algorithm ensures that the match for any man can only improve.

17.3 Path to Stability

We now present an algorithm that gives a path from any proposal vector to a stable marriage vector. Note that depending on the initial proposal vector, both Algorithms α and β may return null. For example, when men and women are equal and the initial vector is greater than or incomparable to woman-optimal vector, then the algorithm α will return null. If the initial vector is incomparable to both man-optimal and woman-optimal proposal vectors, then both algorithms α and β will return null. In RVV setting, we need to combine a downward traversal with an upward traversal to go from an arbitrary proposal vector to a stable matching. There are two choices for combining these traversals, a downward traversal followed by an upward traversal, or vice versa. We will use the former approach. The RVV algorithm introduces men and women one at a time and performs multiple upward and downward traversals.

The Algorithm γ is shown in Fig. Algorithm γ. Given any arbitrary initial vector I, we first perform a downward traversal to get to a proposal vector that is less than or equal to H, the largest possible stable marriage. If to begin with, the initial vector is at most H, then this step is not necessary. H can be computed using the algorithm β using a downward traversal starting from the vector $[m, m, \ldots, m]$. Our goal is to find blocking pairs in G such that, by satisfying them, we get to a proposal vector $\leq H$. In contrast to algorithms in the literature, we pick blocking pairs to satisfy carefully. Specifically, during the downward traversal, we satisfy only those men whose component in the proposal vector is beyond H. To find a sequence from I to G such that $G \leq H$, we first compute a vector K as $\max(H, G)$. Since $G \not\leq H$, we get that K is also not $\leq H$, i.e., there is at least one component i of K such that $K[i] > H[i]$. Since $K > H$, it cannot be a stable marriage. There are two parts of K, the part that matches G and the part that does not match G, i.e., men in the vector K have either proposed to women at H or beyond H (this is the part that matches G), or, they have proposed to women that are strictly ranked better than in H (this is the part that matches H but not G). We denote the part that matches G as K_G and the part that matches H but not G as K_H. We now invoke downward traversal on K using $rForbidden$ function of algorithm β. Since Algorithm β returns the greatest stable marriage less than the initial proposal vector (in our case K), it finds as blocking pair only those men which are part of K_G. By satisfying these men one at a time, we generate a sequence of proposal vectors that makes G less than or equal to H. At this point we start the second phase of the algorithm.

In the second phase, we do upward traversal in which women improve their match. We use the function α to find the least stable marriage that is greater than or equal to G. In this phase, we satisfy the blocking pairs by improving the match of women. Since the input to algorithm α is less than or equal to H, we are guaranteed to get a stable marriage at the end.

Algorithm Algorithm γ: Algorithm γ with $O(n_m^2)$ complexity.

```
input: A stable marriage instance, initial proposal vector $I$
output: a stable marriage $M$
// the following step is independent of $G$ compute $H$ using Algorithm $\beta$ with the initial vector as
  $[m, m, \ldots, m]$;
$G := I$;
// Downward traversal
$K := H \cup G$;
while $(G \not\leq H)$ do
    // $K$ is not less than or equal to $H$
    // $K$ cannot be a stable marriage
    $(i, q)$ := a blocking pair in $K$
    //assert $(i, q)$ is also a blocking pair in $G$
    $G[i]$ := choice that corresponds to $q$;
    print $G$;// the next proposal vector in the sequence
    $K[i] := G[i]$;
end
//Upward traversal
while there exists a man $i$ such that $forbidden(G, i)$ do
    find the next woman $q$ in the list of man $i$
    such that $i$ has the most preferred proposal to $q$ until $G$;
    $G[i]$ := choice that corresponds to woman $q$;
    print $G$;// the next proposal vector in the sequence
end
return $G$;
```

17.4 Hospitals-Residents Problem

In this section, we pursue a slight generalization of the stable matching problem called the Hospitals-Residents problem. There are n hospitals and m residents. Each hospital H_i has a certain integral capacity c_i. Each hospital has a preference list $hpref$, and each resident has a ranking function $rrank$ for all hospitals. Two seats in the same hospital have the same rank for any resident. If the capacity of each hospital is 1, and the number of hospitals is equal to the number of residents, the problem reduces to the problem of stable marriage.

Figure LLP-OptimalHospitalResident gives an algorithm for the hospital residents problem. We use the variable G such that $G[j]$ is the choice number assigned to the seat j. The size of G equals the total number of seats:

$$\sum_{i=1}^{i=n} c_i.$$

We use $hospital(j)$ to refer to the hospital which has the seat j. The variable z refers to the resident that $G[j]$ denotes. The preference list for each of these seats for a particular hospital is identical. The preference list for each resident is based simply on hospitals. If there are multiple seats offered to a resident, then she prefers the least-numbered seat.

In Fig. LLP-OptimalHospitalResident, G is assigned 1 initially. Thus, each seat in every hospital "makes proposal" to the top choice for the hospital. The index j is forbidden if there is another index i such that either that index i also corresponds to the same resident and the resident prefers the hospital assigned to index i, or they are assigned same hospitals and i is less than j. Whenever, an index j is forbidden, it advances to the next choice by incrementing $G[j]$.

Algorithm LLP-OptimalHospitalResident: An Algorithm for Hospital-Optimal Matching for Hospital-Resident Problem

```
1 $P_j$: Code for thread $j$
2 input: $hpref[1..n, 1..m]$: int; // hospital preferences
3     $rrank[1..m][1..n]$: int; // resident ranks
4 init: $G[j] := 1$;
5 always: $z = hpref[hospital(j)][G[j]]$;
6 forbidden:
  $(\exists i : (z = hpref[i][G[i]]) \land ((rrank[z][i] < rrank[z][j]) \lor ((rrank[z][i] = rrank[z][j]) \land (i < j))))$
7     advance: $G[j] := G[j] + 1$;
```

We leave the correctness of the algorithm to the reader. We make the following observations.

- *Generalization of the Stable Marriage Problem*: The algorithm reduces to the stable matching algorithm when the capacity of each hospital is 1, and the number of hospitals is equal to the number of residents. When the total number of seats is more than the number of residents, some of the seats may go unfilled. When the total number of seats is less than the number of residents, some of the residents may remain unassigned.

- *Finite Distributive Lattice of All Stable Assignments*: Just as for stable marriages, we get a finite distributive lattice of all stable assignments.

We invite the reader to design a resident-oriented algorithm for the Hospital-Resident problem.

17.5 Stable Matching with Ties

In this section, we consider the stable matching problem when the list of preferences may have ties. We will assume that the number of men is equal to the number of women in this chapter. The problem of stable marriage with ties is clearly more general than the standard stable matching problem discussed in Chapter 2.

We consider three versions of matching with ties. In the first version, called *weakly stable matching* M, there is no blocking pair of man and woman (m, w) who are not married in M but strictly prefer each other to their partners in M. In the second version, called *superstable matching* M, we require that there is no blocking pair of man and woman (m, w) who are not married in M but either (1) both prefer each other to their partners in M, or (2) one of them prefers the other over his/her partner in M and the other one is indifferent, or (3) both are indifferent to their spouses. In the third version, called *strongly stable matching*, we require that if there is no blocking pair (m, w) such that they are not married in M but either (1) both of them prefer each other to their partners in M, or (2) one of them prefers the other over his/her partner in M and the other is indifferent.

For weakly stable matching, ties can be broken arbitrarily, and any matching that is stable in the resulting instance is also weakly stable for the original problem. Therefore, the Gale-Shapley algorithm is applicable for weakly stable matching. We now derive LLP algorithms for superstable and strongly stable matchings.

Superstable Matching

In many applications, agents (men and women for the stable marriage problem) may not totally order all their choices. Instead, they may be indifferent to some choices [Irv94, Man02]. We generalize $mpref[i][k]$ to a set of women instead of a single woman. Therefore, the *mrank* function is no longer 1-1. Multiple women may have the same rank. Similarly, the *wrank* function is no longer 1-1. Multiple men may have the same rank. We now define the notion of blocking pairs for a matching M with ties [Irv94]. We let $M(m)$ denote the woman matched with the man m and $M(w)$ denote the man matched with the woman w. In the version, called *weakly stable* matching M, there is no blocking pair of man and woman (m, w) who are not married in M but strictly prefer each other to their partners in M. Formally, a pair of man and woman (m, w) is *blocking for a weakly stable matching* M if they are not matched in M and
$(mrank[m][w] < mrank[m][M(m)])\wedge$
$(wrank[w][m] < wrank[w][M(w)]$.

For weakly stable matching, ties can be broken arbitrarily, and any matching that is stable in the resulting instance is also weakly stable for the original problem. Therefore, the Gale-Shapley algorithm is applicable for weakly stable matching [Irv94]. We focus on other forms of stable matching, superstable and strongly stable matchings.

A matching M of men and women is *superstable* if there is no blocking pair (m, w) such that they are not married in M but they either prefer each other to their partners in M or are indifferent with their partners in M. Formally, a pair of man and woman (m, w) is *blocking for a super stable matching* M if they are not matched in M and
$(mrank[m][w] \leq mrank[m][M(m)])\wedge$
$(wrank[w][m] \leq wrank[w][M(w)]$.

The algorithms for superstable marriage have been proposed in [Irv94, Man02]. Our goal is to show that the LLP algorithm is applicable to this problem as well. As before, we will use $G[i]$ to denote the

$mrank$ that the man i is currently considering. Initially, $G[i]$ is 1 for all i, i.e., each man proposes to all his top choices. We say that G has a superstable matching if there exist n women $w_1, w_2, \ldots w_n$ such that $\forall i : w_i \in mpref[i][G[i]]$ and the set (m_i, w_i) is a superstable matching.

We define a bipartite graph $Y(G)$ on the set of men and women with respect to any G as follows. If a woman does not receive any proposal in G, then she is unmatched. If she receives multiple proposals, then there is an edge from that woman to all men in the most preferred rank. We say that $Y(G)$ is a perfect match if every man and woman have exactly one adjacent edge in $Y(G)$,

We claim

Lemma 17.6 *If $Y(G)$ is not a perfect matching, then there is no superstable matching with G as the proposal vector.*

Proof: If there is a man with no adjacent edge in $Y(G)$ then it is clear that G cannot have a superstable matching. Now consider the case where a man has at least two adjacent edges. If all the adjacent women for this man have degree one, then exactly one of them can be matched with this man, and other women will remain unmatched. Therefore, there is at least one woman w who is also adjacent to another man m'. If w is matched to m, then (m', w) is a blocking pair. If w is matched to m', then (m, w) is a blocking pair.

■

We now claim that the predicate $B(G) \equiv$ "$Y(G)$ is a perfect matching$''$ is a lattice-linear predicate.

Lemma 17.7 *If $Y(G)$ is not a perfect matching, then at least one index in G is forbidden.*

Proof: Consider any man i such that there is no edge adjacent to i in $Y(G)$. This happens when all women that the man i has proposed in the state G have rejected him. Consider any H such that $H[i]$ equals $G[i]$. All women had rejected the man i in G. As H is greater than G, these women can only have more choices and will also reject man i in H.

Now suppose that every man has at least one adjacent edge. Let $Z(G)$ be the set of women with degree exactly one. If every woman is in $Z(G)$, then we find that $Y(G)$ is a perfect matching because every man has at least one adjacent edge. If not, consider any man i who is not matched to a woman in $Z(G)$. This means that all the women he is adjacent have degrees strictly greater than one. In H all these women would have better ranked proposals or equally ranked proposals. In either case, the man i would not be matched with any of these women. Hence, i is forbidden.

■

We are now ready to present LLP-ManOptimalSuperStableMarriage. In LLP-ManOptimalSuperStableMarria
we start with the proposal vector G with all components $G[j]$ as 1. Whenever a woman receives multiple proposals, she rejects the proposals of men who are ranked lower than anyone who has proposed to her. We say that a man j is forbidden in G, if every woman z that man j proposes in G is either engaged to or proposed by someone who she prefers to j or is indifferent with respect to j. LLP-ManOptimalSuperStableMarriage is a parallel algorithm because all processes j such that forbidden(j) is true can advance in parallel. In LLP-ManOptimalSuperStableMarriage, we use $Y(j)$ to denote all women than man j has proposed in the state G, i.e., $Y(j)$ equals $mpref[j][G[j]]$.

Let us consider an example on three men and three women. The preference lists for men are:
$m_1 : (w_1 = w_2), w_3$
$m_2 : w_2, (w_1 = w_3)$

$m_3 : w_3, w_1, w_2$

The preference lists for women are:
$w_1 : (m_2 = m_3), m_1$
$w_2 : m_1, m_3, m_2$
$w_3 : (m_1 = m_3), m_2$

The algorithm starts with $G = (1, 1, 1)$. The man m_1 proposes to both w_1 and w_2. The man m_2 proposes to w_2 and m_3 proposes to w_3. Since w_3 receives only the proposal from m_3, she accepts it. Since w_2 receives proposals from m_1 and m_2, she rejects m_2 because she prefers m_1. Since w_1 receives the proposal only from m_1, she also accepts the proposal. The man m_2 is forbidden because all his proposals are rejected and he moves to the next level in his preferences. The man $_2$ now proposes both w_1 and w_3. The woman w_1 prefers m_2 to m_1 so she rejects m_1 and accepts m_2. The woman w_3 prefers m_3 to m_2, so she rejects m_2. At this point, we have m_1 with the accepted proposal from w_2, m_2 with the accepted proposal from w_1 and m_3 with the accepted proposal from w_3. The marriage $(m_1, w_2), (m_2, w_1), (m_3, w_3)$ is the optimal super stable marriage.

Algorithm LLP-ManOptimalSuperStableMarriage: A Parallel Algorithm for Man-Optimal Super Stable Matching

```
1 $P_j$: Code for thread $j$
2 **input**: $mpref[i,k]$: set of int for all $i,k$; $wrank[k][i]$: int for all $k,i$;
3 **init**: $G[j] := 1$;
4 **always**: $Y(j) = mpref[j][G[j]]$;
5 **forbidden($j$)**:
6     $\forall z \in Y(j) : \exists i \neq j : \exists k \leq G[i] : (z \in mpref[i][k]) \wedge (wrank[z][i] \leq wrank[z][j]))$
7 // all women $z$ in the current proposals from $j$ have been proposed by someone who either they
    prefer or are indifferent over $j$.
8     **advance**: $G[j] := G[j] + 1$;
```

Let us verify that this algorithm indeed generalizes the standard stable marriage algorithm. For the standard stable marriage problem, $mpref[i, k]$ is a singleton for all i and k. Hence, $Y(j)$ is also a singleton. Using z for the singleton value in $Y(j)$, we get the expression $\exists i \neq j : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (wrank[z][i] < wrank[z][j]))$ which is identical to the stable marriage problem once we substitute $<$ for $\leq$ for comparing the *wrank* of man i and man j.

When the preference list has a singleton element for each rank as in the classical stable marriage problem, we know that there always exists at least one stable marriage. However, in the presence of ties, there is no guarantee of the existence of a superstable marriage. Consider the case with two men and women, where each one of them does not have any strict preference. Clearly, in this case, there is no superstable marriage.

By symmetry of the problem, one can also get woman-optimal superstable marriage by switching the roles of men and women. Let $mpref[i].length()$ denote the number of equivalence classes of women for man i. When all women are tied for the man i, the number of equivalence classes is equal to 1, and when there are no ties, then it is equal to n. Consider the distributive lattice L defined as the cross product of $mpref[i]$ for each i. We now have the following result.

Theorem 17.8 *The set of superstable marriages, $L_{superstable}$, is a sublattice of the lattice L.*

Proof: From Lemma 17.7, the set of superstable marriages is closed under the meet operation. By the symmetry of men and women, the set is also closed under the join operation.

■

It is already known that the set of superstable marriages forms a distributive lattice [Spi95]. The set of join-irreducible elements of the lattice $L_{superstable}$ forms a partial order (analogous to the rotation poset [GI89]) that can be used to generate all superstable marriages. Various posets to generate all superstable marriages are discussed in [Sco05, HG21].

We note that the algorithm LLP-ManOptimalSuperStableMarriage can also be used to find the constrained superstable marriage. In particular, the following predicates are lattice-linear:

1. The regret of man i is at most the regret of man j.
2. The proposal vector is at least I.

Strongly Stable Matching

A matching M of men and women is *strongly stable* if there is no blocking pair (m, w) such that they are not married in M but (1) both prefer each other to their partners in M, or (2) one of them prefers the other to his/her partner in M and the other is indifferent. Formally, a pair of man and woman (m, w) is *blocking for a strongly stable matching M* if they are not matched in M and

$$\begin{aligned}&((mrank[m][w] \le mrank[m][M(m)])\wedge\\&(wrank[w][m] < wrank[w][M(w)]))\\\vee&((mrank[m][w] < mrank[m][M(m)])\wedge\\&(wrank[w][m] \le wrank[w][M(w)])).\end{aligned}$$

As in the superstable matching algorithm, we let $mpref[i][k]$ denote the set of women ranked k by man i. As before, we will use $G[i]$ to denote the $mrank$ that the man i is currently considering. Initially, $G[i]$ is 1 for all i, i.e., each man proposes to all his top choices. We define a bipartite graph $Y(G)$ on the set of men and women with respect to any G as follows. If a woman does not receive any proposal in G, then she is unmatched. If she receives multiple proposals, then there is an edge from that woman to all men in the most preferred rank. For superstable matching, we require $Y(G)$ to be a perfect matching. For strongly stable matching, we only require $Y(G)$ to contain a perfect matching.

We first note that a strongly stable matching may not exist. The following example is taken from [Irv94].

$m_1 : w_1, w_2$
$m_2 : w_1 = w_2$ (both choices are ties)

$w_1 : m_2, m_1$
$w_2 : m_2, m_1$

The matching $\{(m_1, w_1), (m_2, w_2)\}$ is blocked by the pair (m_2, w_1): w_1 strictly prefers m_2 and m_2 is indifferent between w_1 and w_2. The only other matching is $\{(m_1, w_2), (m_2, w_1)\}$. This matching is blocked by (m_2, w_2): w_2 strictly prefers m_2 and m_2 is indifferent between w_1 and w_2.

Consider any bipartite graph with an equal number of men and women. If there is no perfect matching in the graph, then by Hall's theorem there exists a set of men of size r who collectively are adjacent to fewer than r women. We define *deficiency* of a subset Z of men as $|Z| - N(Z)$ where $N(Z)$ is the *neighborhood* of

Z (the set of vertices that are adjacent to at least one vertex in Z). The deficiency $\delta(G)$ is the maximum deficiency taken over all subsets of men. We call a subset of men Z *critical* if it is maximally deficient and does not contain any maximally deficient proper subset. Our algorithm for finding a strongly stable matching is simple. We start with G as the global state vector with the top choices for all men. If $Y(G)$ has a perfect matching, we are done. The perfect matching in $Y(G)$ is a strongly stable matching. Otherwise, there must be a critical subset of men with the maximum deficiency. This set of men must then advance on their proposal number, if possible. If these men cannot advance, then there does not exist a strongly stable marriage and the algorithm terminates.

Algorithm LLP-ManOptimalStronglyStableMarriage: A Parallel Algorithm for Man-Optimal Strongly Stable Matching

```
P_j: Code for thread $j$
input: $mpref[i,k]$: set of int for all $i,k$; $wrank[k][i]$: int for all $k,i$;
init: $G[j] := 1$;
always: $Y(j) = mpref[j][G[j]]$;
forbidden($j$):
    $j$ is a member of the critical subset of men in the graph $Y(G)$
    advance: $G[j] := G[j] + 1$;
```

LLP-ManOptimalStronglyStableMarriage is the LLP version of the algorithm proposed by Irving and the interested reader is referred to [Irv94] for the details and the proof of correctness. Similar to superstable marriages, we also get the following result.

Theorem 17.9 *The set of strongly stable marriages, $L_{stronglystable}$, is a sublattice of the lattice L.*

Observe that each element in $L_{stronglystable}$ is not a single marriage but a set of marriages. This is in contrast to $L_{superstable}$, where each element corresponds to a single marriage.

17.6 Problems

1. Algorithm LLP-OptimalHospitalResident starts from the initial vector of all top choices. Modify the algorithm to give a hospital-resident matching when the search starts from an arbitrary given initial vector I (whenever such a matching exists).

2. Give a resident-oriented algorithm for the Hospital-Resident problem.

3. Show that the problem of sorting an array can be solved using the stable matching problem. (Hint: Suppose that we have n numbers and n slots. View the numbers as men and slots as women. We assume that all numbers are unique. The numbers prefer the least numbered slot as possible and the slots prefer as small a number as possible. Since stable matching is a matching every number gets unique slot. Furthermore, it cannot happen that a there exists i and j such that slot i prefers number assigned to slot j (the number at slot j is smaller) and the number at slot j prefers i (slot i is less than slot j). The algorithm takes $O(n^2)$ sequential proposals. Easy to show that it takes $O(n)$ parallel rounds as at least one number is fixed in each round.)

4. Give all the super stable marriages for the preference lists given below. The preference lists for men are:
   $m_1 : (w_1 = w_2), w_3$
   $m_2 : w_2, (w_1 = w_3)$
   $m_3 : w_3, w_1, w_2$

   The preference lists for women are:
   $w_1 : (m_2 = m_3), m_1$
   $w_2 : m_1, m_3, m_2$
   $w_3 : (m_1 = m_3), m_2$

5. Show the correctness of the LLP-ManOptimalStronglyStableMarriage.

17.7 Bibliographic Remarks

The notion of superstable matching and strongly stable matching and the associated algorithms are from [Irv94]. The LLP versions of these algorithms are from [Gar23].

Chapter 18

Algorithms in Number Theory

18.1 Introduction

In this chapter, we cover some basic algorithms in number theory. The set of all natural numbers forms a distributive lattice under the usual *divides* relation, that is, we define $x \leq y$ for two natural numbers x and y if x divides y. In this (infinite) lattice, the bottom element is 1 since it divides all numbers. The join of any two elements is given by their *least common multiple,* and the meet of any two elements is given by their *greatest common divisor.*

18.2 Greatest Common Divisor (GCD) Algorithm

Let A be an array of n natural numbers. The set of all common divisors of A forms a distributive lattice. This lattice is never empty because 1 divides all numbers. Our goal is to find the greatest common divisor of these numbers. Instead of finding just one number, we find G, an array of numbers of size n such that each number in G divides all the numbers in A. When G is the greatest such vector, then we have the required GCD.

We start with $G[i]$ equal to $A[i]$ for all i. We assume that all are numbers greater than or equal to 1. The algorithm terminates with all numbers identical and equal to the GCD.

The predicate

$$B \equiv (\forall i : gcd(A) \ divides \ G[i]) \wedge (\forall i, j : G[i] \geq G[j])$$

We first need to show that B is lattice-linear.

Lemma 18.1 *B is lattice-linear.*

Proof: Suppose that B is false. Since the first part is invariant of the program, there exist i and j such that $G[j]$ is strictly greater than $G[i]$. Since G can only decrease, the predicate can never become true unless $G[j]$ is advanced, i.e., decreased. This decrease must be such that the invariant is maintained.

■

Algorithm GCD1: Finding the greatest common divisor of a set of numbers

```
var
    $G$: array[1..$n$] of int initially $\forall i : G[i] = A[i]$;
forbidden(j): $\exists i : G[j] > G[i]$
    advance(j): if $G[i]$ divides $G[j]$ then $G[j] := G[i]$;
        else $G[j] := G[j]$ $mod$ $G[i]$;
```

Suppose that $G[i]$ is greater than $G[j]$ for some j and $G[j]$ does not divide $G[i]$, then we claim that it is safe to reduce $G[i]$ to $G[i] mod\ G[j]$. This step gives us an algorithm. If all numbers are the same, we stop. Otherwise, if $G[i] > G[j]$, we replace $G[i]$ by $G[j]$ if $G[i]$ is a multiple of $G[j]$ and $G[i] mod\ G[j]$ otherwise.

The invariant is that every common divisors of A is also a common divisor of G. The invariant holds initially because G is set to A. After every advance, we only need to consider two cases. If $G[j]$ is greater than $G[i]$ and $G[i]$ divides $G[j]$, then it is sufficient to set $G[j]$ to $G[i]$ because the divisors of $G[i]$ are common divisors to both $G[i]$ and $G[j]$. If $G[i]$ does not divide $G[j]$, then the only common divisors are those that divide $G[i]$ and $G[j] mod\ G[i]$ (prove it!).

It is easy to see that the algorithm terminates. All values are initially non-zero positive integers. They stay non-zero and integral and always decrease; therefore, the algorithm must terminate. Initially, G is the same as the given vector A. It decreases as the algorithm executes, and the least possible value is the vector with all 1's.

We now give an alternative GCD algorithm based on lattice-linearity. Let A be an array of n natural numbers. Our goal is to find the GCD of these numbers. When all elements of A are divided by the GCD, we get a quotient vector that is also a vector of natural numbers. Finding GCD of A is equivalent to finding the minimum vector G such that there exists a number d such that for all i, $A[i] = d * G[i]$. The problem can be formulated as finding the minimum G such that $\forall i : G[i] = A[i]/d$. Equivalently, our goal is to find the minimum G such that $B_{gcd}(G) \equiv \forall i, j : A[i]/G[i] = A[j]/G[j]$, This feasibility predicate, B_{gcd}, is equivalent to $\forall j : G[j] \geq \max_i\{A[j]/A[i] * G[i]\}$. Since the right-hand side is a monotone function, we know that the predicate is lattice-linear. We can apply the LLP algorithm with $forbidden(G, j)$ as $G[j] < \max_i\{A[j]/A[i] * G[i]\}$ and $\alpha(G, j) = \lceil \max_i\{A[j]/A[i] * G[i]\rceil$.

We now give an algorithm to compute the gcd of an array of numbers based on this idea.

Algorithm GCD2: Finding the greatest common divisor of a set of numbers

```
var
    $G$: array[1..$n$] of int initially $\forall i : G[i] = 1$;
        for all $j$ such that $G[j] < \max_i\{A[j]/A[i] * G[i]\}$;
            $G[j] := \lceil \max_i\{A[j]/A[i] * G[i]\rceil$;
        endfor;
    return $A[1]/G[1]$;
```

The above algorithm was mechanically derived from the LLP algorithm. Let us see how it relates to Euclid's algorithm to compute GCD of two numbers: $A[1]$ and $A[2]$.

Suppose that $A[1]$ and $A[2]$ are 10 and 15, respectively. We get the following steps. Initially, $G[1] = 1$ and $G[2] = 1$. Since $G[2]*A[1] < G[1]*A[2]$, we update $G[2]$ to 2, getting $G = [1, 2]$. Now, $G[1]*A[2] < G[2]*A[1]$, we get $G = [2, 2]$. Now $G[2]*A[1] < G[1]*A[2]$, we update $G[2]$ to $\lceil A[2]/(A[1]*G[1]\rceil = 15/10*2 = 3$.

Algorithm GCD3: Algorithm GCD to find the greatest common divisor of a set of numbers

```
var
    $G$: int initially $\forall i : G[i] = 1$;
        while($A[1]/G[2] \neq A[2]/G[1]$) do
            if $G[1] * A[2] < G[2] * A[1]$ then
                $G[1] := \lceil A[1]/A[2] * G[2] \rceil$;
            else
                $G[2] := \lceil A[2]/A[1] * G[1] \rceil$;
        endwhile;
    return $A[1]/G[1]$;
```

We now have $G = [2, 3]$ and the algorithm terminates. It returns $A[1]/G[1]$ which equals 5.

18.3 Chinese Remainder Theorem

We have r coprime numbers $m_1, m_2, \ldots m_r$ and r integers b_i such that $0 \leq b_i < m_i$. Our goal is to find the least number x such that $x \equiv b_i(mod\ m_i)$. There are r equations. We think of this as r processes such that process i is solving the equation i. Furthermore, we want each process to get the same value of x. Initially, we start with $G[i]$ equal to $b[i]$ for all i. We are searching for the least G such that

$$B \equiv (\forall i, j : G[i] \leq G[j]) \wedge (\forall i : G[i] \equiv b_i(mod\ m_i))$$

Since each process P_i can only increase the value of $G[i]$, the predicate $(G[i] \leq G[j])$ is lattice-linear. Furthermore, $(G_i \equiv b_i(mod\ m_i))$ is a local predicate and therefore also lattice-linear. Thus, B is a lattice-linear predicate, and we can use the LLP algorithm to solve the problem. The simplest algorithm will advance on the process P_i by 1 if, either $G[i] < G[j]$ for some j or if $\neg(G[i] \equiv b_i(mod\ m_i)$. Furthermore, the only way the second part would become true is if $G[i]$ is advanced to a state in which $G[i] mod\ m_i$ equals b_i. To simplify matters further, we initialize $G[i]$ to b_i since that is the smallest state in P_i satisfying the local predicate. We maintain the invariant that $\forall i : G[i] = b[i] mod\ m[i]$. With this observation, we get the Figure Par-CRT.

Algorithm Par-CRT: A Parallel Algorithm for Chinese Remainder Problem

```
input: $m : array[1..n]$ of int;
$b : array[1..n]$ of int;
var: $G$: array$[1..n]$ of int ;
init: $\forall j : G[j] := b[i]$;
forbidden(j): $(\exists i : G[j] < G[i]) \vee (G[j] mod\ m[j] \neq b[j])$
advance(j):
    $G[j] := G[j] + min_k\{G[j] + k * m[j] \geq G[i]\}$;
```

Observe that process P_i does not need access to any $b[j]$ or $m[j]$ for $j \neq i$. It needs access to $G[j]$. Hence, this algorithm can be applied in a distributed fashion if G is publicly available even though b and m are private. Applying the Chinese Remainder Theorem to solve this problem requires public access to m.

So far we have not shown that the algorithm terminates, but we know that if the algorithm terminates it gives us the smallest number that satisfies B. For this particular example, we do know that the program terminates due to Chinese Remainder Theorem. However, if instead of linear equations we had arbitrary equations, then one could still apply this program, except that the program may not terminate.

Instead of searching for the solution in increasing order, one can also search it in the decreasing order provided we know where to start the search from. In the above example, let M be the product of all $m[i]$, i.e., $M = m[1]m[2]...m[n]$. Then, from the Chinese Remainder Theorem, we know that the solution exists between $0..M - 1$. It is easy to see that B is also dual lattice-linear. Hence, one can use the following algorithm to find the largest solution that is at most M.

Algorithm Par-CRT2: A Parallel Algorithm for Chinese Remainder Problem

```
1 input: $m : array[1..n]$ of int;
2 $b : array[1..n]$ of int;
3 var: $G$: array[1..$n$] of int ;
4 init: $\forall j : G[j] :=$ greatest number at most $M - 1$ that is congruent to $b[j]$ mod $m[j]$;
5 forbidden(j): $(\exists i : G[j] > G[i]) \vee (G[j] mod\ m[j] \neq b[j])$
6 advance(j):
7     $G[j] := G[j] - min_k\{G[j] - k * m[j] \leq G[i]\}$;
```

18.4 Viewing Numbers as Distributive Lattice

The set of all natural numbers forms a distributive lattice under the divides relation. This distributive lattice (infinite) has the following representation. For every prime p, we have a chain of elements $p^0, p^1, p^2, \ldots$. This infinite poset has as many chains as the number of distinct primes. There is a 1-1 correspondence between the lattice of natural numbers and the cross product of all chains in the poset (by the unique factorization of numbers) We can go from any element n in the number lattice to an element in the cross product by choosing $k + 1$ elements from the chain for the prime p if k is the largest integer such that p^k divides n.

Let us define some arithmetical functions on the set of natural numbers. We are interested in functions that can be defined on the poset representation of natural numbers, i.e., the function is defined only for the prime powers. The value of a function on any number is simply the product of its value on each of the prime powers.

Definition 18.2 (Multiplicative Functions) *We call an arithmetical function f* multiplicative *if it is not a zero function and $f(mn)$ equals $f(m)f(n)$ whenever $gcd(m, n)$ equals 1.*

Note that we require $f(mn)$ equals $f(m)f(n)$ only when $gcd(m, n)$ equals 1. If we drop the requirement of gcd to be 1, then the function is called *completely* multiplicative.

We now introduce some multiplicative functions.

Definition 18.3 (Mobius function) *The Mobius function μ is defined on any prime power p^k as 1 if $k = 0$, -1 if $k = -1$ and 0, otherwise.*

Here, μ is defined to be multiplicative. It follows that if $n = p_1^{a_1} p_2^{a_2}, \dots p_m^{a_k}$, then $\mu(n)$ equals $(-1)^k$ if $\forall i \in [k] : a_i = 1$ and $\mu(n)$ equals 0, otherwise.

We now show a simple formula for the divisor sum of $\mu(n)$.

Theorem 18.4 *For any $n > 1$, $\sum_{d|n} \mu(d) = 0$.*

Proof: Let $n = p_1^{a_1} p_2^{a_2}, \dots p_k^{a_k}$. It is easy to see that d divides n iff $d = p_1^{b_1} p_2^{b_2}, \dots p_m^{b_k}$, where for all $i \in [m] : b_i \leq a_i$. Consider the poset $P(n)$. It is easy to verify that $\sum_{d|n} \mu(d)$ is the sum of all cross-products of chains in $P(n)$. This sum is also equal to the product of sum of $\mu(d)$ where d is a prime power. However, $\sum_{d_i|n, d \text{ is a prime power of } p_i} \mu(d)$ is zero because for any prime p that divides n, $\mu(p^0) = 1, \mu(p^1) = -1$, and $\mu(p^k) = 0$ for $k \geq 2$. Hence, $\sum_{d|n} \mu(d) = 0$.

■

We now show another multiplicative arithmetical function defined on the set of natural numbers.

Definition 18.5 (Euler's Totient function) *The Euler's totient function ϕ is defined on any prime power p^k for $k > 0$ as $p^k - p^{k-1}$. Also, $\phi(1)$ is defined as 1.*

For example, $\phi(3^2) = 3^2 - 3^1 = 9 - 3 = 6$. The function ϕ can be extended to any natural number n by using the multiplicative property. We now show that

Lemma 18.6 *For any $n > 1$, $\phi(n)$ equals the number of positive integers less than or equal to n which are relatively prime to n.*

Proof: Let $n = p_1^{a_1} p_2^{a_2}, \dots p_k^{a_k}$ for some $k \geq 1$. When k equals 1, $n = p_1^{a_1}$. There are $p_1^{a_1}$ positive integers less than or equal to n. Of these, $p_1^{a_1 - 1}$ are less than n and divide n. Therefore, $\phi(n)$ equals the number of positive integers less than or equal to n that are relatively prime to n. Let A_i denote the numbers that are relatively prime to $p_i^{a_i}$. Now, suppose that $k > 1$. From the multiplicative property, $\phi(n) = \phi(p_1^{a_1})\phi(p_2^{a_2}), \dots \phi(p_k^{a_k})$. It can be easily verified that $\phi(n)$ consists of numbers relatively prime to n.

■

We now show that

Theorem 18.7 *For any $n > 1$, $\sum_{d|n} \phi(d) = n$.*

Proof: Let $n = p_1^{a_1} p_2^{a_2}, \dots p_k^{a_k}$ for some $k \geq 1$. The poset representation of $\phi(n)$ is shown in Figure. Clearly, $\sum_{d|n} \phi(d) = \Pi_{i=1}^{k} \sum_{j=0}^{j=k} \phi(p_i^j)$. Since $\phi(p_i^j) = p^j - p^{j-1}$, by the telescoping argument we get $\sum_{j=0}^{j=k} \phi(p_i^j) = p_i^k$. Hence, $\sum_{d|n} \phi(d) = \Pi_{i=1}^{k} p_i^k = n$.

■

Some additional multiplicative functions are defined below.

1. The arithmetical function N is defined as $N(p^k) = p^k$.
2. The arithmetical function *Identity*, I is defined as $I(p^k) = 1$ if $k = 0$ and 0 otherwise.
3. The arithmetical function *Unit*, u is defined as $u(p^k) = 1$ for all p, k.

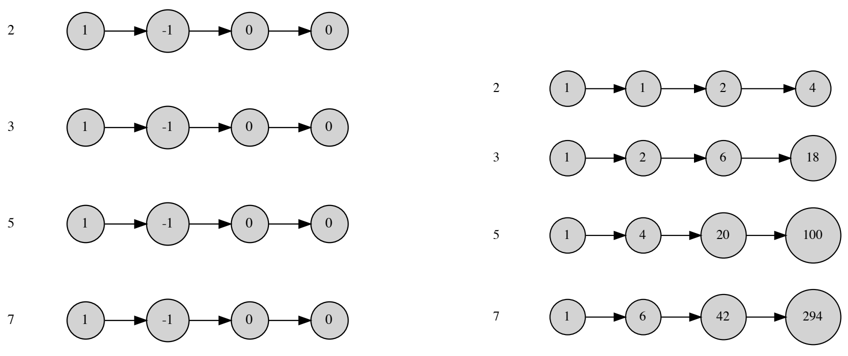


Figure 18.1: Poset representations for μ and ϕ. The value for $\phi(p^k)$ is $p^k - p^{k-1}$. Thus, the third entry for prime 3 is $3^2 - 3^1 = 6$.

4. The arithmetical function λ is defined as $\lambda(p^k) = (-1)^k$ for all k.

The following Lemma gives the summation results for these functions.

Lemma 18.8 *For any $n > 1$,*
(a) $\sum_{d|n} N(d) = \Pi_{p^k|n}(p^{k+1} - 1)/(p-1)$.
(b) $\sum_{d|n} I(d) = 1$.
(c) $\sum_{d|n} u(d) =$ the number of divisors of n.
(d) $\sum_{d|n} \lambda(d) = 1$ if n is a square and 0, otherwise.

Proof: Since all these functions are multiplicative, we will simply compute them for $n = p^k$.
(a) $\sum_{d|p^k} N(d) = (p^{k+1} - 1)/(p-1)$ since we have a geometric series. From multiplicative property, the results follows.
(b) $\sum_{d|p^k} I(d) = 1$ since only $I(1) = 1$. It is 0 for all $k > 1$. By multiplicative property, we get that $\sum_{d|n} I(n) = 1$.
(c) $\sum_{d|p^k} u(d) = k + 1$. By invoking the multiplicative property, we get that $\sum_{d|n} u(d) =$ the number of divisors of n.
(d) $\sum_{d|p^k} \lambda(d) = 1$ if k is even and 0 otherwise. Therefore, $\sum_{d|n} \lambda(d) = 1$ if for each prime the corresponding exponent is even and 0 otherwise. Thus, $\sum_{d|n} \lambda(d) = 1$ if n is a square and 0, otherwise.

■

18.5 Problems

18.6 Bibliographic Remarks

Euclid's algorithm for computing the greatest common divisor is a cornerstone of number theory. [Knu97a] provides a comprehensive analysis of its efficiency and extensions. The Chinese Remainder Theorem solves systems of congruences with coprime moduli. [Gau01] formalized it in modern number theory, and [Ros11] gives an accessible treatment with practical applications.

Chapter 19

Linear Programming

19.1 Introduction

In this chapter, we consider linear programming, which is a general technique for solving many problems. We note here that linear programming has also served as a unifying tool for many combinatorial optimization problems. In particular, the man-optimal stable marriage problem, the shortest path problem, and the least market clearing price can all be modeled as integer linear programs with total unimodularity. However, it is not known whether linear programming is strongly polynomial. In contrast, the LLP method results in algorithms that are close to the best known strongly polynomial algorithms for all these problems. There are other differences in these two approaches. Algorithms for linear programming typically stay inside the feasible space and improve the objective function with every iteration. Our approach keeps the objective function as extremal as possible while making progress towards a feasible solution. The linear programming method uses real variables with linear objective function and linear constraints. Our method assumes that the search is for the infimum element of a feasible space in a distributive lattice such that the feasible space satisfies a lattice-linear predicate.

Linear constraints for linear programming are defined using the conjunction of half-spaces. Lattice-Linear predicates are incomparable to half-spaces. There are half-spaces that are not lattice-linear, and there are lattice-linear predicates that are not half-spaces.

Linear programming is a general technique to solve combinatorial optimization problems. Many optimization problems can be modeled using linear or integer linear programs.

Suppose that there is an objective function which we are trying to minimize,

$7x_1 + x_2 + 5x_3$

subject to constraints,

$x_1 - x_2 + 3x_3 \geq 10$

$5x_1 + 2x_2 - x_3 \geq 6$

where x_1, x_2, x_3 are the decision variables and the assumption is that

$x_1, x_2, x_3 \geq 0$

Suppose that we have a solution, $x = (2, 1, 3)$. On substituting in the objective function, we get the objective value to be $14+1+15 = 30$. The reader can verify that this solution is also feasible, as it satisfies the constraints, 2 - 1 + 3(3) = 10

5(2) + 2(1) - 3 = 9

Trying to find a bound on primal:

$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10$ by comparing coefficients point-wise and $x_i \geq 0$

solution has to be ≥ 10 but we are trying to get a better or greater lower bound

Add up the constraints equations,

$6x_1 + x_2 + 2x_3 \geq 16$

The objective function is still greater than the above equation. Hence new lower bound is 16.

The goal of primal was to minimize the objective function; the goal of its dual will be to maximize its objective function. We have to figure out what combination of each constraint should we add such that it gives the tightest bound on the objective function.

We have n decision variables and m constraints in our primal. So, the dual will have m variables (one variable for each constraint in the primal) and n constraints (since there are n decision variables in the primal).

The objective function is $10y_1 + 6y_2$ subject to constraints,

$y_1 + 5y_2 \leq 7$

$-y_1 + 2y_2 \leq 1$

$3y_1 - y_2 \leq 5$

In a generalized representation of the primal, objective function can be written as

$$\sum_{i=1}^{n} c_i x_i$$

constraints can be written as

$$\sum_{j} A_{ij} x_j \geq b_j$$

where A is a matrix of dimension m*n, c and b are vectors of size n and m respectively.

For dual, b is translated to c, c is translated to b and matrix A becomes A^T.

19.2 Duality in Linear Programming

Strong Duality

Theorem 19.1 *The primal program has finite optimum if and only if its dual has finite optimum. Moreover, if $x*$ and $y*$ are optimal solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

This theorem translates into all many min-max theorems such as max flow = min cut, max matching = min vertex cover and min number of chains to cover a poset = max sized anti-chain.

Weak Duality

Theorem 19.2 *If x and y are feasible solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof:

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j$$

This claim is true because y is feasible and $x \geq 0$

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \geq \sum_{i=1}^{m} b_i y_i$$

This claim is true because x is feasible and $y \geq 0$

Hence proved,

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

■

Complementary Slackness Conditions

Theorem 19.3 *Given that x,y are (P) and (D) feasible solutions. Then x and y are both optimal if and only if the following complementary slackness conditions are satisfied:*

$$\forall j \quad either (x_j = 0) \quad or \quad \sum_{i=1}^{m} a_{ij} y_i = c_j$$

$$\forall i \quad either (y_i = 0) \quad or \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

19.3 Formulating Combinatorial Optimization Problems as Linear Programs

Maximum Matching Problem

Primal:

$$max \sum_{i,j} W_{ij} x_{ij}$$

subject to:

$$\sum_{j} x_{ij} \leq 1$$

and

$$\sum_i x_{ij} \leq 1$$

where x_{ij} is an edge between A and B, i is an index from A and j is an index from B. For each edge, there is a weight W_{ij}. We need to pick the edges such that the sum of their weights is maximum.

Replacing integral constraint by linear constraint,

$$x_{ij} \in \{0,1\} \Rightarrow x_{ij} \geq 0$$

The above transformation is called LP-Relaxation. This can be used only when we can prove that the optimal solution is always going to be integral.

When $|A| = |B| = n$, the number of decision variables equals n^2, and the number of constraints is $2n$.

Dual:

The number of decision variables equals $2n$ and the number of constraints equals n^2.

Objective function (minimizing vertex labels):

$$min \sum u_i + \sum v_j$$

subject to: $u_i + v_j \geq w_{ij}$

$u_i \geq 0 \quad and \quad v_j \geq 0$

where u_i is constraints for side A and v_j is constraints for side B

The Transportation Problem

We have n supply nodes and n demand nodes. The supply at node i is given by a_i and the demand at node j is given be b_j. We assume that the total supply is equal to the total demand. If the supply and demand does not match, we can create fake nodes to ensure the equality.

Let w_{ij} be the cost function when an item is shipped from i to node j. Let the variable x_{ij} be the number of items i shipped from the source node i to the demand node j. Our goal is to choose x_{ij} such that the cost of transportation is minimized. $\sum_{i,j} w_{ij}x_{ij}$ is maximized subject to:

$\sum_j x_{ij} = a_i$ for all i

$\sum_i x_{ij} = b_j$ for all j

and $x_{ij} \geq 0$.

The dual of the relaxed linear program is:

maximize $\sum_i a_i u_i + \sum_j b_j v_j$ subject to

$u_i + v_j \leq w_{ij}$ for all i, j,

$u_i \geq 0$ for all i,

$w_j \geq 0$ for all j.

The Stable Marriage Problem

Let $x_{i,j}$ denote that man i is matched to woman j. We require that $x_{ij} \in \{0,1\}$. The constraint that every man and every woman is matched to a single partner is given as follows. For each man i,

$$\sum_j x_{ij} = 1.$$

Similarly, for each woman j,

$$\sum_i x_{ij} = 1.$$

Furthermore, we would like this assignment to not have any blocking pair. For any man i and woman j, they do not form a blocking pair

This condition can be expressed as follows. For any man i and woman j, if j is matched to a man who is less preferred than i, then man i must be matched to a woman who he prefers to i. Otherwise, (i, j) will form a blocking pair for the matching. Let $Q(j, i)$ be the set of men who are less preferable than i to woman j, i.e.,

$$Q(j,i) = \{i' \mid wrank[j][i'] > rank[j][i]\}$$

Then, the expression $\sum_{i' \in Q(j,i)} x_{i',j}$ is one iff j is matched to someone who is less preferable than i. Similarly, let $R(i, j)$ be the set of women who are more preferable than woman j to man i. Then, the expression $\sum_{j' \in R(i,j)} x_{i,j'}$ is one iff i is matched to someone who is more preferable than woman j. Given these two expressions, we can write the condition that (i, j) is not a blocking pair as

$$\sum_{i' \in Q(j,i)} x_{i',j} - \sum_{j' \in R(i,j)} x_{i,j'} \leq 0.$$

Thus, we have $O(n^2)$ constraints.

The Shortest Path Problem

Let x_i denote the distance from the source vertex x_0 to x_i. Let t be the destination vertex. We assume that there are no incoming edges to x_0 or outgoing edges from x_t.

We would like to maximize $x_t - x_0$ such that for all edge $(i, j) \in E$: $x_j \leq x_i + w[i, j]$. Hence, the LP formulation is as follows.

$$\begin{aligned} \text{maximize} \quad & x_t - x_s \\ \text{subject to} \quad & x_j \leq x_i + w[i,j] \quad \forall (i,j) \in E, \\ & x_i \geq 0 \quad \forall i \end{aligned} \tag{19.1}$$

This formulation is identical to that we used in designing an LLP based algorithm (or equivalently, Bellman-Ford algorithm). We assume that there is no incoming edge to x_0. In this linear program, there are n variables, one for each node, and there are m constraints, one for each edge. The variables x_i are nonnegative.

The dual of this linear program is as follows.

We have m variables, $y_{i,j}$ for each edge $(v_i, v_j) \in E$. The variable $y_{i,j}$ can be viewed as the flow on the edge (v_i, v_j).

We would like to minimize $\sum_{(i,j) \in E} w[i,j] * y[i,j]$ as follows.

minimize $\sum_{(i,j) \in E} w[i,j] * y[i,j]$

subject to

for each node v_i other than v_0 and v_t: $\sum_j y_{j,i} - \sum_k y_{i,k} = 0$.

for v_t, $\sum_j y_{j,t} = 1$.

for v_0, -$\sum_j y_{0,j} = -1$.

for all i, j: $y[i, j] \geq 0$.

19.4 Comparison of Linear Programming with Lattice-Linear Predicates

Linear programming can also be viewed as a search for an optimal feasible solution. However, there are many important differences.

- **Vector Space vs Distributive Lattice**: First, the underlying space in linear programming is the set of real valued vectors whereas the underlying space in the lattice-linear predicate detection method is a distributive lattice.

  In the domain of distributive lattices, we do not have addition or the scalar multiplication as in vector spaces. All of lattice-linear predicate algorithms use the following two operations: meet of the underlying lattice and the "advance" operation. The advance operation maps an element of the lattice to a bigger element in the lattice.

- **Polyhedron vs Meet-Closed Predicates**: In linear programming, the feasible space is characterized by a polyhedron (or the set of vectors x such that $Ax \leq 0$ for some matrix A). There is no lattice structure required on the feasible space. It is not guaranteed that if two vectors are feasible, then their component-wise minimum vector is also feasible. Lattice-linear predicate detection requires the feasible space to be closed under meets.

  Linear programming has its optimization objective as minimization or maximization of a linear cost function. Lattice linear predicate detection simply uses the underlying order operation to define optimization. Since feasible space is closed under meets, the infimum of the feasible space is well-defined.

- **Efficiency of the General Algorithm**: Even though many problems studied in this book can also be solved via linear programming, the algorithms derived in that manner are not as efficient as the LLP algorithm. The following list gives the algorithms that are used for Linear Programming.

  1. **Simplex Method:** Developed by George Dantzig, it is widely used for solving linear programming problems by moving from one vertex of the feasible region to an adjacent one with a higher objective value until the optimum is reached. The *dual Simplex method* is a variant of the simplex method, used when the initial solution is not feasible. It modifies the constraints to move towards feasibility and optimality simultaneously. The worst-case time complexity is exponential, but in practice, it is often efficient.
  2. **Interior Point Methods:** These methods, including Karmarkar's algorithm, approach the optimal solution from within the feasible region. They can be faster than the simplex method for large-scale problems.
  3. **Ellipsoid Method:** Introduced by Khachiyan, this was the first polynomial-time algorithm for linear programming. It's more of theoretical interest as it's outperformed by other methods in practice.
  4. **Cutting Plane Method:** Adds additional linear constraints (cuts) to the problem to exclude regions that do not contain the optimal solution, gradually cutting closer to the optimal point.

5. **Branch and Bound Method:** Often used for mixed-integer linear programming, this method explores branches of possible solutions and bounds them to identify the optimal solution.

- **Parallelism of the General Algorithm**: The LLP algorithm determines which component in G are forbidden and advances on all of those components in parallel. The worst case complexity is given be the height of the lattice.

19.5 Problems

1. A company produces two products, X and Y, with profits of \$3 and \$4 per unit, respectively. Production requires labor and materials, with constraints:
   - Labor: $2X + Y \leq 8$ hours.
   - Materials: $X + 2Y \leq 8$ units.
   - Non-negativity: $X, Y \geq 0$.

   Formulate and solve the linear program to maximize profit, using the graphical method.

2. A factory produces three items, A, B, and C, with profits \$5, \$3, and \$4 per unit. Constraints are:
   - Machine hours: $3A + B + 2C \leq 12$.
   - Labor hours: $A + 2B + C \leq 10$.
   - Non-negativity: $A, B, C \geq 0$.

   Formulate and solve using the simplex method to maximize profit.

3. Formulate the max-flow problem for the network below as a linear program and solve it.

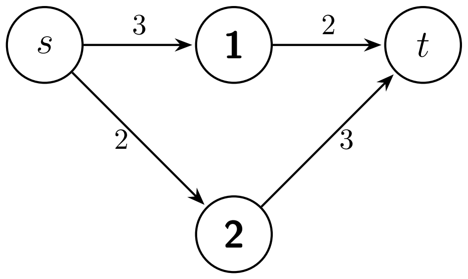


Figure 19.1: Flow network for Problem 3.

19.6 Bibliographic Remarks

Linear programming optimizes a linear objective subject to linear constraints. [Dan63] introduced the simplex method, a cornerstone algorithm, and developed foundational theory. [Kar84] proposed the interior-point method, establishing polynomial-time solvability. For a comprehensive modern treatment, including duality and applications, see [Van20].

Chapter 20

The Predicate Detection Problem

20.1 Introduction

Debugging and testing multithreaded software is widely acknowledged to be a hard task. Sometimes it takes a programmer days to locate a single bug, especially when the bug appears in one thread schedule but not in others. The current debugging and testing method for multithreaded programs is as follows. The programmer tries the program with multiple inputs in the hope of finding a faulty execution. However, the behavior of a multithreaded program depends not only on the external user input, but also the thread schedule and the order in which locks are obtained by the program. It is easy for the testing process to miss a bug that arises with an alternate schedule. One of the fundamental problems in debugging these systems is to check if the user-specified condition exists in *any* global state of the system that can be reached by an alternative thread schedule. This problem, called *predicate detection*, takes a concurrent computation (in an online or offline fashion) and a condition that denotes a bug (for example, violation of a safety constraint), and outputs a schedule of threads that exhibits the bug if possible. Predicate detection is predictive because it generates inferred reachable global states from the computation; an inferred reachable global state might not be observed during the execution of the program, but is possible if the program is executed in a different thread interleaving.

20.2 Detecting Conjunctive Predicates

In this section we describe two parallel algorithms to detect a conjunctive predicate $B = l_1 \wedge l_2 \wedge \cdots \wedge l_n$. To detect B, we need to determine if there exists a consistent global state G such that B is true in G. Note that given a computation on n processes each with m states, there can be as many as m^n possible consistent global states. Therefore, enumerating and checking the condition B for all consistent global states is not feasible. Since B is conjunctive, it is easy to show [GW92] that B is true iff there exists a set of states $s_1, s_2, ..., s_n$ such that (1) for all i, s_i is a state on P_i, (2) for all i, l_i is true on s_i and (3) for all i, j: $s_i \| s_j$. Our detection algorithm will either output such local states or guarantee that it is not possible to find them in the computation. When the global predicate B is true, there may be multiple G such that B holds in G. We are interested in algorithms that return the minimum G that satisfies B.

The minimum G corresponds to the smallest counter-example to a programmer's understanding becuase B typically represents the violation of a safety constraint.

Our parallel algorithm is based on the setting where the execution traces for all processes have been collected at one process. For example, in the centralized algorithm for conjunctive predicate detection, one process serves as a checker and collects the traces. All other processes involved in detecting the conjunctive predicate, referred to as application processes, check for local predicates during the computation. Each process P_i also maintains the vector clock algorithm. Whenever the local predicate of a process becomes true for the *first* time since the most recently sent message (or the beginning of the trace), it generates a debug message containing its local timestamp vector and sends it to the checker process [GW92].

The checker process uses queues of incoming messages to hold incoming local snapshots from application processes. We require that messages from an individual process be received in FIFO order. If the underlying system is non-FIFO, then sequence numbers can be attached with messages to ensure FIFO delivery. At the end of the computation, the checker process has a sequence of local states from each process where its local predicate is true. We now describe a sequential and a parallel algorithm to detect B on these traces. The sequential algorithm is an adaptation of the algorithm from [GW92]. We include it here because it is instrumental in understanding the parallel algorithm. Moreover, the correctness of the parallel algorithm is shown by assuming the correctness of the sequential algorithm.

20.3 A Work-Efficient Parallel Algorithm

The algorithm in Fig. 20.2 takes as input n traces each of size m as shown in Fig. 20.1. Since conjunctive predicates are lattice-linear, we simply use LLP algorithm to detect them. We use G for the consistent global state. A local state $G[j]$ is forbidden if it is less than $G[i]$ for some i. Since we are interested only in global states where the local predicate is true for all $G[i]$, we assume that for any i, we only consider $G[i]$ such that the local predicate is true in $G[i]$.

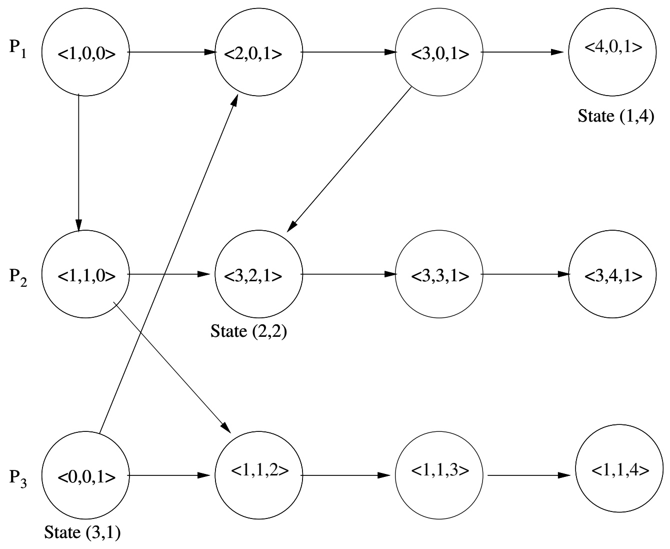


Figure 20.1: State-Based Model of a Distributed Computation

```
function ConjunctiveAlgorithm()
var
    $G$: array[1..$n$] of int initially 1;
    $T = (m_1, m_2, ..., m_n)$; //maximum number of proposals at $P_i$
always
    $forbidden(G, j, B) \equiv \exists i : G[j] \rightarrow G[i]$
while $\exists j : forbidden(G, j, B)$ do
    for all $j$ such that $forbidden(G, j, B)$ in parallel:
        if $(G[j] = T[j])$ then return null;
        else $G[j] := G[j] + 1$;
endwhile;
return $G$; // satisfying global state
```

Figure 20.2: Conjunctive Predicate Detection Algorithm.

20.4 An NC Algorithm for Conjunctive Predicate Detection

We now give another parallel algorithm that has lower time complexity but higher work complexity. Our approach is based on computing a *state rejection graph* for the trace. The state rejection graph is a directed graph with all local states as vertices of the graph. Let state j on process i be denoted as (i, j). The state rejection graph puts a rejection edge from the state (i, j) to (i', j') if the rejection of state (i, j) as a possible component of the consistent cut implies that the state (i', j') will also be rejected.

In Fig. 20.4 we show the state rejection graph of the computation in Fig. 20.1. The first state in P_1 given by the vector clock $\langle 1, 0, 0 \rangle$ will be rejected by the sequential algorithm because it happened before $\langle 1, 1, 0 \rangle$. The sequential algorithm will then move P_1 to the second state $\langle 2, 0, 1 \rangle$ (successor of the first state in P_1). However, this would imply that the state $\langle 0, 0, 1 \rangle$ will be rejected because $\langle 0, 0, 1 \rangle$ happened before $\langle 2, 0, 1 \rangle$. Hence, there is a rejection edge from $\langle 1, 0, 0 \rangle$ to $\langle 0, 0, 1 \rangle$. Similarly, the rejection of $\langle 0, 0, 1 \rangle$ implies that P_3 will move to $\langle 1, 1, 2 \rangle$. However, that move will result in rejection of the state $\langle 1, 1, 0 \rangle$. Therefore, we put a rejection edge from $\langle 0, 0, 1 \rangle$ to $\langle 1, 1, 0 \rangle$. Finally, the rejection of $\langle 1, 1, 0 \rangle$ will result in P_2 moving to $\langle 3, 2, 1 \rangle$ which will result in the rejection of $\langle 2, 0, 1 \rangle$ and $\langle 3, 0, 1 \rangle$. All the rejection edges are shown in dashed arrows in Fig. 20.4. We show how such a graph can be constructed efficiently in parallel. The next step in the algorithm is to compute the transitive closure of this graph. Finally, the algorithm determines the least local state at each process which has not been rejected. In this example, it is the fourth state on P_1, the second state on P_2 and the third state on P_3.

Our parallel algorithm is presented in Fig. 20.3. The input to the algorithm is the same as that of the LLP algorithm: a two-dimensional array of vector clocks so that we can determine the happened-before order between states.

We now explain the steps of the algorithm.

Step 1: We first create F, the set of all initially rejected states. Let I be the global state consisting of each processor's first local state, i.e., $I = \{(i, 1) \mid i \in 1..n\}$. If there are no dependencies between any of these states, we have already reached the first consistent global state. Else, if there is a dependency from one of these states to another, we reject whichever state happened-before the other and add it to F. We represent the set F by a boolean bit array of size n that is indexed by processor. F is initially empty. Then, we set $F[i]$ to 1 whenever there exists a state $(j, 1)$ such that $(j, 1) \rightarrow (i, 1)$.

```
**function** ParallelCut()
Input: $states : array[1 \ldots n][1 \ldots m]$ of vectorClock // Sequence of local states at each process
Output: Consistent Global State as array $cut[1 \ldots n]$

Step 1: Create $F$: set of states rejected in the first round
 **var** $F : array[1 \ldots n]$ of $0 \ldots 1$ **initially** 0;
 **for all** $(i \in 1 \ldots n, j \in 1 \ldots n)$ in parallel do
           **if** $((i,1) \to (j,1))$ **then**
              $F[i] := 1;$

Step 2: Create $R$: State Rejection Graph // Represented as an Adjacency Matrix
 **var** $R : [(1 \ldots n, 1 \ldots m), (1 \ldots n, 1 \ldots m)]$ of $0 \ldots 1$;
 **for all** $(i \in 1 \ldots n, j \in 1 \ldots m)$ in parallel do
      $R[(i,j),(i,j)] := 1;$
 **for all** $(i \in 1 \ldots n, j \in 1 \ldots m-1, i' \in 1 \ldots n, j' \in 1 \ldots m)$
           such that $i \neq i'$ in parallel do
           **if** $((i',j') \ \to (i,j+1))$ **then**
              $R[(i,j),(i',j')] := 1;$
           **else**
              $R[(i,j),(i',j')] := 0;$

Step 3: Create $RT$: transitive closure of $R$
 **var** $RT : array[(1 \ldots n, 1 \ldots m), (1 \ldots n, 1 \ldots m)]$ of $0 \ldots 1$;
 $RT := TransitiveClosure(R);$

Step 4: Create $valid$: replace invalid states by 0
 **var** $valid : array[[1 \ldots n][1 \ldots m]$ of $0 \ldots 1$;
 **for all** $(i \in 1 \ldots n, j \in 1 \ldots m)$ in parallel do
      $valid[i][j] := 1;$
 **for all** $(i \in 1 \ldots n, i' \in 1 \ldots n, j' \in 1 \ldots m)$ in parallel do
      **if** $(F[i] = 1) \wedge (RT[(i,1),(i',j')] = 1)$ **then**
           $valid[i'][j'] := 0;$


Step 5: Create $cut$: First Consistent Global State
 **var** $cut : array[1 \ldots n]$ of $0 \ldots m$ **initially** 0;
 **for all** $(i \in 1 \ldots n, j \in 1 \ldots m)$ in parallel do
           **if** $(valid[i][j] \neq 0)$ **then**
              **if** $(j = 1) \vee ((j > 1) \wedge (valid[i][j-1] = 0)$ **then**
                 $cut[i] := j;$
 **for all** $(i \in 1 \ldots n)$ in parallel do
           **if** $(cut[i] = 0)$ **then**
              output("No satisfying Consistent Cut");

 return $ConsistentCut := cut;$
```

Figure 20.3: The ParallelCut algorithm to find the first consistent cut.

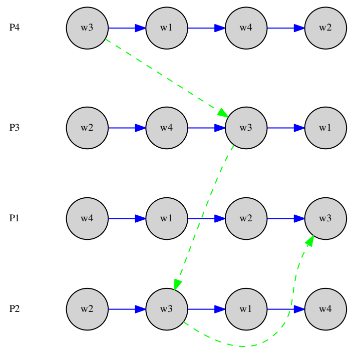


Figure 20.4: State Rejection Graph of a computation shown in dashed arrows

This step can be done in $O(1)$ time in parallel with $O(n^2)$ work by using a separate processor for each value of i and j.

Step 2: In step two, we create the state rejection graph represented as an adjacency matrix. We define a new two-dimensional array called R, which is of size $mn \times mn$, where each row and each column represents a different state. In this directed graph, there is an edge from state (i, j) to another state (i', j') only if we know that once state (i, j) is rejected, state (i', j') will also be rejected. In the adjacency matrix R, this is represented as $R[(i, j), (i', j')] = 1$. Additionally, we make the diagonal of the matrix all 1's. We show that creating this boolean matrix can be done in constant time. First, setting the diagonal to all 1's in R takes constant time. We now discuss how off-diagonal entries are set. This is a crucial step in our algorithm. Suppose that a state (i, j) is rejected. We know that the processor will advance to the next state. Then, the next choice for that processor is $(i, j + 1)$. Thus, the rejection of (i, j) would lead to the rejection of all states (i', j') where $(i', j') \rightarrow (i, j + 1)$. Formally,

$$R[(i, j), (i', j')] = 1 \equiv (i', j') \rightarrow (i, j + 1)$$

By using a separate processor for each tuple (i, j, i', j'), we can set R in $O(1)$ time and $O(m^2n^2)$ work. We represent the state rejection graph as a boolean matrix so that we can compute the transitive closure of this graph by doing matrix multiplications.

Step 3: In step three, we take the transitive closure of R. The transitive closure of R is also represented as an adjacency matrix RT. It is well known that the transitive closure for any directed graph with $|V|$ vertices can be computed in $O(\log |V|)$ time using $O(|V|^3 \log |V|)$ work on the common CRCW PRAM [JáJ92]. Since our graph has $O(mn)$ vertices, this step takes $O(\log mn)$ time using $O(n^3m^3 \log mn)$ operations on the CRCW PRAM. This is the only step in our algorithm that takes more than $O(1)$ time.

Step 4: In step four, we use both F and RT to determine which states will not be part of the first consistent global state. To do this, we create a new two-dimensional array called $valid[1 \dots n][1 \dots m]$ where each entry is a value of $0 \dots 1$. We initialize every entry in $valid$ to 1 in $O(1)$ time with $O(mn)$ work. The algorithm sets rejected states in $valid$ with a 0. We use F and RT for this purpose. For all possible values of i, j, i', and j', in parallel, we check to see if a state (i, j) is an element of F and if there is an edge from (i, j) to another state (i', j'). If the two states (i, j) and (i', j') fit this criteria, then we set $valid[i'][j']$ to 0. In other words, if a state (i, j) is initially rejected, and there is an edge from (i, j) to (i', j') in RT, then we know the state (i', j') will also be rejected.

This step can also be done in $O(1)$ time and $O(m^2n^2)$ work. In our example, we compute the states reachable by $\langle 1, 0, 0\rangle$. In our example, states $\{\langle 0, 0, 1\rangle, \langle 1, 1, 0\rangle, \langle 2, 0, 1\rangle, \langle 3, 0, 1\rangle\}$ are all reachable by $\langle 1, 0, 0\rangle$ by following one or more rejection edges. Thus, we mark the states $(1, 1), (2, 1), (3, 1), (1, 2), (1, 3)$ with $0's$ in *valid*.

Step 5: In step five, we traverse *valid* and construct the set of states that form the consistent global state in a new array called *cut* where $cut[i] = j$ signifies that the tuple (i, j) is a part of the consistent global state. To do this, for every process, in parallel, we simply search for the first entry which is nonzero and either it is the first entry or the entry prior to it is zero.

In our example, we can easily see that the first consistent global state is the set $cut = \{(1, 4), (2, 2), (3, 2)\}$. This step can be done in $O(1)$ time and $O(nm)$ work.

Thus, the entire algorithm takes $O(\log mn)$ time and $O(m^3n^3 \log mn)$ work on a common CRCW PRAM.

Remark: Note that the algorithm to detect a conjunctive predicate can be used to detect a global predicate in Disjunctive Normal Form. A predicate is in Disjunctive Normal Form (DNF) if it is expressed as a disjunction of k pure conjunctions. To detect a predicate in this form it is sufficient to detect each conjunction in parallel.

20.5 Conjunctive Predicate at the Given Level

We now show that, in general, asking for a conjunctive predicate on a particular level is **NP**-complete.

Theorem 20.1 *Given a distributed computation, deciding whether there exists a global state with k events satisfying a given conjunctive predicate is* **NP***-complete.*

Proof: We first show that the problem is in **NP**. The global state itself provides a succinct certificate. We can check that all local predicates are true in that global state and that the global state is at level k.

For hardness, we use the subset sum problem. Given a subset problem on n positive integers, $x_1, x_2, \ldots, x_n$ with the requirement to choose a subset that adds up to k, we create a computation on n processes as follows. Each process P_i has x_i events. The local predicate on P_i is true initially and also after it has executed x_i events. Thus, the local predicate is true at each process exactly twice. The problem asks us if there is a global state with k events in which all local predicates are true. Such a global state, if it exists, would choose for every process either the initial local state or the final local state. All the final states that are chosen correspond to the numbers that have been chosen.

To avoid the expansion of the numbers in binary to unary construction, we encode the representation of events on a process as follows: Since the conjunctive predicates can only be true when the local predicates are true, we keep only local states which satisfy their corresponding local predicate and store the number of local events executed so far with them. This leaves two local states at each process: The initial state with zero events executed until that point, and the final state of the process with the number of events equal to x_i for the i^{th} process. Now, the construction of the computation from the subset sum problem is polynomial in the size of the input.

■

20.6 Complexity of Predicate Detection Problem

In this section, we explore the complexity-theoretic results for predicate detection. It is not surprising that given a parallel program computation and a boolean predicate, it is computationally hard to determine if the execution went through a global state in which the predicate became true. We also show that given a boolean predicate b and an execution, determining whether b is lattice-linear for that execution is co-NP-complete.

Since there are N processes, the total number of global states possible is m^N, where m is the number of state intervals at any process. Consider a boolean predicate B. Even when B is a boolean expression, and processes do not communicate, the problem of detecting *possibly*: B is NP-complete.

We show in this section that the problem of global predicate detection is NP-complete. In fact, we show that it is NP-complete even in the absence of messages between processes.

The global predicate detection problem is a decision problem. It can be written as:

> **Input instance**: a poset S of N sequences, a set of variables X partitioned into N subsets $X_1, \ldots, X_N$, and a predicate B defined on X.
> **Problem:** Determine whether there exists a consistent cut $G \in S$ such that $B(G)$ has the value true.

We now show:

Theorem 20.2 *The global predicate detection problem is NP-complete.*

Proof: First note that the problem is in NP. The verification that the cut is consistent can easily be done in polynomial time (for example, using vector clocks and examining all pairs of states from the cut). Therefore, if the predicate itself can be evaluated in polynomial time, then the detection of that predicate belongs to the set NP.

We show NP-completeness of the simplified predicate detection problem where all program variables are restricted to taking the values "true" or "false", and at most one variable from each X_i can appear in B. We reduce the satisfiability problem of a boolean expression (SAT) to the global predicate detection problem by constructing an appropriate poset.

The poset is constructed as shown in Figure 20.5. For each variable $u_i \in U$, we define a process P_i that hosts variable x_i (i.e., $X_i = \{x_i\}$). Let the sequence S_i consist of exactly two states. In the first state, x_i has the value false. In the second state, x_i has the value true.

It is easily verified that the predicate B is true for some cut in S if and only if the expression is satisfiable.

■

The above result shows that detection of a general global predicate is intractable even for simple distributed computations.

20.7 Recognizing Lattice-Linear and Regular Predicates

As discussed earlier, efficient detection algorithms exist for various classes of predicates. Thus, given a boolean expression B, one would like to determine if it belongs to a tractable subclass, in which case detection of the predicate may be performed efficiently. We first consider the classes of Lattice-Linear

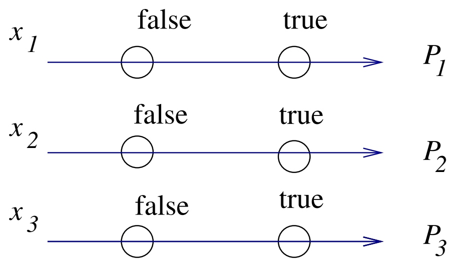


Figure 20.5: Transformation from SAT to global predicate detection

and regular predicates. In this section, we show that determining whether a given boolean expression is Lattice-Linear with respect to a given distributed computation is a co-NP-complete problem. We also show that this problem is co-NP-complete for regular predicates, as well as dual-Lattice-Linear predicates.

We define the decision problems of predicate recognition for Lattice-Linear and regular predicates as follows.

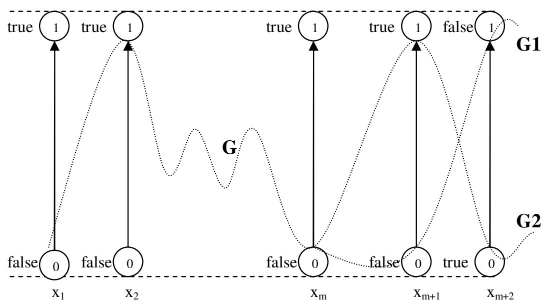


Figure 20.6: Transformation for Theorems 20.3 and 20.4.

Lattice-Linearity: Given a boolean expression b and a program computation, is b a Lattice-Linear predicate?

Regularity: Given a boolean expression b and a program computation, is b a regular predicate?

Theorem 20.3 Lattice-Linearity *is co-NP-complete.*

Proof: *Lattice-Linearity is in co-NP*: Given a pair of candidate global states G and H in which the predicate is true, it can be easily verified in polynomial time that the predicate is false in the global state $G \cap H$. Thus, Lattice-Linearity is in co-NP.

Lattice-Linearity is co-NP-hard: To show co-NP-hardness, we transform an arbitrary instance of TAUTOLOGY to an instance of Lattice-Linearity.

Let b be a boolean expression involving variables $x_1, x_2, ..., x_m.\forall i = 1..m$, we place each x_i on a separate process, P_i. Each of these m processes has two local states, a *true* state and a *false* state, which corresponds to the value taken by the variable x_i in that local state. We also define two new variables, x_{m+1} and x_{m+2}, and place them on processes P_{m+1} and P_{m+2} respectively. Process P_{m+1} has two local states: an initial *false* state and a final *true* state, and process P_{m+2} has two local states: an initial *true* state and a final *false* state. Figure 20.6 shows this transformation. It is evident that this transformation can be achieved in polynomial time.

We define

$$B = b \vee x_{m+1}x_{m+2} \vee \overline{x}_{m+1}\overline{x}_{m+2}$$

We claim that B is Lattice-Linear iff b is a tautology. If b is a tautology, then B is trivially Lattice-Linear, since B will be true for all global states. Conversely, if b is not a tautology, then there exists a subcut involving processes $P_1 \ldots P_m$ in which b evaluates to false. Let us call this subcut G. We can now extend the subcut G to form two cuts, $G1$ and $G2$, in which the predicate B is true, as shown in Figure 20.6.

$$G1 = (G, 0, 1)$$

and

$$G2 = (G, 1, 0)$$

However,

$$G1 \cap G2 = (G, 0, 0)$$

in which the predicate B is false. Thus, B is not Lattice-Linear.

■

Theorem 20.4 Regularity *is co-NP-complete.*

Proof: *Regularity is in co-NP*: Given two candidate global states in which the predicate is true, and their union and intersection such that the predicate is false in either the union or intersection, it can be easily verified in polynomial time that the predicate is not regular. Thus, Regularity is in co-NP.

Regularity is co-NP-hard: The transformation in Theorem 20.3 holds for Regularity as well. That is, b is a tautology iff $B = b \vee x_{m+1}x_{m+2} \vee \overline{x}_{m+1}\overline{x}_{m+2}$ is regular. If b is a tautology, then B is trivially regular since B is true for all global states. Conversely, if b is not a tautology, then B is not regular since both the intersection and union of $G1$ and $G2$ result in a global state in which B is false. Thus, Regularity is co-NP-complete.

■

Note that the above transformation can also be used to show that the problem of deciding whether a given boolean predicate is dual-Lattice-Linear is co-NP-complete.

We stated earlier that efficient detection of Lattice-Linear predicates relies on the assumption that the given predicate satisfies the efficient advancement property, that is, the forbidden state can be identified in polynomial time. The question that arises is, do all Lattice-Linear predicates satisfy the efficient advancement property? If not, then is it possible to efficiently detect Lattice-Linear predicates that do not satisfy this property? We show here that, unless RP=NP, polynomial-time detection cannot be performed for all Lattice-Linear predicates.

We use a result by Valiant and Vazirani [VV85], which states that satisfiability is NP-hard under randomized reductions even for instances that have at most one satisfying assignment (USAT). Valiant and Vazirani's proof uses a randomized polynomial-time algorithm that reduces a given instance of SAT to an instance of USAT.

Theorem 20.5 (Valiant-Vazirani) *If there exists a randomized polynomial-time algorithm for solving instances of SAT having at most one satisfying assignment, then NP=RP.*

We know that a predicate having at most one satisfying assignment is Lattice-Linear, so every instance of USAT is a Lattice-Linear predicate. Given any instance B of USAT, involving variables $x_1, x_2, ..., x_m$, one can create a distributed computation as depicted in Figure 20.5, such that detecting *possibly* : B is equivalent to solving USAT for B. Since we know that Lattice-Linear predicates that satisfy efficient advancement can be detected in polynomial-time, this indicates that Lattice-Linear predicates that do not exhibit efficient advancement may not be detected in polynomial-time, even by a randomized algorithm. Furthermore, since every instance of USAT is also a regular predicate, detection of regular predicates is also NP-hard under randomized reductions.

20.8 Problems

20.9 Bibliographic Remarks

The detection of conjunctive predicates was discussed by Garg and Waldecker in [GW92]. Distributed online algorithms for detecting conjunctive predicates were presented in Garg and Chase [GC95]. Observer-independent predicates were introduced by Charron-Bost, Delporte-Gallet, and Fauconnier [CBDGF95]. Hurfin, Mizuno, Raynal and Singhal [HMRS95] gave a distributed algorithm that does not use any additional messages for predicate detection. Distributed algorithms for offline evaluation of global predicates are also discussed in Venkatesan and Dathan [VD92]. Stoller and Schneider [SS95] have shown how Cooper and Marzullo's algorithm can be integrated with that of Garg and Waldecker's to detect a conjunction of global predicates.

The second algorithm for detecting conjunctive predicates is from [GG19]. The **NP**-completeness of detecting a conjunctive predicate at the level k is shown in [GS24].

The complexity of detecting a boolean predicate is taken from [CG98]. The complexity of checking whether a predicate is lattice-linear (or regular) is from [KG05].

Chapter 21

Enumeration Algorithms

21.1 Introduction

So far, we have been concerned with finding the least feasible elements of the set satisfying the feasibility predicate. What if we wanted to enumerate all feasible elements. For example, we may want to enumerate all satisfying assignments for Horn formulas, all stable marriages, all integral market clearing prices etc. In general the feasible set may be large. For most applications, we consider the set of feasible elements is exponentially larger than the input size. Our goal would be to reduce the complexity of enumeration per element.

21.2 Birkhoff's Theorem

We first define *join-irreducible* elements as follows.

Definition 21.1 (Join-Irreducible Elements) *An element $x \in L$ is join-irreducible if*

1. $x \neq 0$, *and*
2. $\forall a, b \in L : x = a \sqcup b \Rightarrow (x = a) \vee (x = b)$.

Pictorially, in a finite lattice, an element is join-irreducible iff it has exactly one lower cover, that is, there is exactly one edge coming into the element. Figure 21.1(a) shows a distributive lattice with its join-irreducible elements. Intuitively, the set of join-irreducible elements of a distributive lattice are analogous to basis elements of a linear vector space in the sense that the lattice can be generated using join-irreducible elements in the same way as the vector space can be generated by linear combination of basis elements. We denote the set of consistent cuts of any distributed computation $(E, \rightarrow)$ by $C(E)$ ($\rightarrow$ is implicit). We exploit the property that the lattice is distributive to derive the notion of a computation slice. Let $J(L)$ denote the set of join-irreducible elements in L. Now we can state Birkhoff's theorem for finite distributive lattices.

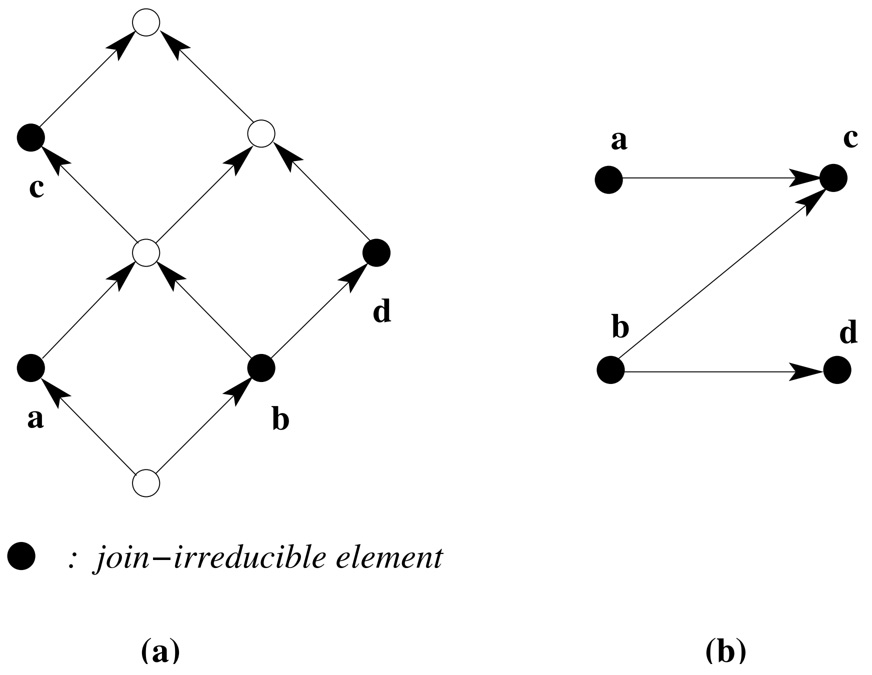


Figure 21.1: (a) An example of a distributive lattice (b) its partial order representation.

Theorem 21.2 (Birkhoff's Representation Theorem) *Let L be a finite distributive lattice. Then the map $f : L \to C(J(L))$ defined by*

$$f(a) = \{x \in J(L) \mid x \leq a\}$$

is an isomorphism of L onto $C(J(L))$. Dually, let P be a finite poset. Then the map $g : P \to J(C(P))$ defined by

$$g(a) = \{x \in P \mid x \leq a\}$$

is an isomorphism of P onto $J(C(P))$.

The above theorem implies one-to-one correspondence between a finite poset and a finite distributive lattice. Given a finite poset, we get the finite distributive lattice by considering its set of down-sets. Given a finite distributive lattice, we can recover the poset by focusing on its join-irreducible elements. Informally, any element of a lattice can be written as a join of a subset of join-irreducible elements of the lattice. For example, Figure 21.1(b) gives the poset corresponding to the lattice in Figure 21.1(a).

From the above discussion it is clear that given any finite distributed computation, the structure *finite distributive lattice* completely characterizes its execution graph.

21.3 Slicing

The notion of consistent global states or ideals of a poset can be extended to graphs in a straightforward manner. A subset of vertices, H, of a directed graph, P, is an *ideal* if it satisfies the following condition: if H contains a vertex v and (u, v) is an edge in the graph, then H also contains u. Observe that an ideal of P either contains all vertices in a strongly connected component or none of them. Let L denote the set of ideals of a directed graph P. Observe that the empty set and the set of all vertices trivially belong to L. We call them *trivial* ideals. It is easy to show that given a directed graph P, $(L; \subseteq)$ forms a distributive lattice.

Since trivial ideals are always part of L, it is more convenient to deal only with nontrivial ideals of a graph. It is easy to convert a graph P to P' such that there is one-to-one correspondence between all ideals of P and all nontrivial ideals of P'. We construct P' from P by adding two additional vertices $\bot$ and $\top$ such that $\bot$ is the smallest vertex and $\top$ is the largest vertex (i.e., there is a path from $\bot$ to every vertex and a path from every vertex to $\top$). It is easy to see that any nontrivial ideal will contain $\bot$ and not contain $\top$. As a result, every ideal of P is a nontrivial ideal of the graph P' and vice versa. We will deal with only nontrivial ideals from now on and an ideal would mean nontrivial ideal unless specified otherwise.

We will deal with only nontrivial ideals from now on and an ideal would mean nontrivial ideal unless specified otherwise. The directed graph representation of Fig. 21.2(a) is shown in Fig. 21.2(c).

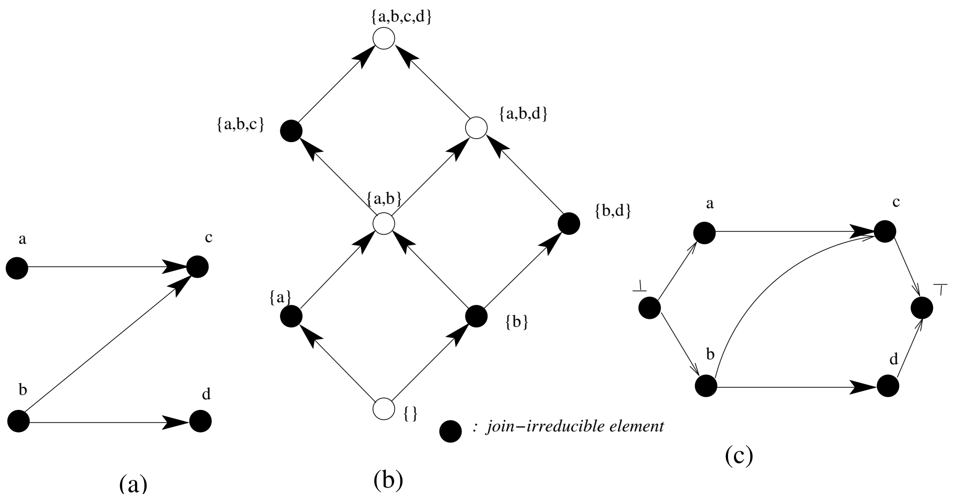


Figure 21.2: (a) P: A partial order (b) the lattice of ideals. (c) the directed graph P'

The slice of a directed graph P with respect to a predicate B (denoted by $slice(P, B)$) is a graph derived from P such that all the ideals in L that satisfy B are included in the ideals of the slice with respect to B. The slice may include some additional ideals which do not satisfy the predicate. Formally,

Definition 21.3 (Slice) *A* slice of a graph P with respect to a predicate B is the directed graph obtained from P by adding edges such that:
(1) it contains all the ideals of P that satisfy B and
(2) of all the graphs that satisfy (1), it has the least number of ideals.

Let L be a finite distributive lattice generated by the graph P. Let L' be any sublattice of L. Then, there exists a graph P' that can be obtained by adding edges to P that generates L'.

21.4 Computing All Stable Matchings

We now consider the problem of computing all stable matchings. Since the number of stable matchings may be exponential in n, instead of keeping all matchings in explicit form, we would like a concise representation of polynomial size that can be used to enumerate all stable matchings. In SMP literature, rotation posets are used to capture all stable matchings. We will use the notion of computation slicing introduced in [MG01b] for this purpose. In particular, we give an efficient algorithm to compute the slice for the SMP computation. A rotation poset [GI89] is a special case of the slice when the set of external constraints is empty.

We first show that the set of stable matchings is a sublattice of the lattice of all men assignments. We have already shown that the predicate "the assignment is a stable matching" is a lattice-linear predicate and therefore closed under meets. We now show that the predicate is also closed under joins. To that end, we use the dual of a lattice-linear predicate called dual-linear predicate. Given any predicate B, we define

$$\text{reverse-forbidden}(G,i) \equiv \forall H \in L : H \subseteq G : (G[i] \neq H[i]) \vee \neg B(H)$$

We define a predicate B to be dual-linear with respect to poset S if for any global state G in the poset, B is false in G implies that G contains a *reverse-forbidden state.* Formally, A boolean predicate B is *dual-linear* with respect to a poset S iff:

$$\forall G \in L : \neg B(G) \Rightarrow \exists i : \text{reverse-forbidden}(G,i)$$

Note that although the concepts of lattice-linear predicate and dual-linear predicates are dual, the proof of dual-linearity is different from that of lattice-linearity. This is because we are computing assignments with respect to men and not women thereby bringing in asymmetry.

Lemma 21.4 *Let G be any assignment such that it is not a stable matching. Then, there exists i such that $G[i]$ is reverse-forbidden.*

Proof: First assume that G is not a matching. We know that there is at least one woman q who is missing in this assignment. Let i be the most preferred man for that woman in the pre-frontier events that correspond to proposal to q. If there is no man in pre-frontier events for that woman who proposes to q, then there is no matching possible because q is not proposed in either a frontier or a pre-frontier event. So, we can assume that i exists. We claim that $G[i]$ is reverse-forbidden. Consider any H such that $H \subseteq G$ and $H[i] = G[i]$. We need to show that H cannot be a stable matching. For H to be a matching, for some k different from i, $H[k].w$ equals q. However, by our choice of i, q prefers man i and man i prefers q to $H[i].w$.

Now, assume that G is a matching, but not stable. Suppose that (j,k) is a blocking pair. Let q be the woman corresponding to $G[j].w$. Of all the men who propose to that woman in pre-frontier events, choose the one who is most preferred for that woman. We know that there exists at least one because (j,k) is a blocking pair. Let that man be i. We claim that $G[i]$ is reverse-forbidden. Consider any H such that $H \subseteq G$ and $H[i] = G[i]$. We need to show that H cannot be a stable matching. Since G is a matching, $G[i].w$ is different from $q = G[j].w$. For H to be a matching, for some l, $H[l].w$ equals $G[j].w$. However, by our choice of i, we know that $H[l].w$ prefers man i to l and man i prefers $H[l].w$ to $G[i].w$.

■

The above lemma allows us to find the man-pessimal stable matching. We start with G such that $G[i]$ equals the last choice proposal for P_i. If G is a stable matching, we are done. Otherwise, from the proof of Lemma 21.4, we can find i such that unless $G[i]$ goes backward, there cannot be any stable matching. By repeating this procedure, we get the men-pessimal stable matching.

Since stable matching is dual-linear, it follows that the set of assignments is closed under joins. Predicates that are both meet-closed and join-closed are called regular predicates [GM01b]. The set of ideals that satisfy a regular predicate forms a sublattice of the lattice of all ideals. Since a sublattice of a distributive lattice is also distributive, the set of ideals that satisfy a regular predicate forms a finite distributive lattice. From Birkhoff's theorem [Bir67] we know that a finite distributive lattice can be equivalently represented using the poset of its join-irreducible elements.

For any regular predicate B, let L_B be the sublattice of consistent global states that satisfy B. Since L_B can have a number of elements that is exponential in n, we would like to have a compact representation of L_B. A slice of a poset P with respect to a predicate B is a graph such that the consistent global states of the graph includes all consistent global states that satisfy B and when B is regular it includes only those. The computation of a slice is shown in Fig. computeSlice. The algorithm requires computation of $J(B, e)$ for all $e \in E$ where $J(B, e)$ be the least consistent global state that satisfies B and includes e.

Algorithm computeSlice: An efficient algorithm to compute the slice for a regular predicate B

```
graph function computeSlice($B$:regular_predicate, $P$: graph)
var $R$: graph initialized to $P$; forall element $e$ in $P$ do
    let $J(B, e)$ be the least global state of $P$ that satisfies $B$ and includes $e$;
    forall $f \in J(B, e)$ do
        add edge $(f, e)$ to $R$;
    end
end
return $R$;
```

Hence, to compute a slice it is sufficient to give a procedure to compute $J(B, e)$. To determine $J(B, e)$ it is sufficient to use the algorithm for detecting lattice-linear predicate by using the following predicate for every e: $B_e(G) \equiv B(G) \wedge (e \in G)$. Since B is a lattice-linear predicate, and the predicate $e \in G$ is also lattice-linear, $B_e(G)$ is also lattice-linear.

21.5 An Algorithm to Determine Join-Irreducible Min Cuts

To identify join-irreducible min cuts:

1. Construct the lattice of all min cuts, ordered by inclusion of S-sets.

2. For each min cut $C = (S, T)$, find its immediate predecessors: min cuts $C' = (S', T')$ where $S' \subset S$ and no S'' exists such that $S' \subset S'' \subset S$.

3. A min cut is join-irreducible if it has exactly one immediate predecessor.

Consider the directed graph in Fig. 21.3 with 4 vertices.

The graph has a minimum cut capacity of 4. The min cuts are: $\{s\}|\{1, 2, t\}$, $\{s, 1\}|\{2, t\}$, $\{s, 2\}|\{1, t\}$, and $\{s, 1, 2\}|\{t\}$. These mincuts form a distributive lattice shown in Fig. 21.4

As mentioned earlier, in the worst case, the lattice may be exponential in the size of the graph. We construct the poset of join-irreducible mincuts which generates the lattice of mincuts. This poset has at most $n - 2$ elements.

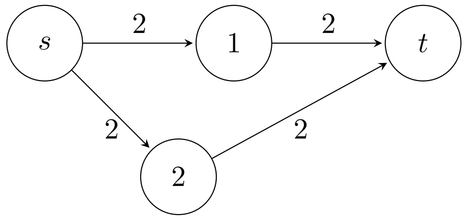


Figure 21.3: Directed graph with capacities (acyclic).

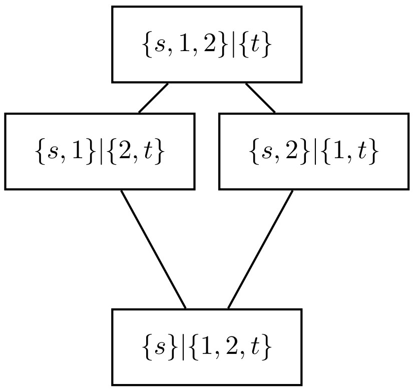


Figure 21.4: Lattice of minimum cuts for the acyclic directed graph.

- $\{s\}|\{1, 2, t\}$: No predecessors (bottom element), so join-irreducible.
- $\{s, 1\}|\{2, t\}$: One predecessor ($\{s\}$), so join-irreducible.
- $\{s, 2\}|\{1, t\}$: One predecessor ($\{s\}$), so join-irreducible.
- $\{s, 1, 2\}|\{t\}$: Two predecessors ($\{s, 1\}$, $\{s, 2\}$), so not join-irreducible.

Thus, the join-irreducible min cuts are $\{s\}|\{1, 2, t\}$, $\{s, 1\}|\{2, t\}$, and $\{s, 2\}|\{1, t\}$.

We now give an efficient algorithm to generate all the join-irreducible elements of the mincut lattice. For all vertices $v \in V$, we let $J(v)$ be the least mincut that includes the vertex v, if it exists. We first show that $J(v)$ is a join-irreducible mincut for all $v \in V - \{t\}$.

Lemma 21.5 *Let $v \in V - \{t\}$ and $J(v)$ be the least mincut that includes the vertex v. Then, $J(v)$ is a join-irreducible mincut.*

Proof: Let $J(v)$ be join of two mincuts X and Y such that $J(v)$ is different from both X and Y. Since the join corresponds to union, and $v \in J(v)$, either $v \in X$ or $v \in Y$. Without loss of generality, suppose that $v \in X$. Since $J(v)$ is the least minimal that includes v, we get $J(v) \subseteq X$. But, $J(v) = X \cup Y$ implies that $X \subseteq J(v)$.

■

We can compute $J(v)$ with two max-flow computations as shown in Fig. LeastJoinIrreducible. We first find the max-flow in the graph (line 1). Let μ be the max-flow. Now, we construct another max-flow graph G' as follows. We add an edge from the source vertex s to v with capacity equal to infinity (line 4).

We again find the max-flow μ' (line 5). If μ' equals μ, then the mincut obtained is the least mincut that includes the vertex v (lines 7-9). If there is no mincut that includes v, then we return null.

Algorithm LeastJoinIrreducible: Generate Least Join-Irreducible Min Cut including v

```
   Input: Directed graph $G = (V, E)$, capacities $c(e)$, source $s$, sink $t$
   Output: Join-irreducible min cut $J(v)$
1  $\mu :=$ min cut capacity of $G$ ;                                  // One max-flow
2  $J(v) := null$ ;                                                    //
3  $G' := G$ ;                                                         // Copy graph
4  Add edge $s \to v$ with capacity $\infty$ in $G'$ ;                  // Force $v$ with $s$
5  $(S_v, T_v) :=$ min cut in $G'$ (via max-flow)
6  $\mu' :=$ capacity of cut $(S_v, T_v)$ in original $G$
7  if $\mu' = \mu$ then
8  |   $J(v) := (S_v, T_v)\}$
9  end
10 return $J(v)$
```

We now extend this algorithm to generate J, the list of all join-irreducible mincuts. We observe here that the algorithm is polynomial in the input size.

Algorithm AllJoinIrreducible1: Generate Join-Irreducible Min Cuts

```
   Input: Directed graph $G = (V, E)$, capacities $c(e)$, source $s$, sink $t$
   Output: Set $J$ of join-irreducible min cuts
1  $J := \emptyset$
2  $Covered := \emptyset$ ;                                            // Vertices in some $S$-set
3  $\mu :=$ min cut capacity of $G$ (via max-flow) $S_0 := \{s\}$, $T_0 := V \setminus \{s\}$
4  $C_0 :=$ capacity of cut $(S_0, T_0)$
5  if $C_0 = \mu$ then  $J := J \cup \{(S_0, T_0)\}$ $Covered := Covered \cup S_0$ ;
6  for $v \in V \setminus \{s, t\}$ do
7  |   if $v \in Covered$ then continue;
8  |   $G' := G$ ;                                                     // Copy graph
9  |   Add edge $s \to v$ with capacity $\infty$ in $G'$ ;              // Force $v$ with $s$
10 |   $(S_v, T_v) :=$ min cut in $G'$ (via max-flow)
11 |   $C_v :=$ capacity of cut $(S_v, T_v)$ in original $G$
12 |   if $C_v = \mu$ then
13 |   |   $J := J \cup \{(S_v, T_v)\}$
14 |   |   $Covered := Covered \cup S_v$
15 |   end
16 end
17 return $J$
```

The algorithm works as follows.

- Compute μ once with a max-flow algorithm.
- Check the minimal cut $\{s\}|V \setminus \{s\}$; add it to J if it is a min cut and mark its vertices as covered.
- For each $v \in V \setminus \{s, t\}$, skip if already covered. Otherwise, compute the least min cut containing v using max-flow.

- Add the cut to J if its capacity is μ, updating *Covered*.
- The number of max-flow computations is $O(k)$, where k is the number of join-irreducible min cuts ($k \leq |V| - 2$).
- Total time complexity: $O(kMF(n,m)$, where $MF(n,m)$ is the time complexity of computing maxflow in a graph with n vertices and m edges. The worst case for k is $O(n)$.

We now show how the computation complexity of the above algorithm can be improved. We show an algorithm that avoids the second computation of max-flow. We assume that we have access to the reduced graph from the first computation of max-flow. Based on the first max-flow, the algorithm computes a reduced graph of *super-vertices*. Each super-vertex corresponds to a subset of vertices such that if u is in the super-vertex and (u,v) is not a staurated edge, then v is also in the supervertex. The reduced graph can be computed in time proportional to the number of edges. The algorithm is inspired from [PQ80]. A difference is that they do not explicitly generate *join-irreducible* mincuts. Furthermore, we later extend this algorithm to find a mincut that satisfies the given predicate B.

Algorithm AllJoinIrreducible2: Generate Join-Irreducible Min Cuts (Directed Graphs)

```
   Input: Directed graph $G = (V,E)$, capacities $c(e)$, source $s$, sink $t$
   Output: Set $J$ of join-irreducible min cuts
1  $J := \emptyset$
2  $Covered := \emptyset$ ;                                   // Vertices in some $S$-set
3  $\mu :=$ min cut capacity of $G$ (via max-flow) ;                    // max-flow
4  $S_0 := \{s\}$, $T_0 := V \setminus \{s\}$
5  $C_0 :=$ capacity of cut $(S_0, T_0)$
6  if $C_0 = \mu$ then $J := J \cup \{(S_0, T_0)\}$ $Covered := Covered \cup S_0$ ;
7  for $v \in V \setminus \{s,t\}$ do
8      if $v \in Covered$ then continue;
9      find strongly connected component $W$ that includes $v$ with all edges in the residual graph;
10     if $t \notin W$ then
11         $S_v := S_0$ union with all vertices reachable from $W$ using edges in the residual graph
12         $T_v := V \setminus S_v$
13         $J := J \cup \{(S_v, T_v)\}$
14         $Covered := Covered \cup S_v$
15     end
16 end
17 return $J$
```

21.6 Problems

21.1. Show that $x \in L$ is join-irreducible iff
(i) $x \neq 0$
(ii) $\forall a, b : (a < x) \wedge (b < x) \Rightarrow ((a \sqcup b) < x)$

21.2. The dual notion of join-irreducible elements is *meet-irreducible* elements. Let $M(L)$ denote the ordered set of meet-irreducible elements of L. Show that the ordered set $J(L)$ is isomorphic to the set $M(L)$.

21.7 Bibliographic Remarks

The notion of a slice of a computation was proposed in Garg and Mittal [GM01a] and later generalized in Mittal and Garg [MG01a].

Chapter 22

Equilevel Predicate Detection

22.1 Introduction

We introduce a new class of predicates called *equilevel predicates* to model and solve many problems in computing. These predicates are defined on the elements of a distributive lattice. The distributive lattice is such that the height is small, but its size is large. This lattice can be viewed as generated from a poset with at most n chains and m elements per chain. In this case, the height of the lattice is bounded by $O(nm)$ but the cardinality can be as many as n^m elements. For efficiency, our methods operate on the underlying poset even though we are interested in finding an element in the lattice satisfying the given predicate. This approach allows us to obtain algorithms with complexities efficient in n and m.

A predicate defined on a lattice is equilevel if all the elements of the lattice that satisfy the predicate are on the same level of the lattice. It can be shown that detecting an equilevel predicate is, in general, a hard problem. For example, the minimum vertex cover problem for a graph can be modeled as detecting an equilevel predicate. Even so, we will see that there are many equilevel predicates which can be detected efficiently, and that this efficiency can be implied by a few simple properties.

We first show that the problem of equilevel predicate is NP-complete in general. For example, we show that slight generalizations of lattice-linear predicate-based problems with efficient solutions, such as the conjunctive predicate detection, are NP-complete. We then define a property on equilevel predicates called the *helpful* property. An equilevel predicate has the *helpful* property with a polynomial time algorithm if the algorithm can return a nonempty set of indices such that advancing on any of them can be used to detect the predicate in polynomial time in an online manner. We apply the helpful property to modeling

Class of Predicates	**Examples**	*Detection Algorithm*
General Equilevel	Vertex Cover, k-conjunctive	Not efficient unless P= NP
With Helpful Property	bipartite matching, basis vectors	P
General Solitary	USAT	Not efficient unless NP= RP
With Efficient Advancement	optimal stable marriage	P

Figure 22.1: Various Classes of Predicates. Equilevel predicates are the ones that are true on elements of a lattice at a single level. Solitary predicates are the ones that are true on a single element in the lattice.

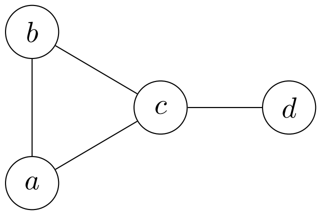


Figure 22.2: A graph $G = (\{a, b, c, d\}, \{(a, b), (b, c), (a, c), (c, d)\})$.

and efficient detection of various problems such as bipartite matching and computing bases which span sets of vectors as equilevel predicates.

Next, a special class of equilevel predicates, called solitary predicates, is identified. These are those predicates that can only be true on a single element in the lattice, and we show that even with this substantial restriction, detecting these predicates is still hard unless NP=RP. However, lattice-linearity implies that solitary predicates can be detected efficiently when they satisfy *efficient advancement*. Observe that for any lattice-linear predicate B we can define a stronger predicate B' that is satisfied only on the least element in the lattice satisfying B. Then B' is a solitary predicate. For example, consider the distributive lattice of assignments for the stable marriage problem [GS62, GI89]. There may be multiple stable marriages; however, there is a unique man-optimal stable marriage. The predicate that an assignment corresponds to the man-optimal stable marriage is a solitary predicate.

22.2 Equilevel Predicates

Let poset P consist of n chains, i.e., P can be decomposed into n chains $P_1, P_2, \ldots P_n$ of distinct elements. Thus, we can assume L is represented this way without loss of generality, and every element $G \in L$ can be viewed as an ideal (or, downset) of P.

Let $G[i]$ denote the number of elements in G from P_i, while $|G|$ denotes the cardinality counting all elements in the ideal representing G. Finally, $\perp$ and $\top$ correspond to the bottom and top elements of the lattice respectively. Then, a predicate on L is *equilevel* if all the satisfying elements are on the same level of the lattice.

Definition 22.1 (Equilevel Predicate) *A Boolean predicate B is* equilevel *with respect to a lattice L iff*

$$\forall G, H \in L : B(G) \wedge B(H) \Rightarrow |G| = |H|.$$

Fig. 22.3 shows a search space (modeled using a distributive lattice) and a predicate and its set of satisfying solutions which are all on the same level of the lattice.

For example, consider the minimum vertex cover problem in an undirected graph (V, E). A subset of vertices V' satisfies B if V' is the vertex cover of minimum size. Here, the lattice we consider is the Boolean lattice of the vertex set. It is clear that B is an equilevel predicate since there may be multiple minimum vertex covers and they must all be of the same size. Then, we observe that detecting an equilevel predicate is NP-complete in general, as the minimum vertex-cover decision problem is well known to be NP-complete.

As another example of an equilevel predicate, suppose that we have a bipartite graph (L, R, E) and that we are interested in subsets of the maximum size of L that can be matched. The size of the maximum-sized

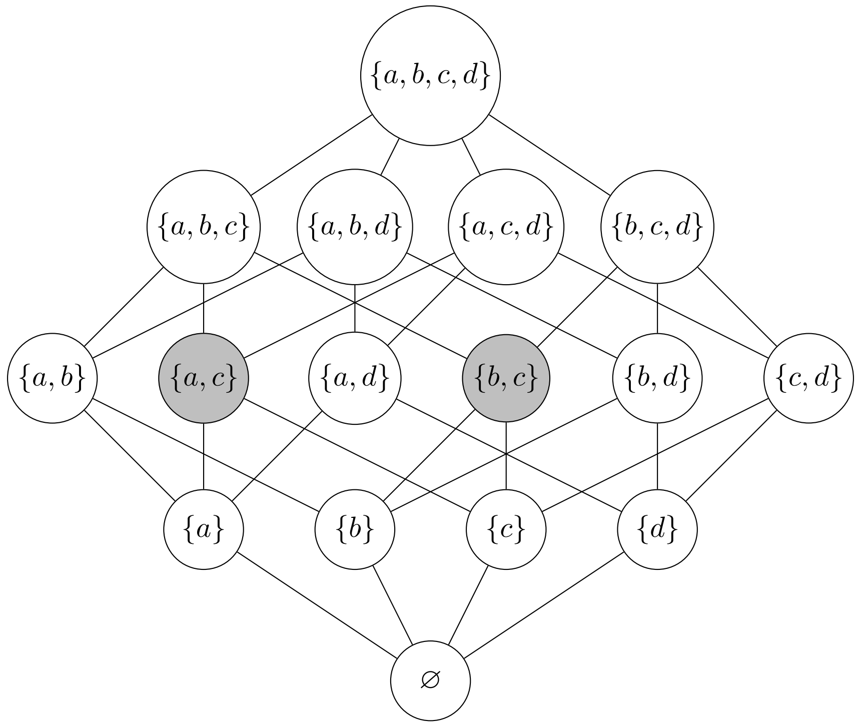


Figure 22.3: Hasse diagram of the vertex powerset, ordered by inclusion.

subsets of L that can be matched to elements in R is constant, although there may be multiple matched sets in L. For example, consider $L = \{x_1, x_2, x_3\}$, $R = \{y_1, y_2, y_3\}$ and $E = \{(x_1, y_1), (x_2, y_1), (x_3, y_3)\}$. Then, we can match $\{x_1, x_3\}$ with $\{y_1, y_3\}$ or match $\{x_2, x_3\}$ with $\{y_1, y_3\}$. Both matching are of size 2.

Equilevel predicates occur in numerous other contexts. Consider the problem of finding a minimum weight spanning tree of a connected undirected graph (V, E) with n vertices. If the edge weights are not unique, then there may be multiple minimum-spanning trees with equal weight. Let the predicate B be "the set of edges form a spanning tree with minimum weight." It is clear that all sets that satisfy the predicate have $n - 1$ edges. Thus, in the inclusion lattice over the graph's edges, the sets satisfying the predicate are all at the same level.

Equilevel predicates are closed under conjunction. If the predicates B_1 and B_2 are true for different levels, then the conjunction is false for all elements of the distributive lattice and the predicate $B_1 \wedge B_2$ is trivially equilevel. Otherwise, if they are true for the same level, then $B_1 \wedge B_2$ is also true only at that level (or possibly always false). However, equilevel predicates are not closed under disjunction or complement because these operations may introduce elements satisfying the new predicate at many different levels of the lattice.

As we have observed, since NP-complete problems such as vertex cover can be modeled as detecting an equilevel predicate, the problem of detecting an arbitrary equilevel predicate is NP-hard. As another example, consider the problem of detecting a Hamiltonian path in an undirected graph. If we consider the Boolean lattice of all edges, then we only need to consider the subset of edges at level $n - 1$ on a graph with n vertices. Thus, the problem of Hamiltonian path can also be modeled as detecting an equilevel predicate.

We now show that some problems in distributed computation with efficient algorithms are equilevel predicates when slightly generalized. For example, consider *conjunctive predicates* which are given by the Boolean formula $l_1 \wedge l_2 \wedge \ldots l_n$ where each l_i is a function only of the state local to the i^{th} process. For this class, there is an efficient algorithm for detecting this predicate in a distributed computation [GW94]. However, if we generalize the problem by asking for the conjunctive predicate on a particular level of the lattice, then the problem can be modeled as an equilevel predicate. We now show that, in general, asking for a conjunctive predicate on a particular level is NP-complete.

Theorem 22.2 *Given a distributed computation, deciding whether there exists a global state with k events satisfying a given conjunctive predicate is NP-complete.*

Proof: We first show that the problem is in NP. The global state itself provides a succinct certificate. We can check that all local predicates are true in that global state and that the global state is at the level k.

For hardness, we use the subset sum problem. Given a subset problem for n positive integers, $x_1, x_2, \ldots, x_n$ with the requirement to choose a subset that adds up to k, we create a computation on n processes as follows. Each process P_i has x_i events. The local predicate on P_i is true initially and also after it has executed x_i events. Thus, the local predicate is true for each process exactly twice. The problem asks us if there is a global state with k events in which all local predicates are true. Such a global state, if it exists, would choose for every process either the initial local state or the final local state. All the final states that are chosen correspond to the numbers that have been chosen.

To avoid the expansion of the numbers in binary to unary construction, we encode the representation of events on a process as follows: Since the conjunctive predicates can only be true when the local predicates are true, we keep only local states which satisfy their corresponding local predicate and store the number of local events executed so far with them. This way of modeling gives us two local states at each process: The initial state with zero events executed to that point and the final state of the process with the number of events equal to x_i for the i^{th} process. Now, the construction of the computation from the subset sum problem is polynomial in the size of the input.

■

In what follows, we restrict our attention to a class of equilevel predicates to one that can be detected efficiently.

22.3 Equilevel Predicates with The Helpful Property

One class of equilevel predicates that we can efficiently detect is the set of predicates that satisfy the *helpful* property. An equilevel predicate satisfies the helpful property if, whenever it is false in a global state G, we are guaranteed to be able to compute in polynomial time the indices of the local states which are *helpful*. Unless the global state G is advanced on *at least one* helpful local state, the predicate can never become true.

Before we define the helpful property, we introduce the notation $B(L)$ to mean that there exists $W \in L$ such that $B(W)$. Now, we define the helpful property.

Definition 22.3 (Helpful Property) *A Boolean predicate B has the* helpful *property with a polynomial time algorithm $\mathcal{A}$ on a finite distributive lattice L if, for all $G \in L$,*

$$\neg B(G) \Rightarrow \mathcal{A}(G) \neq \varnothing \wedge \forall i \in \mathcal{A}(G) : helpful(i, G, B).$$

where $helpful(i, G, B)$ is defined as

$$B(L) \Rightarrow \exists H > G : (H[i] > G[i]) \wedge B(H).$$

The definition states that whenever the predicate is not true in a global state G, the polynomial-time algorithm $\mathcal{A}$ can return a nonempty set of indices $\mathcal{A}(G)$ such that advancing on any of the corresponding local states can be used to detect the predicate. The i^{th} local state is helpful in the global state G if we can advance the global state G on the index i whenever $\neg B(G)$ holds without the risk of missing a satisfying global state H. If there is no global state that satisfies B, then $helpful(i, G, B)$ holds vacuously.

The helpful property allows us to efficiently detect an equilevel predicate satisfying the property. In all our examples, we also require that the algorithm $\mathcal{A}$ be *online* in the sense that the algorithm $\mathcal{A}$ has access to the poset only in states up to G. Therefore, the computation of helpful indices is based only on the poset of states less than or equal to G. This allows us to use a sequential procedure, given in Algorithm Equilevel, to advance on helpful indices towards a satisfying global state if one exists.

The Algorithm Equilevel works as follows. It starts with the global state G initialized to the least global state corresponding to the bottom element of the lattice L. If the predicate B is true in the initial global state, then we are done and we can return G. If not, we first check whether G corresponds to the top element of the lattice. If G is the top element of the lattice, we can safely return that there is no element in L that satisfies B. Otherwise, we advance on the index i where $helpful(i, G, B)$ holds.

Algorithm Equilevel: A Sequential Algorithm to find a state satisfying B

```
function GetSatisfying($B$: predicate, $L$: Lattice)
var $G$: array$[1 \ldots m]$ of int initially $\forall i : G[i] = 0$;
while $\neg B(G)$ do
    if ($G$ is the top element of $L$) then  return null;;
    else  $G[i] := G[i] + 1$ where $helpful(i, G, B)$; ;
end
return $G$; // an optimal solution
```

Theorem 22.4 gives the correctness of Algorithm Equilevel.

Theorem 22.4 *Let B be any equilevel predicate that satisfies a helpful property with a polynomial-time algorithm $\mathcal{A}$. Then B can be detected online in polynomial time.*

Proof: Algorithm Equilevel starts with the least element in the lattice L. The *while* loop can execute at most mn times where n is the number of chains and m is the maximum height of any chain. The algorithm $\mathcal{A}$ to compute $helpful$ has polynomial complexity. Hence, Algorithm Equilevel also has polynomial complexity. The algorithm $\mathcal{A}$ has access to the lattice L only up to G. Since B is an equilevel predicate, the algorithm is guaranteed to return a minimal state that satisfies B.

■

By *online* detection of the predicate, we mean that the algorithm has access to the lattice only up to the current global state G. For example, when we consider the stable marriage problem, the vector G denotes the current assignment of men to women. The algorithm needs to make its decisions based only on the information on men preferences and women preferences up to that point. This interpretation is different from the online algorithm in the standard algorithm literature, where the information about the problem comes incrementally.

We show detection of equilevel predicates through the following examples:

Example 1: Maximum Cardinality Bipartite Matching. Consider the problem of bipartite matching in a graph (L, R, E). We are interested in finding a subset $V \subseteq L$ of maximum size such that V can be matched with vertices in R. We use the Boolean lattice on the vertices L. A vertex set $V \subseteq L$ satisfies B if $|V|$ is the maximum number of vertices that can be matched. It is clear that B is an equilevel predicate. Suppose that we have a set of vertices $V \subseteq L$ which is not of the maximum size. Then, we can find a set of helpful set of vertices W such that any element of W can be added to V. It can be checked efficiently that V does not satisfy B (for example, by checking if there exists an alternating path from a vertex in $L - V$ to an unmatched vertex in R). The algorithm to find helpful vertices is simple: any vertex in $L - V$ that has an alternating path to an uncovered vertex in R is a helpful vertex. For efficiency, the algorithm may maintain the set of matched vertices in R. One can use Algorithm Equilevel to find the largest set in L with matching in R. The predicate $helpful(i, G, B)$ holds whenever the vertex i can be added to the current set of matched vertices G.

Example 2: Minimum Spanning Tree in an Undirected Graph. Let B be true on a set of edges of an undirected graph if the edge set forms a spanning tree of the graph. Given a connected undirected graph with n vertices, it is well-known that any edge set that is acyclic and has $n - 1$ edges satisfies B. This predicate is equilevel because all the elements that satisfy the predicate B are at the level $n - 1$ in the Boolean lattice of all edges. Given a set of edges, one can efficiently compute whether the set is acyclic. Hence, one can use Algorithm Equilevel to find a spanning tree as follows: The predicate $helpful(i, G, B)$ holds whenever the edge e_i does not form a cycle with edges in G, so at each step Algorithm Equilevel adds any edge which does not make a cycle in the current solution.

Example 3: Basis Set of Vectors. Suppose that we are given a set of vectors S of the same dimension. Our goal is to compute a basis, i.e. find a subset of S of the biggest size containing linearly independent vectors. The predicate B is true on a set G if all the vectors in G are linearly independent, and there does not exist any vector in $S - G$ that is linearly independent with vectors in G. Here $helpful(i, G, B)$ holds if the vector v_i is independent of all the vectors in G.

Lattice-linear predicates are associated with the concept of *forbidden* local states. In particular, for any G which does not satisfy a lattice-linear predicate B, there exists at least one local state s such that no global state that satisfies B can have the same local state s. This means that one can advance on all forbidden local states of a lattice-linear predicate unconditionally. In particular, if there are multiple local states that are forbidden then the global state can be advanced on all of them in parallel as forbidden local states remain forbidden until they are advanced. In contrast, when it comes to equilevel predicates, advancing on one helpful local state may make the local state on some other process "unhelpful." For example, in computing a spanning tree we may be able to add one of edge a or edge b to our current solution, but adding both of them could create a cycle.

An equilevel predicate may not have the helpful property as shown next.

Theorem 22.5 *There exist equilevel predicates that do not have any helpful property unless $P = NP$.*

Proof: We have earlier shown that the problem of finding the vertex cover of the least size is an equilevel predicate. If there exists any helpful property for this problem, we would have a polynomial time algorithm to find the vertex cover. Similarly, detecting a conjunctive predicate at a particular level of the lattice of global states also cannot have the helpful property unless $P = NP$.

■

22.4 Solitary Predicates

A *solitary* predicate is one which is false on all the elements of the lattice or is true on a single element in the lattice.

Definition 22.6 (Solitary Predicate) *A Boolean predicate B is* solitary *with respect to a lattice L iff*

$$\forall G, H \in L : B(G) \wedge B(H) \Rightarrow (G = H).$$

The above definition states that if the predicate is true for two elements G and H in the lattice, then G must be equal to H. This definition also includes the empty predicate that is not true in any element of the lattice.

Solitary predicates are closely related to the problem of Unique SAT. Unique SAT (USAT) [VV85] asks for the satisfiability of a Boolean expression given the promise that it has either a single satisfying assignment or none. Let $x_1, x_2, \ldots, x_n$ be n Boolean variables. Let B be a Boolean expression on these variables. The USAT problem requires the algorithm to return 1, if the Boolean expression has a unique satisfying assignment, 0, if the Boolean expression is not satisfiable and either 0 or 1, if it has multiple satisfying assignments. Valiant and Vazirani have shown the following result.

Theorem 22.7 *(***Valiant-Vazirani***[VV85]) If there exists a randomized polynomial-time algorithm for solving instances of SAT with at most one satisfying assignment, then NP=RP.*

As a direct application, we get the following result by using the construction in [KG05] as shown below.

Theorem 22.8 *Given any finite distributive lattice L generated from a poset P, and a solitary predicate B, there does not exist a randomized polynomial time algorithm to detect B unless NP=RP.*

Proof: Suppose, if possible, there exists a randomized polynomial time algorithm $\mathcal{A}$ to detect a solitary predicate. Given any instance of USAT on n variables $x_1, x_2, \ldots, x_n$, we construct a poset with n events on n processes, $P_1, P_2, \ldots, P_n$ with one event per process. Process P_i hosts the variable x_i which is initially set to false. When the event is executed, the variable is set to true. The distributive lattice generated from this poset has 2^n elements — one for every truth assignment. If the given instance of USAT has zero satisfying assignments, then $\mathcal{A}$ is guaranteed to return $false$. If the given instance of USAT has exactly one satisfying assignment, then $\mathcal{A}$ is guaranteed to return $true$ with a satisfying assignment. Otherwise, we do not care what $\mathcal{A}$ returns. Therefore, any randomized polynomial time algorithm to detect a solitary predicate can be used to solve USAT in polynomial time.

■

Although detecting a general solitary predicate is hard, we can detect the predicate B, whenever it satisfies the efficient advancement property.

Now consider lattice-linear predicates that are true on multiple elements in the lattice. Let B be any lattice-linear predicate. We know that B is closed under the meet operation of the lattice. Suppose that B becomes true in the lattice and the least element that satisfies B is G_{min}. We derive another predicate B' from B that holds only for the element G_{min} whenever it exists. Thus, B' holds for G iff $B(G)$ and for all elements H, if $B(H)$, then $G \leq H$. Therefore, $B'(G)$ is false for all other elements besides G_{min}.

Theorem 22.9 *Let B be any lattice-linear predicate. Let $B'(G)$ be defined as follows:*

$$B'(G) \equiv B(G) \wedge (\forall H : B(H) \Rightarrow G \leq H).$$

Then, B' is a solitary predicate.

Proof: First, consider the case when B does not hold for any element in the lattice. This implies that B' also does not hold for any element in the lattice and is, therefore, solitary. Now, suppose that B is true in the lattice. Since B is an LLP predicate, it is closed under meets. Therefore, there is the smallest element, G_{min}, in the lattice such that $B(G_{min})$ holds. It is readily verified that $B'(G)$ holds iff $G = G_{min}$.

■

Thus, all the instances of lattice-linear predicates provide us examples of solitary predicates when we focus on the least element.

Example 1: Man-Optimal Stable Marriage. In this problem, we are given as input n men and n women. We are also given a list of men preferences as $mpref$ where $mpref[i][k]$ denotes k^{th} top choice of man i. The women preferences are more convenient to express as a $rank$ array where $rank[i][j]$ is the rank of man j by woman i. A matching between man and woman is stable if there is no *blocking pair*, i.e., a pair of woman and man such that they are not matched and prefer each other to their spouses. The underlying lattice for this example is the set of all n dimensional vectors of $1..n$. We let $G[i]$ be the choice number that man i has proposed to. Initially, $G[i]$ is 1 for all men.

If we now focus on the man-optimal stable marriage, then the predicate "the assignment is a man-optimal stable marriage" is a solitary predicate. The predicate $B_{stableMarriage}$ for the stable marriage is given by,

$$B_{stableMarriage} \equiv \forall j : \neg forbidden(j, G)$$

where $forbidden(j, G)$ is defined as

$$(\exists i : \exists k \leq G[i] : (z = mpref[i][k]) \wedge (rank[z][i] < rank[z][j])),$$

with $z = mpref[j][G[j]]$.

The predicate says that a marriage given by the vector G is stable if none of its index j is forbidden. The index j is forbidden if the woman z corresponding to man j's preference in $G[j]$ is also equal to the preference of man i in G, or a global state before G, and the woman z prefers i to j.

We now define the predicate for the man-optimal stable marriage, $B_{mosm}(G)$ as

$$B_{stableMarriage}(G) \wedge (\forall H : B_{stableMarriage}(H) \Rightarrow G \leq H).$$

By definition, it is clear that B_{mosm} is a solitary predicate. Given a lattice L, we can search for the element satisfying B_{mosm} by searching for $B_{stableMarriage}$.

22.5 Problems

1. Consider the stable marriage problem. We say that a marriage is a k-level marriage if it is at the level k of the lattice of all assignments. Consider the predicate B to hold on an assignment if it is a stable marriage at level k. Show that B is an equilevel predicate. What is the time complexity of detecting B?

2. Consider the Independent Set problem in an undirected graph (V, E) Consider the predicate B to hold on a subset $V' \subseteq V$ iff $(|V'| = k)$ and there is no edge between any pair of vertices in V'. Show that B is an equilevel predicate. What is the time complexity of detecting B?

22.6 Bibliographic Remarks

The notion of equilevel predicates and solitary predicates is from reference [GS24].

Chapter 23

Appendix: List of Lattice Linear Programs in Java

```java
import java.util.Arrays;
public class JobScheduling extends LLP {
   final int[] time;
   final int[][] prerequisites;
   int[] G;
   public JobScheduling(int[] time, int[][] prerequisites) {
      super(time.length);
      this.time = time;
      this.prerequisites = prerequisites;
      G = Arrays.copyOf(time, time.length);
   }
   public boolean forbidden(int j) {
      for (int i : prerequisites[j])
         if (G[j] < G[i] + time[j]) return true;
      return false;
   }
   public void advance(int j) {
      int newStart = -1;
      for (int i : prerequisites[j])
         newStart = Math.max(newStart, G[i] + time[j]);
      G[j] = newStart;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.1: Job Scheduling Program

```java
import java.util.Arrays;
public class ParallelReduce extends LLP {
   final int n;
   final int[] A;
   int[] G;
   public ParallelReduce(int[] A) {
      super(A.length);
      this.n = A.length;
      this.A = A;
      G = new int[n];
      Arrays.fill(G, Integer.MIN_VALUE);
      G[0] = 0;
   }
   public boolean forbidden(int j) {
      return ((1 <= j && j < n/2 && G[j] < G[2*j] + G[2*j+1]) ||
            (n/2 <= j && j < n && G[j] < A[2*j-n] + A[2*j-n+1]));
   }
   public void advance(int j) {
      if (1 <= j && j < n/2) G[j] = G[2*j] + G[2*j+1];
      if (n/2 <= j && j < n) G[j] = A[2*j-n] + A[2*j-n+1];
   }
   public int[] getSolution() {
      int[] reduce = new int[n - 1];
      for (int i = 1; i < n; i++)
         reduce[i-1] = G[i];
      return reduce;
   }
}
```

Figure 23.2: Parallel Reduce Program

```
import java.util.Arrays;
public class ParallelPrefix extends LLP {
   final int n;
   final int[] A;
   final int[] S;
   int[] G;
   public ParallelPrefix(int[] A, int[] S) {
      super(A.length * 2);
      this.n = A.length;
      this.A = A;
      this.S = S;
      G = new int[2 * n];
      Arrays.fill(G, Integer.MIN_VALUE);
      G[0] = -1;
   }
   public boolean forbidden(int j) {
      return ((j == 1 && G[j] < 0) ||
            (j % 2 == 0 && G[j] < G[j/2]) ||
            (j % 2 == 1 && j > 1 && j < n && G[j] < S[j-2] + G[j/2]) ||
            (j % 2 == 1 && j > 1 && j > n && G[j] < A[j-n-1] + G[j/2]));
   }
   public void advance(int j) {
      if (j == 1) G[j] = 0;
      else if (j % 2 == 0) G[j] = G[j/2];
      else if (j % 2 == 1 && j < n) G[j] = S[j-2] + G[j/2];
      else if (j % 2 == 1 && j > n) G[j] = A[j-n-1] + G[j/2];
   }
   public int[] getSolution() {
      int[] prefix = new int[n];
      for (int i = n; i < 2*n; i++)
         prefix[i-n] = G[i];
      return prefix;
   }
}
```

Figure 23.3: Parallel Prefix Program

```java
public class StableMarriage extends LLP {
   final int m, w;
   final int[][] mprefs;
   final int[][] wprefs;
   int[] G;
   public StableMarriage(int[][] mprefs, int[][] wprefs) {
      super(mprefs.length);
      this.m = mprefs.length;
      this.w = wprefs.length;
      this.mprefs = mprefs;
      this.wprefs = wprefs;
      this.G = new int[m];
   }
   public boolean forbidden(int j) {
      int z = mprefs[j][G[j]];
      int[] wpref = wprefs[z];
      for (int i = 0; i < m; i++) {
         if (i==j) continue;//only for different men
         if (z != mprefs[i][G[i]]) continue;
         if (wpref[i] < wpref[j]) return true;
      }
      return false;
   }
   public void advance(int j) {
      G[j]++;
   }
   public int[] getSolution() {//Return the partner for each man as a list
      int[] assignment = new int[m];
      for (int i = 0; i < m; i++)
         assignment[i] = mprefs[i][G[i]];
      return assignment;
   }
}
```

Figure 23.4: Stable Marriage Program

```
public class ListRank extends LLP {
   public class Node {
      int dist;
      int next;
      public Node(int dist, int next) {
         this.dist = dist;
         this.next = next;
      }
   }
   Node[] G;
   int r;
   int n;
   public ListRank(int[] parent) {
      super(parent.length);
      this.n = parent.length;
      this.G = new Node[n];
      for (int i = 0; i < n; i++)
         if (parent[i] == -1) {
            this.r = i;
            G[i] = new Node(0, i);
         } else G[i] = new Node(1, parent[i]);
   }
   public boolean forbidden(int j) {
      return G[j].next != this.r;
   }
   public void advance(int j) {
      while (true) {
         int dist = G[G[j].next].dist;
         int next = G[G[j].next].next;
         // Ensure the values haven't changed in between reads.
         if (G[G[j].next].dist == dist) {
            G[j].dist += dist;
            G[j].next = next;
            break;
         }
      }
   }
   public int[] getSolution() {//Return the distance of every node to the root
      int[] distances = new int[n];
      for (int i = 0; i < n; i++)
         distances[i] = G[i].dist;
      return distances;
   }
}
```

Figure 23.5: List Rank Program

```java
public class Traversal extends LLP {
   int n; //number of vertices
   final int[][] edge;
   int[] visited;
   int[] G;
   public Traversal(int n, int[][] edge, int source) {
      super(n);
      this.edge = edge;
      this.G = new int[n];
      visited = new int[n];
      visited[source] = 1;
   }
   public boolean forbidden(int j) {
      if ((visited[j] == 1) && (G[j] == 0)) return true;
      else return false;
   }
   public void advance(int j) {
      G[j] = 1;
      for (int k: edge[j]) visited[k] = 1;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.6: Graph Traversal Program

```
import java.util.*;
public class Traversal2 {
   int n; //number of vertices
   final int[][] edge;
   boolean[] visited;
   boolean[] G;
   int source;
   LinkedList<Integer> H;
   public Traversal2(int n, int[][] edge, int source) {
      this.n = n;
      this.edge = edge;
      this.source = source;
      this.G = new boolean[n];
      visited = new boolean[n];
   }
   public boolean forbidden(int j) {
      if (visited[j] && !G[j] ) return true;
      else return false;
   }
   public void advance(int j) {
      G[j] = true;
      for (int k: edge[j]) {
            visited[k] = true;
            if (!G[k]) H.addLast(k);
      }
   }
   public boolean[] getSolution() {
      H = new LinkedList<Integer>();
      H.add(source);
      visited[source] = true;
      while (!H.isEmpty()) {
         int j = H.poll();
         if (forbidden(j)) advance(j);

      }
      return G;
        }
}
```

Figure 23.7: Graph Traversal Program

```java
public class TraversalBFS extends LLP {
   final int[][] pre;
   int[] G;
   public TraversalBFS(int n, int[][] pre, int source) {
      super(n);
      this.pre = pre;
      G = new int[n];
      for (int i=0; i<n; i++)
         G[i] = n+1; //equivalent to infinity
      G[source] = 0;
   }
   public boolean ensure(int j) {
      boolean changed = false;
      for (int i: pre[j])
         if (G[i] < n)
            if (G[j] > G[i]+1) {
               changed = true;
               G[j] = G[i]+1;
            }
      return changed;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.8: Graph BFS Traversal Program

```
public class Layering extends LLP {
   final int[][] prerequisites;
   int[] G;
   boolean[] fixed;
   public Layering(boolean[] fixed, int[][] prerequisites) {
      super(fixed.length);
      this.fixed = fixed;
      this.prerequisites = prerequisites;
      this.G = new int[fixed.length];
      fixed[0] = true;
      G[0] = 0;
   }
   public boolean forbidden(int j) {
      if (fixed[j]) return false;
      for (int i : prerequisites[j])
         if (!fixed[i]) return false;
      return true;
   }
   public void advance(int j) {
      int newStart = 0;
      for (int i : prerequisites[j])
         newStart = Math.max(newStart, G[i]+1);
      G[j] = newStart;
      fixed[j] = true;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.9: Graph Layering Program

```java
public class SlowComponent extends LLP {
   final int[][] adj;
   int[] G;
   int num;
   boolean debug = true;
   public SlowComponent(int num, int[][] adj) {
      super(num);
      this.adj = adj;
      this.G = new int[num];
      for (int i=0; i<num; i++) G[i] = i;
   }
   public boolean ensure(int j) {
      boolean changed = false;
      for (int i: adj[j])
         if (G[j] < G[i]) { G[j] = G[i]; changed = true; }
      return changed;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.10: Slow Component Program

```java
public class FastComponent extends LLP {
   final int[][] adj;
   int[] G;
   int num;
   boolean debug = true;
   public FastComponent(int num, int[][] adj) {
      super(num);
      this.adj = adj;
      this.G = new int[num];
      for (int i=0; i<num; i++) G[i] = i;
   }
   public boolean ensure(int j) {
      boolean changed = false;
      if (G[j] != G[G[j]]) { G[j] = G[G[j]]; changed = true; }
      else
         for (int i: adj[j])
            if (G[j] < G[i]) { G[j] = G[i]; changed = true; }
      return changed;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.11: Fast Component Program

```java
import java.util.*;
class Node implements Comparable<Node>{
   int index;
   int distance;
   public Node(int index, int distance) {
      this.index = index;
      this.distance = distance;
   }
   public int compareTo(Node other) { return Integer.compare(distance,
       other.distance); }
}
public class Dijkstra extends LLP {
   final List<List<Node>> adjacency;
   final int[][] w;
   int[] G;
   int num;
   boolean debug = false;
   boolean[] fixed;
   PriorityQueue<Node> H;
   public Dijkstra(int num, List<List<Node>> adjacency, int[][] w) {
      super(num);
      this.adjacency = adjacency;
      this.w = w;
      G = new int[num];
      fixed = new boolean[num];
      Arrays.fill(G, Integer.MAX_VALUE);
      Arrays.fill(fixed, false);
      int start = 0;
      G[start] = 0;
      H = new PriorityQueue<Node>();
      H.add(new Node(start, 0));
   }
   public boolean forbidden(int j) {
      if (H.isEmpty()) return false;
      Node temp = H.peek();
      if (temp.index != j) return false;
      else return true;
   }
   public void advance(int j) {
      Node t = H.poll();
      if (fixed[t.index]) return;
      fixed[t.index] = true;
      for (Node k : adjacency.get(t.index))
         if (G[t.index] + k.distance < G[k.index]) {
            G[k.index] = G[t.index] + k.distance;
            H.add(new Node(k.index, G[k.index]));
         }
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.12: Dijkstra's Shortest Path Program

```java
public class BellmanFord extends LLP {
   final int[][] pre;
   final int[][] w;
   int[] G;
   int num;
   public BellmanFord(int num, int[][] pre, int[][] w) {
      super(num); this.pre = pre; this.w = w;
      G = new int[num];
      for (int i=0; i<num; i++)
         G[i] = 1000; // Integer.MAX_VALUE;
      G[0] = 0;
   }
   public boolean ensure(int j) {
      boolean changed = false;
      for (int i: pre[j])
         if (G[j] > G[i] + w[i][j]) {
            G[j] = G[i] + w[i][j];
            changed = true;
         }
      return changed;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.13: Bellman-Ford Shortest Path Program

```java
public class Johnson extends LLP {
   final int[][] pre;
   final int[][] w;
   int[] G;
   int num;
   public Johnson(int num, int[][] pre, int[][] w) {
      super(num); this.pre = pre; this.w = w;
      G = new int[num];
      for (int i=0; i<num; i++) G[i] = 0;
   }
   public boolean ensure(int j) {
      boolean changed = false;
      for (int i: pre[j])
         if (G[j] < G[i] - w[i][j]) {
            G[j] = G[i] - w[i][j];
            changed = true;
         }
      return changed;
   }
   public int[] getSolution() { return G; }
}
```

Figure 23.14: Johnson Shortest Path Program