Spatial Reuse and Fairness of Mobile Ad-Hoc Networks with Channel-Aware CSMA Protocols

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Abstract

We investigate the benefits of channel-aware (opportunistic) scheduling of transmissions in ad-hoc networks. The key challenge in optimizing the performance of such systems is finding a good compromise among three interdependent quantities: the density and channel quality of scheduled transmitters, and the resulting interference at receivers. We propose two new channel-aware slotted CSMA protocols: opportunistic CSMA (O-CSMA) and quantile-based CSMA (QT-CSMA) and develop stochastic geometric models allowing us to quantify their performance in terms of spatial reuse and spatial fairness. When properly optimized these protocols offer substantial improvements in performance relative to CSMA – particularly when the density of nodes is moderate to high. Moreover, we show that a simple version of Q-CSMA can achieve robust performance gains without requiring careful parameter optimization. The quantitative results in this paper suggest that channel-aware scheduling in ad-hoc networks can provide substantial benefits which might far outweigh the associated implementation overheads.

I. Introduction

Evaluating and optimizing the capacity of wireless ad-hoc networks has been one of the goals of the networking and information theory research communities for a last decade. Due to the inherent randomness in such networks, e.g., locations of nodes, wireless channels, and node interactions governed by protocols, researchers have developed stochastic models that can parsimoniously capture the uncertainty of such environments while still giving insight on system performance and optimization. Work based on stochastic geometric models have perhaps been the most successful in terms of providing reasonably realistic, yet mathematically tractable, results, see e.g., [5], [44], [20]. This paper leverages this line of work to study the performance of networks operated under two channel-aware slotted CSMA type protocols.

One of the important factors determining the capacity of a wireless network is the *degree of spatial reuse;* this is mainly determined by the associated medium access control protocol. The following basic protocols: ALOHA, Opportunistic ALOHA (O-ALOHA), and CSMA have been studied in detail in literature. We discuss these briefly below.

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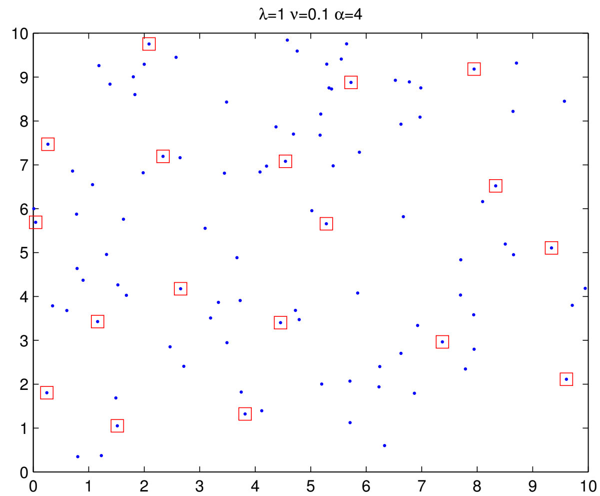


Figure 1: A realization of modified Matérn hardcore process : randomly distributed points are the realization of marked Poisson point process where each point has an independent identically distributed mark denoting its timer value in $[0, 1]$. If a point has the smallest timer value in its neighborhood (neighborhood is not shown here but formally defined later in (3)), then, it is selected as a CSMA transmitter. Selected CSMA transmitters were drawn inside boxes.

ALOHA is a basic MAC protocol in which spatially distributed nodes simply transmit with some probability p. A mathematical model for a spatial version of an ALOHA based wireless ad-hoc network is detailed in [5]; various extensions capturing the impact of modulation techniques on the transmission capacity have been studied, see e.g., [44]. Because transmitters contend independently, the transmission probability p should be properly chosen as a function of node density so as to achieve a high spatial reuse. This involves finding a compromise between a high density of transmitters and associated excessive interference which deteriorates the quality of transmissions and accordingly leads to low spatial reuse.

In [2], [43], the performance of an opportunistic version of spatial ALOHA (O-ALOHA)[1] was evaluated. In their models, only *qualified* transmitters, namely nodes whose channel quality to their associated receivers exceeds a threshold γ, can transmit with probability p. The resulting spatial reuse is thus affected by both parameters. When properly tuned, this simple channel-aware MAC can increase spatial reuse by roughly 40% relative to simple ALOHA.

Although O-ALOHA can dramatically increase spatial reuse by qualifying nodes seeing good channels, it still suffers from collisions which limits its performance. Unlike (O-)ALOHA, carrier sense based medium access (CSMA) protocols achieve high spatial reuse by coordinating transmissions. In [28], [3], a modified Matérn hardcore process model for a spatial slotted CSMA protocol was introduced. Each node contends with its 'neighbors' via a uniformly distributed contention timer. The node with the earliest timeout wins. As a result the transmitters end up being nicely separated, see e.g., Fig. 1. Based on this model, CSMA is shown to increase spatial reuse by roughly 25% over basic ALOHA.

[1]The ALOHA considering channel state information (a.k.a opportunistic ALOHA) in single hop network was introduced and studied in [1], [34].

In this paper, we extend the CSMA ad-hoc network model introduced in [3] to study two simple channel-aware MAC protocols. In the first scheme, named opportunistic CSMA (O-CSMA), we use a channel quality threshold γ, as introduced in [2], [43], to qualify nodes to participate in the CSMA contention process. Optimizing performance of such networks requires selecting γ as a function of node density and channel variation distributions. In the second scheme called quantile-based CSMA (QT-CSMA), nodes contend based on the quantile of the channel quality to their associated receivers. Doing so allows nodes to transmit when their channel is the 'best' in their neighborhood. This also ensures that each node gets a fair share of access opportunities among the nodes in its neighborhood, and circumvents the problem of choosing a density dependent qualification threshold. This is particularly desirable if channel statistics seen across nodes are heterogeneous. Quantile-based scheduling approaches for downlinks in cellular networks were introduced and studied in [33], [32], [31], [6] and in the wireless LAN setting in [21], [11].

The performance metrics considered in this paper are *spatial* averages of network performance, which means the performance metric captures an average over possible realizations of nodes' locations. This is particularly meaningful, since in real world scenarios nodes are irregularly placed and/or motion might make a performance metric which is a function of nodes' location is less informative. To that end, we characterize the performance as seen by a *typical* node using tools from stochastic geometry together with analytical/numerical computation methods.

a) Contributions: This paper makes the following four contributions.

First, to our knowledge, this is the first attempt to evaluate CSMA-based opportunistic MAC protocols in ad-hoc networks, namely O-CSMA and QT-CSMA. Our approach captures the delicate interactions between the channel gains and interference statistics underlying the performance of opportunistically scheduled nodes in ad-hoc networks.

Second, we evaluate the sensitivity of spatial reuse to various protocol parameters, showing a clear advantage of QT-CSMA over O-CSMA which in turn have substantially better performance than simple ALOHA based schemes. To that end, we characterize the interplay between the density of active transmitters and the quality of transmissions as the function of qualification threshold γ and carrier sensing threshold ν. In addition, we explore the maximum achievable spatial reuse under O/QT-CSMA in asymptotically dense networks by scaling γ as the function of node density. We show that both O/QT-CSMA exhibit a 'phase transition' phenomenon for spatial reuse depending on the scaling of γ.

Third, this paper is the first to evaluate the spatial fairness realized by these protocols and to find that QT-CSMA can achieve better fairness than CSMA. Specifically we introduce and quantify a spatial fairness index among sets of nodes sharing the same number of neighbors, and which captures the impact of random nodes' placements.

Finally, we study tradeoffs between spatial fairness and spatial reuse, and compare the Pareto-frontier of O-CSMA and that of QT-CSMA. In particular, we show that quantile-based CSMA without a qualification step (QT_0-CSMA) achieves a performance comparable to that of O/QT-CSMA in terms of both fairness and the density of successful transmissions, which is then a robust and attractive choice from an engineering perspective. We present some initial discussion of implementation consideration for such a protocol.

b) Related Work: The two-hop wireless network studied in [18] is perhaps the first analytical model for a multi-hop wireless network. A similar model was used in [37], [38] to study the performance of slotted ALOHA and CSMA protocols respectively. Subsequently [39] and [7] have considered general multi-hop wireless networks.

The Markovian model for a multi-hop CSMA wireless network introduced in [7] has been the basis of much subsequent work. The network is modeled as a graph, where each node denotes a transmitter and if a pair of transmitters can interfere with each other's receivers they are connected by an edge. Thus viable sets of transmitters correspond to independent sets. A MAC protocol can be modeled as a Markov chain over independent sets, whose stationary distribution captures the long run performance seen by nodes, but is hard to evaluate.

This idealized model was later extended and widely used to show various insights on system behavior and performance [42][13][12][41][40][14]. In [42], various throughput approximations for CSMA/CA based networks are developed, and various fairness driven scheduling methods are proposed and evaluated. The authors show that on a simple linear network with three nodes, CSMA/CA can be very unfair when nodes are aggressively accessing the medium. This problem arises due to location dependent contention in the multi-hop network setting, and can be serious since it can lead to node starvation [26][16]. Based on the Markov chain model in [7] and [42], [13] studies the impact of asymmetry or so-called border effects and carrier sensing ranges on fairness in a linear topology. The authors confirm that unfairness is due to asymmetry in the network topology, which implies that unfairness can be removed, to a large extent, by either increasing the size of 1D networks or making carrier sense range larger than the receive range. However, it turns out that, in 2D grid networks, a phase-transition like phenomenon occurs whereby unfairness in a large network arises sharply if the intensity of nodes' access is sufficiently high [12]. The emergence of this phenomenon is a result of the regular structure of the grid-network and it vanishes as the network becomes irregular. Still CSMA-based networks exhibit various degrees of unfairness. To explore how to make the CSMA network fair, [41] and [40] introduce a Markov chain model, with node-specific access intensities which equalize the per-node throughput. This approach was also used in [27], which introduced a framework translating various fairness objectives to corresponding contention resolution algorithms. The above described Markovian model is simple enough and somewhat tractable, so widely accepted. However, the Markovian model is too idealized to incorporate various PHY/MAC parameters and random factors such as node locations, fading channels[2] and aggregate interference at receivers, all of which have a substantial impact on the performance of individual nodes and the overall network.

With the introduction of the IEEE 802.11 protocol, several researchers have attempted to analyze multi-hop wireless networks using the IEEE 802.11. [10] was one of the early efforts which provided an analytical model for a given fixed network and computed the lower bound on the sum throughput of transmitter-receiver pairs for a given network. However, the model's simplified physical layer, so-called protocol model, does not take into account the impact of aggregate interference. Later, [9] provided a more sophisticated model which takes into account various PHY and MAC layer parameters. The authors linearly approximated the access probability of individual nodes as a function of its success probability

[2]Note that random fading is crucial for studying opportunistic scheduling.

and found a linear system like relationship between success probabilities and transmission probabilities of nodes in a given network. This gives a reasonable approximation of the per-node throughput, however, the work does not reveal how the system is affected by various system parameter selections or the inherent randomness in wireless environment. Furthermore performance is evaluated for a *given* fixed network, which is less informative considering mobile nature of wireless nodes.

The above limitations - i.e., not taking into account random fading channel, random node locations, impact of aggregate interference, and capture effect[3] - are naturally addressed in research based on stochastic geometric models, see e.g., [20], [5], [36], [44], on which our work is based. In this line of work, the performance metrics of interest is an average over random environments (including fading, node locations, protocols, etc), which can be more informative in terms of representing typical behavior. Specifically the CSMA related work of [3] and [28] used a spatial point process to model spatially distributed wireless nodes using a CSMA-like MAC protocol. These works successfully approximated the statistics of the aggregate interference resulting from CSMA-like MAC nodes by those of a non-homogeneous Poisson point process of interferers. The approximation was validated via simulation and was shown to match well. However, characterizing the exact interference statistics is still very hard and has remained an open problem. As a response to this, subsequent work in [17], [15] suggested an alternative approximation for the performance of CSMA nodes which is accurate for asymptotically sparse networks. The carrier sense mechanism models have also been successfully used to study cognitive radio networking scenarios in [24], [23], [29].

Our work is different from the above work in the following aspects. First, we build upon the CSMA model in [3] incorporating *opportunistic scheduling* schemes. We consider the dependency between the channel gain of a scheduled node and the activity of the surrounding nodes (or accordingly the statistics of interference), which has to our knowledge not been explored before. Second, we consider the fairness for *slotted* (or synchronized) CSMA networks. In particular, we study how system parameters and opportunistic CSMA protocols can change the fairness characteristics of the slotted CSMA network.

A. Organization

In Section II, we describe our system model, including details for our two proposed opportunistic MAC protocols. In Section III, the transmission and success probability of a typical node under the two MAC protocols are derived. These will be used later to compute the two performance metrics. In Section IV, we compare the numerical results for the spatial reuse of O-CSMA and QT-CSMA networks under three node density regimes, and in Section V, the fairness of such networks is evaluated and tradeoffs between spatial reuse and fairness are considered under various parameter values. We describe possible implementations of proposed protocols in Section VI and conclude in Section VII.

[3]If two transmitters happen to send their packets to the same receiver, the one with a higher signal strength can be received with non-zero probability. This is called as a capture effect.

II. System Model

A. Node Distribution and Channel Model

We model an ad-hoc wireless network as a set of transmitters and their corresponding receivers. Transmitters are distributed in $\mathbb{R}^2$ as an independently marked homogeneous Poisson Point Process (PPP) $\Psi = \{X_i, E_i, T_i, \mathbf{F}_i, \mathbf{F}'_i\}$, where $\Phi \equiv \{X_i\}_{i \geq 1}$ is the PPP with density λ denoting the set of transmitters or their locations in $\mathbb{R}^2$ and E_i is an indicator function which is equal to 1 if a node X_i transmits and 0 otherwise. The value of E_i is governed by the medium access protocol used and the activity of other nodes $\{X_j\}_{j \neq i}$. We assume that the receiver of each transmitter is located r meters away from the transmitter. The direction from a transmitter to its receiver is randomly distributed, i.e., uniformly on $[0, 2\pi]$. Throughout this paper, we only consider the performance as seen by a typical receiver.

Let $\mathbf{F}_i = (F_{ij} : j)$ be a vector of random variables F_{ij} denoting the fast fading channel gains between the ith *transmitter* and the *receiver* associated with jth transmitter. In particular, F_{ii} denotes the channel gain from ith transmitter to its associated receiver. We assume that F_{ij} are identically distributed (i.i.d.) with mean μ^{-1}, i.e., $F_{ij} \sim F$ (For two random variables A and B having the same distribution, we write $A \sim B$.), with cumulative distribution function (cdf) $G(x) = P(F \leq x)$. Let $\mathbf{F}'_i = \left(F'_{ij} : j\right)$ be the vector with coordinates F'_{ij} where F'_{ij} is the random variable denoting the fast fading gain between the ith *transmitter* and the jth *transmitter*. The random variable F'_{ij} are assumed symmetric[4] and i.i.d, i.e., $F'_{ij} = F'_{ji}$ and $F'_{ij} \sim F$. In this paper, we only consider the Rayleigh fading case where F has an exponential distribution with cdf $G(x) = 1 - \exp\{-\mu x\}$ for $x \geq 0$, but other fading models could be considered. Let $\|x\|$ denote the norm of the vector $x \in \mathbb{R}^2$ and $l(\|x - y\|) = \|x - y\|^\alpha$ be the path loss between two locations $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ with pathloss exponent $\alpha > 2$. Then, the interference power that the jth receiver at location y experiences from the ith transmitter at location x is $F_{ij}/l(\|x - y\|)$.

B. Signal to Interference Ratio Model

The performance of a receiver is governed by its signal to interference plus noise ratio (SINR). Under the model given above, the SINR seen at the i-th receiver is

$$\mathrm{SINR}_i = \frac{F_{ii}/l(r)}{I_{\Phi \setminus \{X_i\}} + W}, \tag{1}$$

where $I_{\Phi \setminus \{X_i\}} = \sum_{X_j \in \Phi \setminus \{X_i\}} E_j F_{ji}/l(\|X_i - X_j\|)$ is the aggregate interference power, or so-called shot noise, and W is the thermal noise. We shall focus on interference limited networks, when the impact of thermal noise is comparatively negligible. In this paper we focus on such a regime and let $W = 0$. The reception model we consider is the so-called *outage reception model*, where a receiver can successfully decode a transmission if its received SINR exceeds a decoding threshold t, i.e., the i-th receiver gets $\log(1 + t)$ bits per second (bps) per transmission if $\mathrm{SINR}_i > t$ and zero otherwise[5].

[4]Unlike F'_{ij}, F_{ij} is not symmetric, i.e., $F_{ij} \neq F_{ji}$.

[5]Instead of outage model, one can consider SINR model, where the performance of a typical receiver is given as $\mathbb{E}^0[\log(1 + SINR_0)]$ bits per second where $SINR_0$ is the SINR of a typical receiver [2]. This is an appropriate model for wireless devices using adaptive modulation and coding technique. In this paper, we consider the outage model for simplicity, but it can be easily extended to the adaptive modulation model.

C. Carrier Sense Multiple Access Protocols

We consider a *slotted* CSMA network, where nodes compete with each other to access a shared medium. Carrier sensing is followed by data transmission at each slot. Each node contends with its 'neighboring' nodes using a (uniformly distributed on $[0,1]$) timer value. The timer value is *independent* of everything else and each node transmits if it has the smallest timer value in its neighborhood and defers otherwise. CSMA provides a way to resolve contentions among nodes but does not take advantage of channel variations. In what follows, we introduce two distributed *opportunistic* CSMA protocols which take advantage of channel variations amongst transmitters and their receivers: opportunistic CSMA (O-CSMA) and Quantile-based CSMA (QT-CSMA).

Under O-CSMA, nodes whose channel gains are higher than a fixed threshold γ qualify to contend; we call this the *qualification process*. We assumed that channel feedback from each receiver is available to its associated transmitter each slot. Qualified nodes in turn, contend for transmission with their neighbors on that slot. Specifically, let $\Phi^{\gamma} = \{X_i \in \Phi \mid F_{ii} > \gamma\}$ denote the set of qualified nodes or *contenders*. Note that Φ^{γ} is a subset of Φ which is generated by independent marks with probability

$$p_{\gamma} = \mathbb{P}(F > \gamma), \tag{2}$$

so it is a homogeneous PPP with density $\lambda^{\gamma} \equiv \lambda p_{\gamma}$. Each contender $X_i \in \Phi^{\gamma}$ has a set of qualified nodes with which it contends. We say two transmitters X_i and X_j *contend* if the received interference they see from each other is larger than the carrier sensing threshold ν, i.e., if $F'_{ij}/l(\|X_i - X_j\|) > \nu$ and by symmetry $F'_{ji}/l(\|X_i - X_j\|) > \nu$. We call the set of contenders of a node its *neighborhood* and denote it by

$$\mathcal{N}_i^{\gamma} = \left\{X_j \in \Phi^{\gamma} \text{ s.t. } F'_{ji}/l(\|X_i - X_j\|) > \nu, j \neq i\right\}. \tag{3}$$

Contending nodes are not allowed to transmit simultaneously since they can potentially interfere with each other. To avoid collisions, every slot each node X_j in Φ^{γ} picks a random timer value T_j which is uniformly distributed on $[0,1]$. At the start of each time slot node X_j starts its own timer which expires in T_j seconds. Each node senses the medium until its own timer expires. If no node (in its neighborhood) begins transmitting prior to that time, then, it starts transmitting, otherwise it defers. Under this mechanism, a node transmits only if the node's timer value is the minimum in its neighborhood, i.e., when T_i is equal to $\min_{j:X_j \in \mathcal{N}_i^{\gamma} \cup \{X_i\}} T_j$.

Note that the qualification process is a mechanism selecting nodes with high channel gains. Thus, all qualified nodes have channel gains larger than γ. The posterior channel distribution after qualification is F conditioned on that $F > \gamma$, so given by a shifted exponential distribution

$$G_{\gamma}(x) \equiv \mathbb{P}(F < x | F > \gamma) = \left(1 - \exp\{-\mu(x - \gamma)\}\right) \mathbf{1}\{x \geq \gamma\}. \tag{4}$$

The qualification process not only increases the signal strength but also reduces the amount of interference, and therefore we can expect successful transmissions. However, the parameter γ should be chosen judiciously; otherwise there will either be too many transmitting nodes generating too much interference or too few transmitting nodes resulting in low spatial reuse. Neither case is desirable. Note that when $\gamma = 0$ this model corresponds to the standard CSMA one proposed in [3].

Under QT-CSMA one also has a qualification process with threshold γ. However, the active transmitters in a neighborhood are selected based on the *quantile of their current channel gain*; we refer to this as *quantile scheduling*. Specifically, we assume that channel quality F_{ii} is available to the transmitter X_i, and at each slot a qualified transmitter X_i computes its channel quantile $Q_i = G_\gamma\left(F_{ii}\right)$ using its channel gain F_{ii} (conditioned on that $F_{ii} > \gamma$). This transforms the channel distribution to a uniform distribution on $[0, 1]$, which serves both as a relative indicator for channel quality and to determine the timer for collision avoidance. More precisely under QT-CSMA X_i sets its timer value, say T_i, to $1 - Q_i$ and starts sensing the medium until its timer expires. If no transmitting node is detected prior T_i, then, the node accesses the medium, otherwise it defers. In other words, node X_i transmits only if it has the highest quantile in its neighborhood, i.e., when $Q_i = Q_i^{\max}$ where $Q_i^{\max} \equiv \max_{j:X_j\in\mathcal{N}_i^\gamma\cup\{X_i\}} Q_j$. Let $F_{i,\gamma}^{\max} = G_\gamma^{-1}\left(Q_i^{\max}\right)$ be the channel fade of a transmitting node X_i or the channel fade given node X_i transmits, where $G_\gamma^{-1}(\cdot)$ is the inverse function of $G_\gamma(\cdot)$. Let $N_i^\gamma = |\mathcal{N}_i^\gamma|$ for simplicity; then $F_{i,\gamma}^{\max}$ is a $N_i^\gamma + 1$st order statistic, i.e.,

$$F_{i,\gamma}^{\max} = \max\left[F_{1,\gamma}, F_{2,\gamma}, \cdots, F_{N_i^\gamma+1,\gamma}\right], \tag{5}$$

whose distribution conditioned on $N_i^\gamma = n$ is given by

$$\mathbb{P}\left(F_{i,\gamma}^{\max} \leq x | N_i^\gamma = n\right) = \left(1 - \exp\{-\mu(x-\gamma)\}\right)^{n+1} \mathbf{1}\{x \geq \gamma\}. \tag{6}$$

Note that QT-CSMA further exploits opportunism beyond the qualification process. Unlike O-CSMA, a QT-CSMA node transmits only when it has the best channel condition in its neighborhood, which should further improves its likelihood of successful transmission. One may surmise that QT-CSMA may work well even without qualification process since the quantile scheduling will fully take advantage of opportunistic gain from many nodes (so-called multi-user diversity). This will be explored later. For that purpose, we shall denote QT-CSMA with $\gamma = 0$ by QT$_0$-CSMA.

D. Notation

For a random variable I, let $\mathcal{L}_I\left(s\right) = \mathbb{E}\left[e^{-sI}\right]$ be the Laplace transform of I. Let $\|x\|$ be the magnitude of $x \in \mathbb{R}^2$. Given a countable set C, let $|C|$ be the cardinality of C. Let $\mathbf{1}\{\cdot\}$ denote the indicator function and let $B_l \equiv b(0, l)$ denote a ball centered at the origin with radius l. $\mathbb{R}_+$ denotes the set of non-negative real numbers. Let Φ be a stationary point process and $\mathcal{Y}$ be a property of Φ. We will use below the reduced Palm probability $\mathbb{P}^{!0}$ of Φ. Intuitively, the probability that Φ satisfies the property $\mathcal{Y}$ under $\mathbb{P}^{!0}$ is the conditional probability that $\Phi \setminus \{0\}$ satisfies property $\mathcal{Y}$ given that Φ has a point at 0. This will be denoted as follows: $\mathbb{P}^0(\Phi\setminus\{0\} \in \mathcal{Y}) = \mathbb{P}^{0!}(\Phi \in \mathcal{Y})$. We define Φ^0 as a point process Φ given $0 \in \Phi$. $\mathbb{E}^0$ denotes Palm expectation, which is interpreted as the conditional expectation conditioned on a node at the origin [36], [4].

III. Transmission Performance Analysis

In this section, we derive expressions for the access and transmission success probabilities which in turn are used to compute the density of successful transmissions for our opportunistic scheduling schemes.

Table I: Summary of notations

X_i	i-th transmitter or its location in $\mathbb{R}^2$
Φ	Poisson point process denoting the set of transmitters $\{X_i\}$.
λ	Density of nodes in Φ
F	Generic exponential random variable with mean $1/\mu$ denoting short term fading gain
F_{ij}	Short term fading gain between transmitter X_i and the receiver associated with X_j
$F'_{ij}(=F'_{ji})$	Short term fading gain between transmitter X_i and the transmitter X_j
γ	Qualification threshold
p_γ	Probability that a transmitter to qualify $(=\mathbb{P}(F>\gamma))$
Φ^γ	Set of qualified transmitters
F_γ	Fading gain of O-CSMA
$G_\gamma(\cdot)$	Cdf of random variable F_γ
Φ_M^γ	Set of active transmitters using CSMA protocol
$\Phi_M^{\gamma 0}$	Set of active transmitters Φ_M^γ given $0\in\Phi_M^\gamma$
ν	Carrier sense threshold
t	Decoding threshold
λ^γ	Density of qualified transmitters $(=\lambda p_\gamma)$
N_0^γ	Size of neighborhood of a typical node
$F_{0,\gamma}^{\max}(N_0^\gamma+1)$	Fading gain of a typical QT-CSMA transmitter when the size of its neighborhood is N_0^γ
r	Distance between a transmitter and its associated receiver
$N_{s,0}^\gamma$	Size of neighborhood of typical node under the assumption $F'_{ij}=\mathbb{E}[F]$ in Section V-A
$I_{\Phi_M^\gamma\backslash\{0\}}$	Aggregate interference power from transmitters in $\Phi_M^\gamma\backslash\{0\}$
$I^{n,x}_{\Phi_M^\gamma\backslash\{0\}}$	Aggregate interference power from transmitters in $\Phi_M^\gamma\backslash\{0\}$ conditioned on that the associated typical node has $N_0^\gamma=n$ contenders and it has channel gain $F_{0,\gamma}^{\max}(N_0^\gamma+1)=x$
λ_{dens}	Asymptotic density of active transmitters
Φ_M^{dens}	Point process of active CSMA transmitters with density λ_{dens}
$I_{\Phi_M^{dens}\backslash\{0\}}$	Aggregate interference power from transmitters in $\Phi_M^{dens}\backslash\{0\}$

We begin by defining performance metric and restating some known results from [2], [3] modified to fit to our setting.

A. Spatial Reuse

As a measure of spatial reuse, we will use *the density of successful transmissions* which is defined as the mean number of nodes that successfully transmit per square meter. This is given by

$$d_{suc}=\lambda p_{tx}p_{suc}, \tag{7}$$

where λ denotes the density of transmitters, p_{tx} denotes the transmission probability of a typical transmitter, and p_{suc} denotes the transmission success probability. This metric not only measures the *level of spatial packing* through λp_{tx} but also the *quality of transmissions* through p_{suc}, which captures the interactions (though interference) among spatially distributed nodes.

Other relevant metrics, such as transmission capacity [44] given by $\log(1+t)\lambda p_{tx}p_{suc}$ or throughput density [2] given as $\mathbb{E}[\log(1+\text{SINR})]\lambda p_{tx}$, could be used instead of (7), however we focus on (7) for simplicity.

B. Previous Results

Proposition 1. *(Laplace Transform of Shot-Noise for Non-homogeneous Poisson Field) [3] Let* $\Phi_h = \{X_i, F_i\}$ *be an independently marked non-homogeneous PPP in* $\mathbb{R}^2$ *with spatial density* $h(x)dx$*. Then, the Laplace transform of the associated shot-noise interference* $I_{\Phi_h}(r) = \sum_{X_i:(X_i,F_i)\in\Phi_h} F_i/l(\|X_i - r\|)$ *at location* $r \in \mathbb{R}^2$ *is given by*

$$\mathcal{L}_{I_{\Phi_h}(r)}(s) = \mathbb{E}\left[e^{-sI_{\Phi_h}(r)}\right] = \exp\left\{-\int_{\mathbb{R}^2}\left(1 - \mathcal{L}_F\left(\frac{s}{l(\|x-r\|)}\right)\right)h(x)dx\right\}. \tag{8}$$

In particular, if $F_i \sim F$ *is an exponential random variable with rate* μ*, we have*

$$\mathcal{L}_{I_{\Phi_h}(r)}(s) = \exp\left\{-\int_{\mathbb{R}^2}\frac{h(x)}{1+\frac{\mu}{s}l(\|x-s\|)}dx\right\}. \tag{9}$$

Proposition 2. *(Mean Neighborhood Size) [3] The number of neighbors of a typical node is Poisson with mean*

$$\bar{N}_0^\gamma = \mathbb{E}\left[N_0^\gamma\right] = \mathbb{E}^0\left[\sum_{X_i\in\Phi^\gamma\setminus\{0\}}\mathbf{1}\left\{F_i > \nu l(||X_i||)\right\}\right] = \lambda^\gamma\int_{\mathbb{R}^2}\exp\left\{-\nu\mu l\left(||x||\right)\right\}dx = \frac{2\pi\lambda^\gamma\Gamma(2/\alpha)}{\alpha(\nu\mu)^{2/\alpha}}. \tag{10}$$

Proposition 3. *(Conditional Transmission Probability under CSMA Protocol) [3] For the O-CSMA model given in Section II with qualified transmitter density* λ^γ*, the probability that a qualified node* $x_1 \in \mathbb{R}^2$ *transmits given there is a transmitter* $x_0 \in \mathbb{R}^2$ *with* $|x_1 - x_0| = \tau$ *which transmits (i.e., wins its contention), i.e.,* $P(E_1 = 1|E_0 = 1, \{x_0, x_1\} \subset \Phi^\gamma, |x_1 - x_0| = \tau) \equiv h(\tau, \lambda^\gamma)$*, is*

$$h\left(\tau,\lambda^\gamma\right) = \frac{\frac{2}{b(\tau,\lambda^\gamma)-\bar{N}_0^\gamma}\left(\frac{1-e^{-\bar{N}_0^\gamma}}{\bar{N}_0^\gamma} - \frac{1-e^{-b(\tau,\lambda^\gamma)}}{b(\tau,\lambda^\gamma)}\right)\left(1-e^{-\nu\mu l(\tau)}\right)}{\frac{1-e^{-\bar{N}_0^\gamma}}{\bar{N}_0^\gamma} - e^{-\nu\mu l(\tau)}\left(\frac{1-e^{-\bar{N}_0^\gamma}}{\left(\bar{N}_0^\gamma\right)^2} - \frac{e^{-\bar{N}_0^\gamma}}{\bar{N}_0^\gamma}\right)}, \tag{11}$$

where

$$b\left(\tau,\lambda^\gamma\right) \;=\; 2\bar{N}_0^\gamma - \lambda^\gamma\int_0^\infty\int_0^{2\pi}e^{-\nu\mu\left(l(x)+l\left(\sqrt{\tau^2+x^2-2\tau x\cos\theta}\right)\right)}xd\theta dx. \tag{12}$$

Proposition 4. *(Plancherel-Parseval Theorem) C3.3 of [8] If* σ_1 *and* σ_2 *are square integrable complex functions, i.e.,* $\int_{\mathbb{R}}|\sigma_i(x)|^2 dx < \infty$ *for* $i = 1, 2$*, then*

$$\int_{\mathbb{R}}\sigma_1(x)\sigma_2^*(x)dx = \int_{\mathbb{R}}\hat{\sigma}_1(s)\hat{\sigma}_2^*(s)ds, \tag{13}$$

where $\hat{\sigma_i}(s) = \int_{\mathbb{R}}\sigma_i(x)e^{-2j\pi sx}dx$ *is the Fourier transform of* σ_i *and* σ_i^* *is the complex conjugate of* σ_i*.*

C. O-CSMA

Access Probability of a Typical Transmitter: The access probability is the probability that a typical node transmits. As described earlier, under O-CSMA, only nodes who qualify can contend, so the network after the qualification process is indeed equivalent to a network with node density λ^γ. The channel distribution function of a qualified node, say X_i is given by (4). Let $E_i = \mathbf{1}\{F_{ii} > \gamma, T_i < \min_{j:X_j\in\mathcal{N}_i^\gamma} T_j\}$ be the transmission indicator for $X_i \in \Phi$, i.e., that it qualifies and wins the contentions process in its neighborhood, and $\Phi_M^\gamma = \{X_i \in \Phi | E_i = 1\}$ be the set of active transmitters. We define the transmission probability of the typical node (at the origin) as

$$p_{tx}^{op}(\lambda, \gamma, \nu) = \mathbb{P}^0\left(F_{00} > \gamma, T_0 < \min_{j: X_j \in \mathcal{N}_0^\gamma} T_j\right). \tag{14}$$

Note that the two events in (14) are independent. To compute the probability of the second event, we condition on T_0, i.e.,

$$\mathbb{P}^0\left(T_0 < \min_{j: X_j \in \mathcal{N}_0^\gamma} T_j\right) = \mathbb{E}^0\left[\mathbb{P}^0\left(T_0 < \min_{j: X_j \in \mathcal{N}_0^\gamma} T_j \mid T_0\right)\right]. \tag{15}$$

The conditional probability within the above expectation is the probability that X_0 has no neighboring node whose timer value is less than T_0, i.e.,

$$\mathbb{P}^0\left(\{X_j \in \Phi \backslash \{X_0\} \text{ s.t. } F_{jj} > \gamma, F'_{j0} > \nu l\left(\|X_j - X_0\|\right), j \neq 0, T_j < T_0\} = \emptyset \mid T_0\right). \tag{16}$$

The density measure of such nodes at location $x \in \mathbb{R}^2$ with $F_{jj} = f_1$,$F_{j0} = f_2$ and $T_j = m$ is

$$\Lambda(dm, df_1, df_2, dx) = \mathbf{1}\{m < T_0\} dm \mathbf{1}\{f_1 > \gamma\} G(df_1) \mathbf{1}\{f_2 > \nu l(\|x\|)\} G(df_2) \lambda dx.$$

Thus, the conditional void probability of such nodes, i.e., (16), corresponds to

$$\begin{aligned}\exp\left\{-\int_{\mathbf{R}^2}\int_0^\infty\int_0^\infty\int_0^1 \Lambda(dm, df_1, df_2, dx)\right\} &= \exp\left\{-M_0 p_\gamma \int_{\mathbb{R}^2} 1 - G\left(\nu l\|x\|\right)\lambda dx\right\} \\ &= \exp\left\{-M_0 p_\gamma \bar{N}_0\right\}. \end{aligned} \tag{17}$$

Substituting (17) into (15) gives

$$p_{tx}^{op}(\lambda, \gamma, \nu) = \frac{1 - \exp\left\{-p_\gamma \bar{N}_0\right\}}{\bar{N}_0}. \tag{18}$$

Note that the spatial mean number of contenders for a typical node under O-CSMA is given by $p_\gamma \bar{N}_0$ since individual nodes qualify with probability p_γ. The case with $\gamma = 0$ (or $p_\gamma = 1$) corresponds to the pure CSMA scheme without a qualification step.

Transmission Success Probability of a Typical Receiver: Next, we compute the transmission success probability of a receiver associated with a typical active transmitter. This is equivalent to the success probability of the receiver of transmitter $X_0 = 0$ given $X_0 \in \Phi_M^\gamma$:

$$p_{suc}^{op}(\lambda, \gamma, \nu, t) = \mathbb{P}^0\left(\frac{F_{00}/l(r)}{I_{\Phi_M^\gamma \backslash \{0\}}} > t \mid F_{00} > \gamma\right), \tag{19}$$

where

$$I_{\Phi_M^\gamma \backslash \{0\}} \equiv \sum_{X_j \in \Phi_M^\gamma \backslash \{0\}} F_{j0}/l\left(\|X_j - (0, r)\|\right) \tag{20}$$

is shot noise interference seen at the receiver of the transmitter $X_0 \in \Phi_M^\gamma$. When we refer the shot noise without $\mathbb{P}^0(\cdot)$, we will use $I_{\Phi_M^{\gamma 0} \backslash \{0\}}$ instead of $I_{\Phi_M^\gamma}$ to explicitely denote $X_0 \in \Phi_M^\gamma$. Then, the shot noise of interest can be written as

$$I_{\Phi_M^{\gamma 0}\backslash\{0\}} \equiv \sum_{X_j \in \Phi_M^{\gamma 0}\backslash\{0\}} F_{j0}/l\left(\|X_j - (0,r)\|\right). \tag{21}$$

For notational symplicity let F_γ be a random variable with the distribution function (4) which is indenpendent of F_{00}. Then, (19) can be rewritten as follows by conditioning on F_γ:

$$\mathbb{P}^0\left(F_\gamma > tl(r) I_{\Phi_M^{\gamma}\backslash\{0\}}\right) = \mathbb{E}\left[\mathbb{P}^0\left(F_\gamma > tl(r) I_{\Phi_M^{\gamma}\backslash\{0\}} \mid F_\gamma\right)\right]. \tag{22}$$

Note that it is hard to compute (22) since $\Phi_M^{\gamma 0}\backslash\{0\}$ is a point process induced by the qualification process followed by the CSMA protocol, which has dependency among node locations. It is called as a Matérn CSMA process [3]. Thus, following [3], we approximate the shot noise $I_{\Phi_M^{\gamma 0}\backslash\{0\}}$ with $I_{\Phi_h^{\gamma 0}\backslash\{0\}} = \sum_{X_j \in \Phi_h^{\gamma 0}\backslash\{0\}} F_{j0}/l(||X_j - (0,r)||)$ which is a shot noise seen at the receiver of X_0 in a non-homogeneous PPP Φ_h^γ with density $\lambda^\gamma h(\tau, \lambda^\gamma)$ for $\tau > 0$, where $\lambda^\gamma \equiv p_\gamma \lambda$ and $h(\tau,\lambda)$ is the conditional probability that a CSMA transmitter at distance τ from the origin be active conditioned on an active CSMA transmitter at the origin with the density of nodes being λ^γ, see (11). Since h is a function which converges to 0 as $\tau \to 0$, and converges to p_{tx}^{op} as $\tau \to \infty$, it captures well the modification of the interference due to the presence of the transmitter at the origin. The h for a certain parameter sets is shown in Fig. 9a. Then, we have

$$\mathbb{P}^0\left(F_\gamma > tl(r) I_{\Phi_M^{\gamma}\backslash\{0\}}\right) \approx \mathbb{E}\left[\mathbb{P}^0\left(F_\gamma > tl(r) I_{\Phi_h^{\gamma}\backslash\{0\}} \mid F_\gamma\right)\right]. \tag{23}$$

Let $\xi_h(x)$ be the probability density function of $I_{\Phi_h^{\gamma 0}\backslash\{0\}}$. Then, using an indicator function, we can rewrite right hand side of (23) as

$$\mathbb{E}\left[\int_{-\infty}^{\infty} \xi_h(x) \mathbf{1}\{0 < x < \frac{F_\gamma}{tl(r)}\} dx\right]. \tag{24}$$

Clearly $\mathbf{1}\{0 < x < \frac{F_\gamma}{tl(r)}\}$ is square integrable for $r > 0$ and $t > 0$, and $\xi_h(x)$ is square integrable[6]. We can apply Plancherel-Parseval Theorem in (13) to (24) followed by a change of variables to get

$$\int_{-\infty}^{\infty} \mathcal{L}_{I_{\Phi_h^{\gamma 0}\backslash\{0\}}}\left(2i\pi tl(r)s\right) \frac{\mathcal{L}_{F_\gamma}\left(-2i\pi s\right) - 1}{2\pi i s} ds. \tag{25}$$

Noting that $\mathcal{L}_{F_\gamma}(s) = \frac{\mu}{\mu+s} e^{-s\gamma}$, we get

$$p_{suc}^{op}\left(\lambda,\gamma,\nu,t\right) \approx \int_{-\infty}^{\infty} \mathcal{L}_{I_{\Phi_h^{\gamma 0}\backslash\{0\}}}\left(2i\pi l\left(r\right) ts\right) \frac{\frac{\mu}{\mu - 2i\pi s}\exp\left\{2i\pi s\gamma\right\} - 1}{2i\pi s} ds. \tag{26}$$

The last step is to compute the Laplace transform $\mathcal{L}_{I_{\Phi_h^{\gamma 0}\backslash\{0\}}}(s)$ which is given as

$$\mathcal{L}_{I_{\Phi_h^{\gamma 0}\backslash\{0\}}}(s) = \exp\left\{-\lambda^\gamma \int_0^\infty \int_0^{2\pi} \frac{h\left(\tau,\lambda^\gamma\right)\tau d\theta d\tau}{1 + \mu f\left(\tau, r, \theta\right)/s}\right\}, \tag{27}$$

[6]Note that the pdf of Poisson shot noise interference from Poisson transmitters with finite density is square integrable, see [2]. The existence of a Poisson point process of which shot noise dominates $I_{\Phi_h^{\gamma 0}\backslash\{0\}}$ implies that the pdf of $I_{\Phi_h^{\gamma 0}\backslash\{0\}}$ is square integrable.

where $f(\tau, r, \theta) = l\left(\sqrt{\tau^2 + r^2 - 2\tau r\cos\theta}\right)$. Replacing (27) into (26) gives a numerically computable integral form for the outage probability.

D. QT-CSMA

Access Probability of a Typical Transmitter: Computing the access probability of a typical QT-CSMA node is not much different from that of an O-CSMA node. Under QT-CSMA, a node can transmit if its timer expires first or equivalently it has the highest quantile in its neighborhood. Let E_i be the transmission indicator of node $X_i \in \Phi$, i.e., $E_i = \mathbf{1}\left\{F_{ii} > \gamma, Q_i > \max_{j:X_j \in \mathcal{N}_i^\gamma} Q_j\right\}$. Let $\Phi_M^\gamma = \{X_i \in \Phi \text{ s.t. } E_i = 1\}$ be a thinned version of Φ containing only active transmitters. Then, using a similar technique as above, the access probability of a typical node X_0 at the origin under QT-CSMA is computed as follows:

$$p_{tx}^{qt}(\lambda, \gamma, \nu) = \mathbb{E}^0\left[\frac{p_\gamma}{N_0^\gamma + 1}\right] = \frac{1 - \exp\left\{-p_\gamma \bar{N}_0\right\}}{\bar{N}_0}. \tag{28}$$

Since all Q_is in (28) are uniform random variables, the result is the same as (18).

Transmission Success Probability of a Typical Receiver: Next we compute the transmission success probability of a receiver associated with a typical transmitter X_0 at the origin. To determine the success probability, we need to characterize the fading gain $F_{0,\gamma}^{\max}$ and the interference power that the receiver experiences. We shall explicitly denote the fact that $F_{0,\gamma}^{\max}$ depends on $N_0^\gamma + 1$ by writing $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$ in what follows. The aggregate interference from concurrent active transmitters in $\Phi_M^{\gamma 0} \backslash \{0\}$ to the receiver of X_0 is given by $I_{\Phi_M^{\gamma_0} \backslash \{0\}}$ as (21). Then, the success probability of a typical QT-CSMA receiver is written as

$$p_{suc}^{qt}(\lambda, \gamma, \nu, t) = \mathbb{P}^0(F_{0,\gamma}^{\max}(N_0^\gamma + 1) > tl(r) I_{\Phi_M^\gamma \backslash \{0\}}). \tag{29}$$

Unlike the case in (22), the $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$ is no longer independent of $I_{\Phi_M^{\gamma_0} \backslash \{0\}}$. To see this intuitively, consider two extreme cases. First, suppose $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$ has a very small value, say ϵ, then, this implies the channel gains of X_0's neighbors are concentrated within the small interval $[0, \epsilon]$; so, the neighbors of X_0's neighbors are not likely to defer their transmissions, which in turn means X_0's receiver would experience somewhat stronger interference. By contrast, if $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$ has a large value, say ω, then, the fading gains of X_0's neighbors would be distributed on $[0, \omega]$, which is more likely to cause their neighbors to defer. This on average makes the interference level seen at the receiver smaller than in the previous case.

That is, $I_{\Phi_M^{\gamma_0} \backslash \{0\}}$ depends on both N_0^γ and $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$. By conditioning on N_0^γ and $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$, (29) can be written as

$$\mathbb{E}^0\left[\mathbb{P}^0\left(F_{0,\gamma}^{\max}(N_0^\gamma + 1) > tl(r) I_{\Phi_M^\gamma \backslash \{0\}} \mid N_0^\gamma, F_{0,\gamma}^{\max}(N_0^\gamma + 1)\right)\right]. \tag{30}$$

As in (22), we approximate $I_{\Phi_M^{\gamma_0} \backslash \{0\}}$ for a given $N_0^\gamma = n$ and $F_{0,\gamma}^{\max}(N_0^\gamma + 1) = x$ by a non-homogeneous Poisson point process Φ_u^γ with density $\lambda^\gamma u(n, x, \tau, \lambda, \gamma)$, where $u(n, x, \tau, \lambda, \gamma)$ is the conditional probability that a node y_1 transmits conditioned on following facts: 1) y_0 transmits, i.e., $E_0 = 1$, 2) $N_0^\gamma = n$, 3) $F_{0,\gamma}^{\max}(N_0^\gamma + 1) = x$ or equivalently y_0's timer value T_0 is given as $t_0 = 1 - G_\gamma(x)$, 4) both y_0 and

y_1 belong to Φ^γ, and 5) y_1 is τ meter away from y_0. This can be written as

$$u(n,x,\tau,\lambda,\gamma) = \mathbb{P}(E_1 = 1 | E_0 = 1, N_0^\gamma = n, F_{0,\gamma}^{\max}(N_0^\gamma + 1) = x, \{y_0, y_1\} \subset \Phi^\gamma, |y_0 - y_1| = \tau). \quad (31)$$

Using the fact that $1 - G_\gamma(x)$ is one-to-one mapping from $[\gamma, \infty]$ to $[0,1]$, we can rewrite (31) as

$$u(n,x,\tau,\lambda,\gamma) = \mathbb{P}(E_1 = 1 | E_0 = 1, N_0^\gamma = n, T_0 = t_0, \{y_0, y_1\} \subset \Phi^\gamma, |y_0 - y_1| = \tau). \quad (32)$$

Note that the probability (32) is a function of n, t_0, τ and λ^γ; so it is convenient to use the function u' such that

$$u(n,x,\tau,\lambda,\gamma) = u'(n, 1 - G_\gamma(x), \tau, \lambda^\gamma).$$

It is shown in Appendix A that this function is given by

$$u'(n,t_0,\tau,\lambda) = \frac{\bar{N}_0 G(\nu l(\tau))}{n + (\bar{N}_0 - n) G(\nu l(\tau))} \left(\frac{(1 - e^{-t_0 \bar{N}_0 (1-p_s)})}{\bar{N}_0 (1 - p_s)} + \right.$$
$$\left. + (1 - t_0) e^{-\bar{N}_0(1-p_s)} \sum_{k=0}^{n} \frac{k!}{\eta^{k+1}} \left(1 - e^{-\eta} \sum_{j=0}^{k} \frac{\eta^j}{j!} \right) \binom{n}{k} p_s^k (1-p_s)^{n-k} \right), \quad (33)$$

with $p_s = p_s(\tau,\lambda) = 2 - \frac{b(\tau,\lambda)}{\bar{N}_0}$, and $\eta = \bar{N}_0(1-p_s)(t_0 - 1)$. Then, (30) can be approximated with

$$\mathbb{E}^0 \left[\mathbb{P}^0 \left(F_{0,\gamma}^{\max}(N_0^\gamma + 1) > tl(r) I_{\Phi_u^\gamma \setminus \{0\}} \mid N_0^\gamma, F_{0,\gamma}^{\max}(N_0^\gamma + 1) \right) \right]. \quad (34)$$

Let $\xi_u^{n,x}$ be the conditional pdf of $I_{\Phi_u^{\gamma^0} \setminus \{0\}}$ given $N_0^\gamma = n$ and $F_0^{\max}(N_0^\gamma + 1) = x$. Then, (34) can be rewritten as

$$\mathbb{E}^0 \left[\int_{-\infty}^{\infty} \xi_u^{N_0^\gamma, F_{0,\gamma}^{\max}(N_0^\gamma+1)}(y) \mathbf{1} \left\{ 0 \le y \le \frac{F_{0,\gamma}^{\max}(N_0^\gamma + 1)}{tl(r)} \right\} dy \right], \quad (35)$$

where $\mathbf{1} \left\{ 0 \le y \le \frac{F_{0,\gamma}^{\max}(N_0^\gamma + 1)}{tl(r)} \right\}$ and $\xi_u^{n,x}$ are both square integrable, see [2]. Applying the Plancherel-Parseval Theorem in (13) and performing the change of variables gives

$$p_{suc}^{qt}(\lambda,\gamma,\nu,t) \approx \mathbb{E}^0 \left[\int_{-\infty}^{\infty} \mathcal{L}_{I_{\Phi_u^\gamma \setminus \{0\}}^{N_0^\gamma, F_{0,\gamma}^{\max}(N_0^\gamma+1)}} (2i\pi l(r) ts) \frac{\exp\left\{ 2i\pi s F_{0,\gamma}^{\max}(N_0^\gamma + 1) \right\} - 1}{2i\pi s} ds \right]. \quad (36)$$

Note that the expectation in (36) is with respect to N_0^γ and $F_{0,\gamma}^{\max}(N_0^\gamma + 1)$, and $I_{\Phi_u^{\gamma^0} \setminus \{0\}}^{n,x}$ is a random variable with cdf $\mathbb{P}^0(I_{\Phi_u^\gamma \setminus \{0\}} < z | N_0^\gamma = n, F_0^{\max}(N_0^\gamma + 1) = x)$. We have

$$\mathcal{L}_{I_{\Phi_u^{\gamma^0} \setminus \{0\}}^{n,x}}(s) = \exp\left\{ -\lambda^\gamma \int_0^\infty \int_0^{2\pi} \frac{u'(n, 1 - G_\gamma(x), \tau, \lambda^\gamma) \tau d\theta d\tau}{1 + \mu f(\tau, r, \theta)/s} \right\}. \quad (37)$$

Replacing (37) into (36) gives the numerically computable approximation of p_{suc}^{qt}.

IV. Spatial Reuse

In this section, we compare the spatial reuse achieved by O-CSMA vs QT-CSMA in three different node density regimes. To better understand the results or the behavior of protocols as a function of λ, γ, and ν, we first study how transmission probability and success probability change as the functions of the parameters, and then we compare the performance of O-CSMA and QT-CSMA. A brief performance comparison between O-ALOHA and O-CSMA follows.

A. System Behavior in Function of System Parameters

Density of Active Transmitters λp_{tx}: In Fig 2, we show the density of active transmitters λp_{tx} as a function λ. As λ increases, a higher number of active transmitters is achieved, which saturates to a value we will call the asymptotic density of active transmitters.

Definition 5. (Asymptotic density of active transmitters) For a given carrier sensing threshold ν, the asymptotic density of active transmitters $\lambda_{dens}(\nu)$ is defined as

$$\lambda_{dens}(\nu) \equiv \lim_{\lambda \to \infty} \lambda p_{tx}^{op}(\lambda, \gamma, \nu) = \lim_{\lambda \to \infty} \lambda p_{tx}^{qt}(\lambda, \gamma, \nu). \tag{38}$$

Note that $\lambda_{dens}(\nu)$ is not the function of γ, since numerator $\exp\{-p_\gamma \bar{N}_0\}$ in $p_{tx}^{op/qt}(\lambda, \gamma, \nu)$ vanishes as $\lambda \to \infty$, see (18) and (28). It is easy to show that $\lambda_{dens}(\nu) = 1/\hat{N}_0$, where $\hat{N}_0 = \bar{N}_0^\gamma/\lambda^\gamma = \mathbb{E}[\int_{\mathbb{R}^2} \mathbf{1}\{F' > \nu l(\|x\|)\}\, dx]$ is the *mean neighborhood area* of a typical transmitter. Note that since each active transmitter occupies the area of average size $\hat{N}_0$, intuitively, we can have at most $\frac{1}{\hat{N}_0}$ active transmitters per unit space in the asymptotically dense network. Note that both O-CSMA and QT-CSMA have the same asymptotic density of transmitters $\lambda_{dens}(\nu)$ due to the transmitter selection process of the CSMA protocol.

As γ increases, the density of qualified transmitters, λp_γ, reduces, which accordingly decreases λp_{tx}, but the limiting value $\lambda_{dens}(\nu)$ is not affected. As ν increases, the mean neighborhood area $\hat{N}_0$ gets smaller which allows a higher density of active transmitters, and accordingly $\lambda_{dens}(\nu)$ increases as a function of ν.

Success Probability of O-CSMA: Fig. 3a shows the success probability $p_{suc}^{op}(\lambda, \gamma, \nu, t)$ as a function of λ for various γ and ν values. The general behavior of $p_{suc}^{op}(\lambda, \gamma, \nu, t)$ is as follows. As γ increases, the signal quality at receivers improves and at the same time the density of active transmitters goes down, which results in reduced interference at the receiver. Thus, increasing γ increases SINR at receivers, and thus increases success probability. If ν increases, the mean neighborhood area goes down resulting in a higher number of active transmitters, which accordingly generate a stronger aggregate interference. Thus both the received SINR and success probability are decreased. Regarding the behavior of $p_{suc}^{op}(\lambda, \gamma, \nu, t)$ as a function of λ, we have the following proposition.

Proposition 6. *If $\lambda \to \infty$ while $\nu, \gamma < \infty$ are fixed, we have that*

$$\lim_{\lambda \to \infty} p_{suc}^{op}(\lambda, \gamma, \nu, t) = \lim_{\lambda \to \infty} \mathbb{P}^0 \left(F_\gamma > tl(r) I_{\Phi_M^\gamma \setminus \{0\}} \right) < 1. \tag{39}$$

This is because F_γ is exponentially distributed with an infinite support and $I_{\Phi_M^{\gamma 0} \setminus \{0\}}$ converges in distribution to a random variable $I_{\Phi_M^{dens0} \setminus \{0\}} \equiv \sum_{X_i \in \Phi_M^{dens0} \setminus \{0\}} F_{i0}/l(|X_i|)$, where Φ_M^{dens0} is a Matérn CSMA point process with a density $\lambda_{dens}(\nu)$ given an active transmitter at the origin, see Appendix B. Since both random variables have infinite support in $\mathbb{R}_+$, $\mathbb{P}^0 \left(F_\gamma > tl(r) I_{\Phi_M^\gamma \setminus \{0\}} \right)$ converges to a positive value between 0 and 1. It is not easy to find the limit since this would require characterizing $I_{\Phi_M^{dens0} \setminus \{0\}}$.

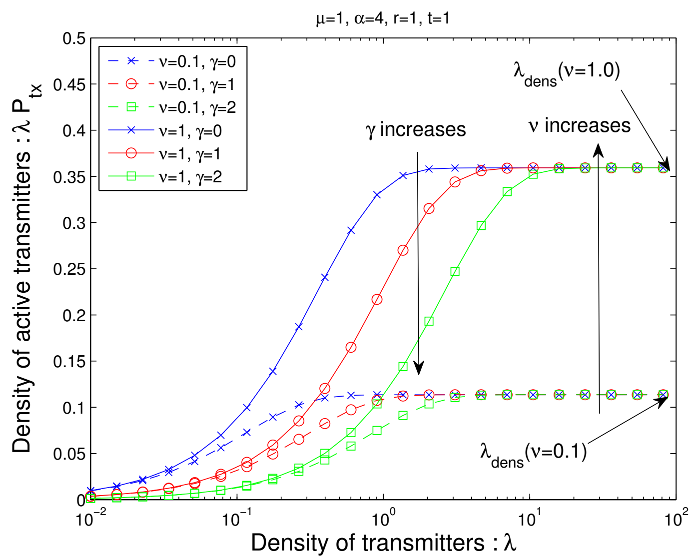


Figure 2: The density of active transmitters for O/QT-CSMA increases and saturates as λ increases due to the carrier sensing in CSMA protocol. Increasing the qualification threshold γ reduces the density of qualified transmitters without affecting the asymptotic density of active transmitters $\lambda_{dens}(\nu)$; so the effect is a shift of the curves to the right hand side. Increasing carrier sensing threshold ν increases $\lambda_{dens}(\nu)$ since it makes the mean size of a typical transmitter's neighborhood smaller.

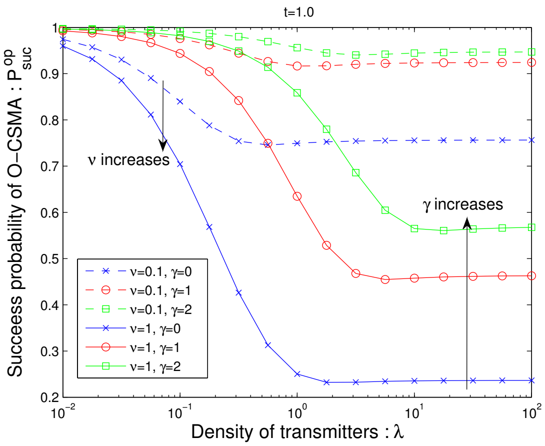
Success Probability of QT-CSMA: Fig. 3b shows the success probability $p^{qt}_{suc}(\lambda, \gamma, \nu, t)$ as a function of λ for various γ and ν values. The general behavior of $p^{qt}_{suc}(\lambda, \gamma, \nu, t)$ is as follows. As γ increases, the interference seen at the receiver decreases due to the reduced density of active transmitters. However it is not clear how the received signal strength would change. Indeed increasing γ, should shift F_γ to the right hand side (improving the signal strength) but, at the same time, it decreases the size of neighborhood, thus reducing the opportunistic gain from picking the node with the best channel. Fig. 3b suggests that the positive effect is larger than the negative effect. And, as ν increases, $p^{qt}_{suc}(\lambda, \gamma, \nu, t)$ decreases due to the increased interference. One thing to note is that if the density λ becomes large enough, then, the success probability increases and eventually converges to 1 due to the increasing opportunistic gain; this is summarized in the following proposition.

Proposition 7. *If $\lambda \to \infty$ while $\nu, \gamma < \infty$ are kept fixed, we have*

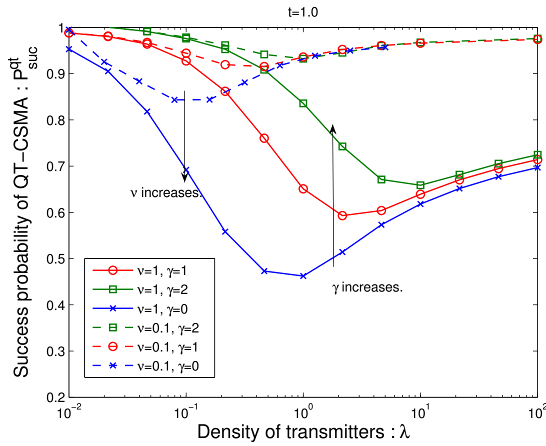
$$\lim_{\lambda\to\infty} p^{qt}_{suc}(\lambda, \gamma, \nu, t) = \lim_{\lambda\to\infty} \mathbb{P}^0 \left(F^{\max}_\gamma (N^\gamma_0 + 1) > tl(r) I_{\Phi^\gamma_M \setminus \{0\}} \right) = 1. \tag{40}$$

This result can be intuitively understood as follows. As λ increases, N^γ_0 and $F^{\max}_\gamma(N^\gamma_0 + 1)$ increase (meaning $\lim_{\lambda\to\infty} \mathbb{P}(N^\gamma_0 > x) = 1$ and $\lim_{\lambda\to\infty} \mathbb{P}(F^{\max}_\gamma(N^\gamma_0 + 1) > x) = 1$ for all fixed $x > 0$), and $I_{\Phi^{\gamma_0}_M \setminus \{0\}}$ converges in distribution to a random variable $I_{\Phi^{dens0}_M \setminus \{0\}}$ defined in Proposition 6. The success probability of O-CSMA and QT-CSMA are compared in the following proposition.

Proposition 8. *Under the same parameter set t, γ, ν, and λ, the success probability of QT-CSMA is always larger than O-CSMA, i.e., $p^{qt}_{suc}(\lambda, \gamma, \nu, t) \geq p^{op}_{suc}(\lambda, \gamma, \nu, t)$.*



(a) The success probability of O-CSMA decreases as λ increases, but converges to a value between 0 and 1 since interference $I_{\Phi_M^{\gamma 0}\setminus\{0\}}$ converges to $I_{\Phi_M^{dens0}\setminus\{0\}}$ in distribution. If the qualification threshold γ increases, it increases F_γ so the success probability increases, and the limiting value $\lim_{\lambda\to\infty} p_{suc}^{op}(\lambda,\gamma,\nu,t)$ also increases. While, if carrier sensing threshold ν increases, it increases the density of active transmitters, which accordingly increases interference, which deteriorates the success probability.



(b) As λ increases, the success probability of QT-CSMA decreases at first, but bounces and converges to 1 due to the increasing opportunistic gain. As the qualification threshold γ increases, the success probability increases due to increased opportunistic gain. While, if carrier sensing range ν increases, the success probability decreases due to increased aggregate interference power.

Figure 3: The success probability versus the density of transmitters for various ν and γ.

This follows from a stochastic ordering relation : $F_\gamma^{\max} \geq^{st} F_\gamma$*, see (5). Note that this implies that the density of successful transmissions of QT-CSMA is always higher than that of O-CSMA, i.e.,* $d_{suc}^{qt}(\lambda,\gamma,\nu,t) \geq d_{suc}^{op}(\lambda,\gamma,\nu,t)$ *for a given parameter set* t,γ,ν *and* λ*.*

Remark 9. The above observations suggest that the effects of adjusting γ and ν are similar in that both control the amount of interference in the network versus the opportunistic gain which are achieved. However, this does not imply that O-CSMA can optimize its performance by tweaking only one of them while fixing the other, but interestingly this seems to work for QT-CSMA. In the following sections, we will further explore this idea of reducing the number of parameters for QT-CSMA.

B. Performance Comparison of O-CSMA and QT-CSMA

We consider networks in three different density regimes : a network with an intermediate density, an asymptotically dense network, and an asymptotically sparse network. By asymptotically dense (sparse) networks, we mean networks whose node density λ keeps increasing to ∞ (decreasing to 0). Dense and sparse, networks are particularly interesting since the former gives maximum performance limits for O-CSMA and QT-CSMA networks under a given parameter set, and the latter allows us to evaluate the performance of an individual node because it is a regime where interactions with other nodes vanish. In addition to the importance of the two regimes, we have p_{suc}^{op} and p_{suc}^{qt} converging to 1 for an appropriately scaled or chosen γ in these two regimes, which allows a simple analysis. While, in the intermediate density regime, p_{suc}^{op} and p_{suc}^{qt} are strictly less than 1, so we can only compare their performance through numerical computations.

Networks in an intermediate density regime: We evaluate the performance of a network in an intermediate density regime for various γ and ν values. Note that it is no surprise to find that QT-CSMA always does better than O-CSMA under the same parameter set, see Proposition 8. Thus, we focus instead on the comparison between QT_0-CSMA (QT-CSMA with $\gamma = 0$) and O-CSMA.

Case $\nu = 1$**:** Fig.4a shows the density of successful transmissions for QT_0-CSMA and O-CSMA as a function of λ for various values of γ and for $\nu = 1$, $t = 1$, and $\mu = 1$. For comparison, ALOHA, CSMA and O-ALOHA are also plotted using our model. The figure confirms the behavior of ALOHA for increasing node density. Unlike ALOHA, the density of successful transmissions of CSMA does not converge to 0 as λ increase due to carrier sensing and controlled network interference. To take advantage of channel variations from many users, ALOHA can use channel threshold method, which we call O-ALOHA.

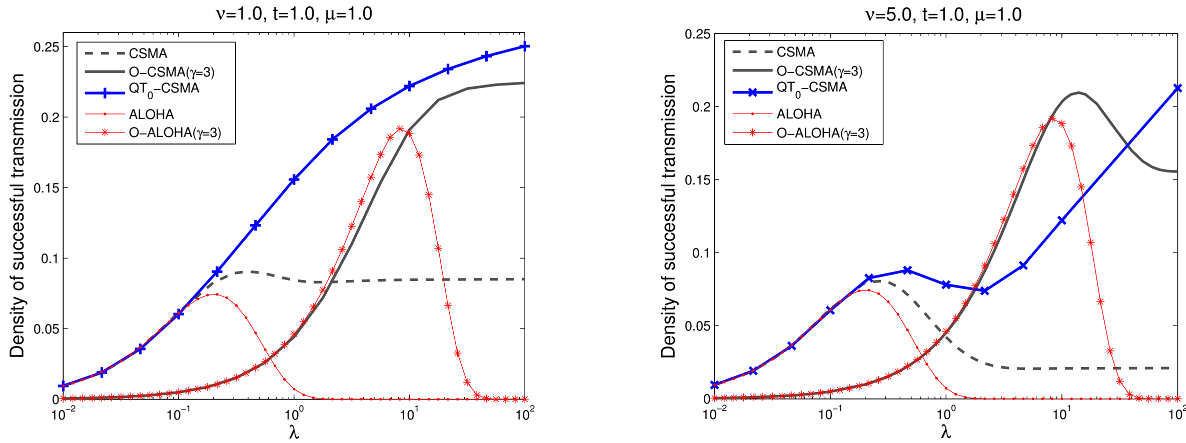
O-CSMA with similar threshold mechanism works as follows. As λ gets larger, $d_{suc}^{op}(\gamma, \lambda)$ increases as a result of the increasing density of active transmitters; however they converge to fixed values since both the density of active transmitters and success probability converge. If λ gets large, d_{suc}^{op} increases and converges to a value less than λ_{dens}. While if λ is small, d_{suc}^{op} decreases as γ increases because the losses coming from the decreased density of active transmitters are not compensated by the gain from the increased quality of transmissions.

The performance of QT_0-CSMA is also shown. Interestingly, the performance of QT_0-CSMA seems to be better than O-CSMA for all λ values. This proves that quantile scheduling without qualification can fully take advantage of opportunistic gains for the given parameter set. The trends are similar when $\nu = 0.1$, which are not shown here.

Case $\nu = 5$ **:** Fig. 4b shows three interesting phenomena. First, as λ increases, the density of successful transmissions of O-CSMA peaks and then decreases to converge to its limiting value. The peak happens since $\lambda p_{tx}^{op}(\lambda)$ converges earlier than $p_{suc}^{op}(\lambda)$. Roughly speaking, if $d_{suc}^{op}(\lambda)$ hits its peak, $\lambda p_{tx}^{op}(\lambda)$ is very close to its limiting value and from that point it increases very slowly, while $p_{suc}^{op}(\lambda)$ keeps decreasing and converges at a larger λ value. Note that this phenomenon implies that the interference suppression from carrier sensing capability is not working well in a high density network. In other words, the carrier sensing threshold $\nu = 5$ is too large (or neighborhood of a node to narrowly defined) making the system less robust to changes in node density. The monotonic behavior shown in Fig. 4a seems to be more desirable since it is predictable for system designer/operators.

The second interesting phenomenon is that the density of successful transmissions for QT_0-CSMA is worse than that of O-CSMA when $2 \lessapprox \lambda \lessapprox 3 \times 10$. This is again due to ν too large. If ν is too large, the size of neighborhood become too small for QT_0-CSMA to take advantage of opportunistic gains and the small neighborhood induces a dense packing which accordingly results in strong interference. Considering ν as a parameter controlling both interference and opportunistic gain, the smaller values are desirable.

Third, the density of successful transmissions for QT_0-CSMA keeps increasing as λ increases. This happens because the size of neighborhood keeps increasing as λ increases. Indeed, the limiting value is even higher than the maximum value which O-CSMA can achieve. This will be further explored later when we consider a dense network.



(a) For $\nu = 1$, the density of successful transmission of QT_0-CSMA is uniformly higher than that of O-CSMA for all node densities $\lambda > 0$ and qualification thresholds $\gamma > 0$. Appropriately chosen ν (or neighborhood size) both increases opportunistic gain of QT_0-CSMA and controls the amount of aggregate interference effectively even for large λ.

(b) If ν is set to too large (small neighborhood), the CSMA protocol allows too many active transmitters which generate too strong aggregate interference for increasing λ. Furthermore due to the small neighborhood size, QT_0-CSMA cannot fully take advantage of opportunism. While O-CSMA can increase γ to select only nodes with high channel gains. This corresponds to the case where ν is inappropriately chosen for QT_0-CSMA.

Figure 4: The density of successful transmissions in a network with intermediate density

Remark 10. The above results show that performance is highly dependent on the selected parameters. For O-CSMA, both ν and γ should be chosen appropriately. However, for QT_0-CSMA, only ν needs to be selected, which is the key advantage of using QT_0-CSMA. As shown above if ν is properly chosen, QT_0-CSMA provides a more robust[7] performance than O-CSMA. Considering λ is usually an uncontrollable parameter, this kind of robust property is very desirable.

Asymptotically Dense Networks: The maximum achievable performance of O/QT-CSMA is obtained when $\lambda \to \infty$ since this is the regime where the space can be packed with a maximum number of active transmitters. To study this regime, we fix ν for both O-CSMA and QT-CSMA and study how the selection of γ affects the density of successful transmissions. Intuitively, for both O-CSMA and QT-CSMA to achieve high performance in dense networks, one should select γ to take advantage of nodes' high channel gains. Recall that increasing γ makes the received signal power stronger, which results in higher success probability, but at the same time it makes it harder for nodes to qualify. Thus, the question is how to scale γ as a function of λ.

In the sequel, we will show that γ should be increased no faster than as a logarithmic function of λ to achieve maximal performance, otherwise the network degenerates and behaves like a sparse network. We will show that *a "phase transition"* occur for the density of successful transmissions depending on the scaling speed of γ. To that end, we consider following fact.

[7]By "robust" we mean QT_0-CSMA gives better performance than O-CSMA in all λ values.

Proposition 11. *If both $\lambda \to \infty$ and $\gamma \to \infty$, then,*

$$\lim_{\lambda,\gamma\to\infty} p_{suc}^{op}(\lambda,\gamma,\nu,t) = 1. \tag{41}$$

This is because, for O-CSMA, F_γ keeps increasing as γ increases (meaning that $\lim_{\gamma\to\infty} P(F_\gamma > x) = 1$ for any fixed $x > 0$) while $I_{\Phi_M^{\gamma 0}\setminus\{0\}}$ converges in distribution to a limiting random variable $I_{\Phi_M^{dens0}\setminus\{0\}}$, see Appendix B.

Following theorem provides our main result on the performance O-CSMA in the asymptotically dense regime.

Theorem 12. *For $\lambda \to \infty$ and fixed $\nu > 0$, the asymptotic density of successful transmissions is upper bounded by the asymptotic density of transmitters, i.e.,*

$$\lim_{\lambda\to\infty} d_{suc}^{op}(\lambda,\gamma,\nu,t) \leq \lambda_{dens}(\nu), \tag{42}$$

where equality holds when $\gamma(\lambda) = c\log(\lambda/\lambda_q)$ with constants $c < \frac{1}{\mu}$ and $\lambda_q > 0$.

Proof: We have a few cases to consider depending on the scaling of γ. First, if γ is a fixed constant, we have that

$$\lim_{\lambda\to\infty} \lambda p_{tx}^{op} = \lim_{\lambda\to\infty} \frac{1-\exp\{-e^{-\mu\gamma}\lambda\hat{N}_0\}}{\hat{N}_0} = \lambda_{dens}(\nu) \tag{43}$$

and $0 < \lim_{\lambda\to\infty} p_{suc}^{op}(\lambda,\gamma,\nu,t) < 1$ by Proposition 6. Thus, we have $\lim_{\lambda\to\infty} d_{suc}^{op}(\lambda,\gamma,\nu,t) < \lambda_{dens}(\nu)$. While if $\gamma = c\log(\lambda/\lambda_q)$ for $c > 0$ and $\lambda_q > 0$, we have

$$\lim_{\lambda\to\infty} \lambda p_{tx}^{op} = \lim_{\lambda\to\infty} \frac{1-\exp\{-\lambda^{1-\mu c}\lambda_q^{\mu c}\hat{N}_0\}}{\hat{N}_0} = \begin{cases} \lambda_{dens} & \text{if } c < \frac{1}{\mu} \\ \lambda_{dens}(1-\exp\{-\lambda_q\hat{N}_0\}) & \text{if } c = \frac{1}{\mu} \\ 0 & \text{if } \frac{1}{\mu} < c, \end{cases} \tag{44}$$

and $\lim_{\lambda\to\infty} p_{suc}^{op}(\lambda,\gamma,\nu,t) = 1$ by Proposition 11. This completes the proof. ∎

Theorem 12 says that $\gamma(\lambda)$ needs to be scaled no faster than as a logarithmic function of λ, specifically $\gamma(\lambda) = c\log(\lambda/\lambda_q)$, for an O-CSMA network to achieve its maximum asymptotic density of successful transmissions. In doing so, a logarithmic increase is the fastest, but increasing speed should not be too fast. Precisely, it should be slow enough to have a sufficient number of contending nodes so as to increase spatial reuse. More specifically, we see a sharp performance change or phase transition phenomenon depending on the value of speed parameter c as shown in (44), see Fig.5. The phase transition phenomenon can be explained by observing how the density of qualified transmitters

$$\lambda p_{\gamma(\lambda)} = \lambda e^{-\mu\gamma(\lambda)} = \lambda^{1-\mu c}\lambda_q^{\mu c} \tag{45}$$

behaves as a function c when $\lambda \to \infty$. Depending on the value of c we have following three cases.

- If $c < \frac{1}{\mu}$, then we have $\lim_{\lambda\to\infty} \lambda p_{\gamma(\lambda)} = \infty$, which implies there exists enough qualified contenders to achieve high spatial reuse.
- If $c = \frac{1}{\mu}$, then the we have $\lim_{\lambda\to\infty} \lambda p_{\gamma(\lambda)} = \lambda_q$. The constant density implies that the space can not be fully reused due to a lack of qualified nodes to compete and fill the space.

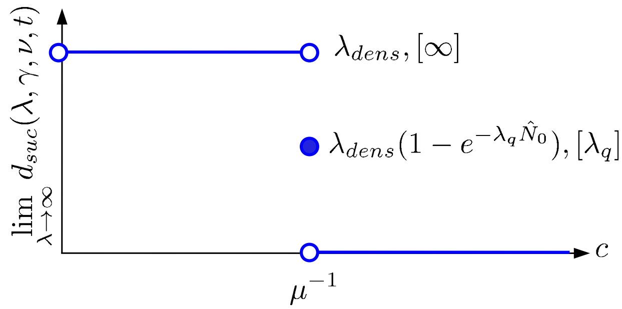


Figure 5: The asymptotic density of successful transmissions for O/QT-CSMA exhibits a phase transition phenomenon depending on the value of scaling speed constant c when qualification threshold γ is selected as the logarithm function of node density λ: $\gamma = c\log(\lambda/\lambda_q)$ with $c > 0$ and $a > 0$. The quantities in the square bracket denote the associated density of qualified nodes $\lim_{\lambda\to\infty} \lambda p_{\gamma(\lambda)}$ which are always larger than their associated asymptotic density of successful transmissions.

- If $c > \frac{1}{\mu}$, then we have $\lim_{\lambda\to\infty} \lambda p_{\gamma(\lambda)} = 0$ because of γ increasing too fast compared to λ. This means that the density of qualified nodes decreases as λ increases.

Note that the gap between λ_{dens} and $\lambda_{dens}(1-\exp\{-\lambda_q \hat{N}_0\})$ in (44) can be made arbitrarily small by selecting λ_q large.

The asymptotic performance of QT-CSMA can also be analyzed in a similar way. Theorem 12 implies that if γ is scaled as a logarithmic function of λ, QT-CSMA will also experience the same phase transition phenomenon. While QT-CSMA can achieve $\lambda_{dens}(\nu)$ for any non-negative constant γ.

From the above discussion, both O and QT-CSMA with $\gamma = c\log(\lambda/\lambda_q)$ with $0 < c < \mu^{-1}$ and QT-CSMA with $\gamma = c\log(\lambda/\lambda_q)$ with $0 \le c < \mu^{-1}$ can achieve *the same maximum* asymptotic performance $\lambda_{dens}(\nu)$ in dense networks. However, *for a given parameter set (t,γ,ν,λ), QT-CSMA is always better than or equal to O-CSMA.*

QT_0-CSMA's performance gain compared to O-CSMA is given in following corollary.

Corollary 13. *For a given $\nu > 0$, if γ is scaled like $\gamma(\lambda) = c\log(\lambda/\lambda_q)$, then, the performance gain of QT_0-CSMA to O-CSMA is given by*

$$\lim_{\lambda\to\infty} \frac{d^{qt}_{suc}(\lambda,0,\nu,t)}{d^{op}_{suc}(\lambda,\gamma,\nu,t)} = \lim_{\lambda\to 0} \frac{p^{qt}_{tx}}{p^{op}_{tx}} = \begin{cases} 1 & \text{if } c < \frac{1}{\mu} \\ \frac{1}{1-e^{-\lambda_q \hat{N}_0}} & \text{if } c = \frac{1}{\mu} \\ \infty & \text{if } \frac{1}{\mu} < c. \end{cases} \tag{46}$$

Asymptotically Sparse Networks: In this section, we consider asymptotically sparse networks, where $\lambda \to 0$. Note that the density of successful transmissions in this case goes to 0, but the comparison (performance ratio) is still meaningful since it is equivalent to comparing the performance of an individual transmitter-receiver pair experiencing no interference. For a fixed $\gamma > 0$, the O-CSMA and QT-CSMA have the same asymptotic performance: $\lim_{\lambda\to 0} \frac{d^{qt}_{suc}(\lambda,\gamma,\nu,t)}{d^{op}_{suc}(\lambda,\gamma,\nu,t)} = 1$ since $\lim_{\lambda\to 0} \frac{p^{qt}_{suc}(\lambda,\gamma,\nu,t)}{p^{op}_{suc}(\lambda,\gamma,\nu,t)} = 1$. Whereas QT_0-CSMA has an exponential gain (as a function of γ) versus O-CSMA. This is captured in the next result.

Theorem 14. *For $\lambda \to 0$ and $\nu > 0$, the asymptotic performance gain of QT$_0$-CSMA to O-CSMA with a fixed $\gamma \geq 0$ is given by $e^{\mu\gamma}$.*

Proof: Using $\lim_{\lambda\to 0} \frac{p^{qt}_{suc}(\lambda,0,\nu,t)}{p^{op}_{suc}(\lambda,\gamma,\nu,t)} = 1$, it is straightforward to compute the gain. Applying L'Hopital's rule, we have

$$\lim_{\lambda\to 0} \frac{d^{qt}_{suc}(\lambda,0,\nu,t)}{d^{op}_{suc}(\lambda,\gamma,\nu,t)} = \lim_{\lambda\to 0} \frac{p^{qt}_{tx}}{p^{op}_{tx}} = \lim_{\lambda\to 0} \frac{1-\exp\{-\lambda\hat{N}_0\}}{1-\exp\{-e^{-\mu\gamma}\lambda\hat{N}_0\}} = \lim_{\lambda\to 0} \frac{\exp\{-\lambda\hat{N}_0(1-e^{-\mu\gamma})\}}{e^{-\mu\gamma}} = e^{\mu\gamma}. \tag{47}$$

This is an expected result since O-CSMA nodes qualify with probability $p_\gamma = e^{-\mu\gamma}$ and qualified nodes in the sparse network will transmit with almost no contention. This shows that in sparse networks O-CSMA should select $\gamma = 0$ to get the same performance as QT$_0$-CSMA. ■

Note that in the above two cases (asymptotically dense and sparse networks) *it turned out that O-CSMA needs to adapt its γ value as a function of node density λ to maximize its performance.* This is a big disadvantage for O-CSMA versus QT$_0$-CSMA since it is hard to estimate λ in practice.

C. Performance Comparison of O-CSMA and O-ALOHA

In this section, we compare the asymptotic performance of O-CSMA and O-ALOHA in [2], [43] for which the density of successful transmissions is given by $d^{oA}_{suc} = \lambda p^{oA}_{tx} p^{oA}_{suc}$, where p^{oA}_{tx} is the transmission probability of each node and p^{oA}_{suc} is the success probability of a typical node. By selecting $p^{oA}_{tx} = p_\gamma$, we can make the density of active transmitters in O-ALOHA network be equal to the density of qualified transmitters in O-CSMA. We have following result for the performance of O-CSMA and O-ALOHA.

Theorem 15. *Under the above setting $p^{oA}_{tx} = p_\gamma$ and $\gamma = c\log(\lambda/\lambda_q)$ with $\lambda_q > 0$ and $c > 0$, we have following asymptotic performance ratio depending on the value of c:*

$$\lim_{\lambda\to\infty} \frac{d^{op}_{suc}}{d^{oA}_{suc}} = \lim_{\lambda\to\infty} \frac{\lambda p^{op}_{tx} p^{op}_{suc}}{\lambda p_\gamma p^{oA}_{suc}(\lambda,\gamma)} = \begin{cases} \infty & \text{if } c < \frac{1}{\mu} \\ \frac{\lambda_{csma}(\nu)(1-\exp\{-\lambda_q\hat{N}_0\})}{\lambda_q p^{oA}_{suc}(\lambda,\gamma)} & \text{if } c = \frac{1}{\mu} \\ 1 & \text{if } c > \frac{1}{\mu}. \end{cases} \tag{48}$$

where $p^{oA}_{suc}(\lambda,\gamma) = \int_{-\infty}^{\infty} \mathcal{L}_I(2i\pi l(r)ts)\frac{\mathcal{L}_{F_\gamma}(-2i\pi s)-1}{2i\pi s}ds$ *with* $\mathcal{L}_I(s) = \exp\left\{-\lambda p_\gamma 2\pi \int_0^\infty \frac{\tau}{1+\frac{\mu\tau^\alpha}{s}}d\tau\right\}$ *and* $\mathcal{L}_{F_\gamma}(s) = e^{-s\gamma}\frac{\mu}{\mu+s}$, *see [2].*

Proof: Depending on the value of c, we have three cases. First, if $c < \frac{1}{\mu}$, the density of transmitters for O-ALOHA keeps increasing as $\lambda \to \infty$, which makes a typical receiver using O-ALOHA experience a success probability of 0 due to increasing interference which is unbounded. While d^{op}_{suc} converges to a non-zero value, so we have infinite gain in this case. Second, if $c = \frac{1}{\mu}$, the density of qualified transmitters for both O-CSMA and O-ALOHA converge to finite numbers, so the amount of interference is limited, accordingly this results in non-zero performance. Third, if $c > \frac{1}{\mu}$, the densities of qualified transmitters in O-CSMA and transmitters in O-ALOHA decrease to 0 and both p^{op}_{suc} and $p^{oA}_{suc} \to 1$ as $\lambda \to \infty$. Applying L'Hopital's rule, we have ratio 1. ■

V. Spatial Fairness

A. Unfairness in CSMA Networks

In this section, we compare the spatial fairness performance of the O/QT-CSMA protocols. It has been reported that *non-slotted* CSMA networks are unfair [12], [42]. The two main reasons are the irregular topology of network and the combination of carrier sensing mechanism and binary exponential backoff which can cause starvation. There have been efforts towards improving fairness by tuning protocols, for example, adjusting carrier sensing range [13] or using node specific access intensity [42], [41], [40].

In *slotted* systems, unfairness partially disappears simply due to *slotting*. Indeed in a slotted system, all nodes' contention windows are reset every slot, which prevents starvation. Accordingly, fairness improves significantly. However, unfairness due to irregular topologies remains. We will show in this section that our opportunistic scheduling scheme can improve fairness.

B. Spatial Fairness

We define two *spatial* fairness indice which capture a fairness of the long-term (*time-averaged)* performance across nodes in space. The first captures the heterogeneity in performance due to nodes' locations. Recall that the performance of node, say X_i, is affected by the remaining nodes and their locations, i.e. $\Phi\backslash\{X_i\}$ and channel gains $\mathbf{F}_i$ and $\mathbf{F}'_i$, where $\mathbf{F}'_i = (F'_{ij} : j \neq i)$. Let $f_i(\Phi, \mathbf{F}_i, \mathbf{F}'_i)$ be a finite value associated with $X_i \in \Phi$ denoting its performance. Then, $\mathbb{E}\left[f_i\left(\Phi, \mathbf{F}_i, \mathbf{F}'_i\right) \mid \Phi = \phi\right]$ denotes the time-averaged (or equivalently, the average w.r.t. $\mathbf{F}_i$ and $\mathbf{F}'_i$ of the) performance for X_i given $\Phi = \phi$. To evaluate the fairness of $\mathbb{E}\left[f_i\left(\Phi, \mathbf{F}_i, \mathbf{F}'_i\right) \mid \Phi = \phi\right]$ across nodes $X_i \in \Phi = \phi$ in space we introduce Jain's fairness index [8], where

$$\tilde{\text{FI}} = \lim_{l\to\infty} \frac{\left(\sum_{X_i\in\phi\cap B_l} \mathbb{E}\left[f_i\left(\Phi, \mathbf{F}_i, \mathbf{F}'_i\right) \mid \Phi = \phi\right]\right)^2}{|\phi \cap B_l| \sum_{X_i\in\phi\cap B_l} \left(\mathbb{E}\left[f_i\left(\Phi, \mathbf{F}_i, \mathbf{F}'_i\right) \mid \Phi = \phi\right]\right)^2}. \tag{49}$$

Given the spatial ergodicity of homogeneous PPPs, see [4], and simple algebra, it is easy to see that (49) becomes

$$\tilde{\text{FI}} = \frac{\left(\mathbb{E}^0\left[\mathbb{E}\left[f_0\left(\Phi, \mathbf{F}_0, \mathbf{F}'_0\right) \mid \Phi\right]\right]\right)^2}{\mathbb{E}^0[\left(\mathbb{E}\left[f_0\left(\Phi, \mathbf{F}_0, \mathbf{F}'_0\right) \mid \Phi\right]\right)^2]}, \tag{50}$$

where $\mathbf{F}_0$ and $\mathbf{F}'_0$ denote the channel fading of a typical node at the origin and accordingly $\mathbb{E}\left[f_0\left(\Phi, \mathbf{F}_0, \mathbf{F}'_0\right) | \Phi\right]$ denotes the performance seen by the node X_0.

The second fairness index captures the heterogeneity in performance across the sets of nodes with the same size of neighborhoods. We, let $\tilde{f}_i(N_i, \mathbf{F}_i, \mathbf{F}'_i)$ be a finite performance metric associated with X_i, where N_i is the number of neighbors of X_i. Then, $\mathbb{E}\left[\tilde{f}_i(N_i, \mathbf{F}_i, \mathbf{F}'_i) | N_i = n\right]$ denotes the time-averaged (or $\mathbf{F}_i$ and $\mathbf{F}_i$'-averaged) value associated with X_i given X_i has a neighborhood of size $N_i = n$. The

[8]Jain's fairness index for a given positive allocation $x = (x_i : i = 1, \cdots, n)$ is given as $\text{FI}_x = \frac{(\sum_{i=1}^n x_i)^2}{n \sum_{i=1}^n x_i^2}$. Note that the maximum value of FI is 1 which is achieved when all x_is have the same value. If total resource $b = \sum_{i=1}^n x_i$ is allocate equally only to k entities out of n, e.g, $x_i = \frac{b}{k}$ for $i = 1, \cdots, k$ and $x_i = 0$ for $i = k+1, \cdots, n$, then, we have $\text{FI}_x = k/n$. See [22].

corresponding Jain's fairness index is given as

$$\text{FI} = \frac{\left(\mathbb{E}^0\left[\mathbb{E}\left[\tilde{f}_0\left(N_0, \mathbf{F}_0, \mathbf{F}_0'\right) \mid N_0\right]\right]\right)^2}{\mathbb{E}^0\left[\left(\mathbb{E}\left[\tilde{f}_0\left(N_0, \mathbf{F}_0, \mathbf{F}_0'\right) \mid N_0\right]\right)^2\right]}. \tag{51}$$

Unlike (50), (51) does not capture a performance variability across nodes with the same number of contenders. However, (51) is a useful metric which is computable in many cases. Depending on the performance metric $f()$ of interest, we sometimes have $\text{FI} = \tilde{\text{FI}}$. In the sequel, we will focus on $\tilde{\text{FI}}$ as our measure of spatial fairness.

C. Spatial Fairness for Access Frequency

We first evaluate spatial fairness in terms of the fraction of time that each node can access the medium. We will show how nodes' random locations impact this metric. We need the following assumption.

Assumption 1. We assume that the contenders of node X_i is the set of nodes located in the disc $b(X_i, (\nu\mu)^{-\alpha})$ or equivalently $F'_{ij} = \frac{1}{\mu} = \mathbb{E}[F]$ with probability 1.

Under this assumption, the neighbors of a node is *not* affected by fading, so the size of a node's neighborhood stays fixed, e.g., might be based on the average channel gain. This might be a reasonable assumption in a system, e.g., where each node's contending neighbors are dynamically maintained based on their average fading gains to the node. Note that F_{ij} is still a random variable. Let

$$N_{s,i}^{\gamma} = N_{s,i}^{\gamma}(\Phi) = |\{X_j \in \Phi : 1/\mu l(\|X_i - X_j\|) > \nu, i \neq j\}|$$

be a random variable denoting the size of X_i's neighborhood under the static fading assumption[9], it corresponds to the number of nodes inside a disk $b(X_i, (\nu\mu)^{-\frac{1}{\alpha}})$. This is a Poisson random variable with mean $\lambda\pi(\nu\mu)^{-\frac{2}{\alpha}}$. Recall that a node with n contenders accesses the channel with probability $\frac{p_\gamma}{n+1}$. This corresponds to the *fraction of time* the node accesses the channel. We will call this quantity as the *access frequency* of the node to differentiate it from the access probability (e.g., p_{tx}^{op} or p_{tx}^{qt}) which is interpreted as the fraction of nodes transmitting in space in a typical slot. Note that since the access frequency depends only on $N_{s,i}^{\gamma}$, so we have $\mathbb{E}[f_i(\Phi, \mathbf{F}_i, \mathbf{F}'_i)|\Phi] = \mathbb{E}[\tilde{f}_i(N_{s,i}^{\gamma}, \mathbf{F}_i, \mathbf{F}'_i)|N_{s,i}^{\gamma}] = \frac{p_\gamma}{N_{s,i}^{\gamma}+1}$. Thus we have following lemman regarding spatial fairness index on access frequency:

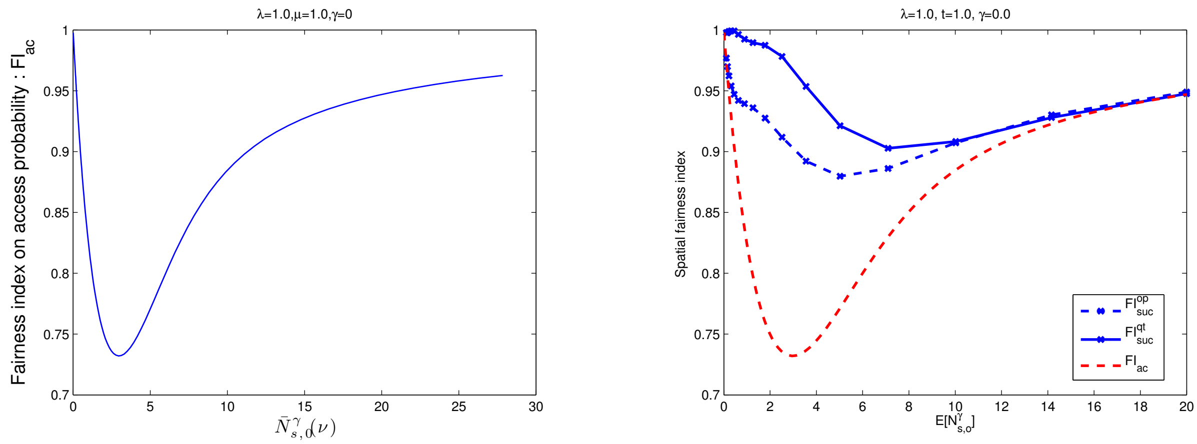
Lemma 16. *If $\mathbb{E}[\tilde{f}_i(N_{s,i}^{\gamma}, \mathbf{F}_i, \mathbf{F}'_i)|N_{s,i}^{\gamma}]$ is given as $\frac{p_\gamma}{N_{s,i}^{\gamma}+1}$, the spatial fairness index is given as follows :*

$$FI_{ac}(\bar{N}_{s,0}^{\gamma}) = \frac{\left(\mathbb{E}\left[\frac{p_\gamma}{N_{s,0}^{\gamma}+1}\right]\right)^2}{\mathbb{E}\left[\left(\frac{p_\gamma}{N_{s,0}^{\gamma}+1}\right)^2\right]} = \frac{e^{\bar{N}_{s,0}^{\gamma}} + e^{-\bar{N}_{s,0}^{\gamma}} - 2}{\bar{N}_{s,0}^{\gamma}\left(Ei(\bar{N}_{s,0}^{\gamma}) - \log \bar{N}_{s,0}^{\gamma} - \eta\right)}, \tag{52}$$

where $\bar{N}_{s,0}^{\gamma} = \mathbb{E}[N_{s,0}^{\gamma}]$, $Ei\,(x) = -\int_{-x}^{\infty} t^{-1}e^{-t}dt$ is the exponential integral function, and $\eta = 0.5772\ldots.$ is the Euler-Mascheroni constant.

Proof: It is straightforwad to compute using $N_{s,0}^{\gamma} \sim \text{Poisson}(\bar{N}_{s,0}^{\gamma})$ and $Ei(x) - \log x - \eta = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$ (see 8.214.2 in page 884 of [19]). ■

[9]Note the difference between $N_{s,i}^{\gamma}$ and N_i^{γ}, where the latter is the number of neighbors without the static fading assumption.



(a) The fairness index on access frequency decreases as the mean number of contenders $\bar{N}^{\gamma}_{s,0}$ decreases, but it rises soon as $\bar{N}^{\gamma}_{s,0}$ increases. The fairness has minimum value ~ 0.73 when $\bar{N}^{\gamma}_{s,0} \approx 3$.

(b) The fairness index on successful transmissions of O-CSMA is equal to the fairness index on access frequency of O-CSMA. While, QT-CSMA's quantile scheduling increases fairness significantly because under QT-CSMA, nodes with larger neighborhood size has higher success probability, which compensates its low access frequency.

Figure 6: Fairness index on access frequency versus the mean number of contenders $\bar{N}^{\gamma}_{s,0}(\nu)$ under fading vector assumption.

Fig.6a shows the fairness index of access frequency for O/QT-CSMA versus $\bar{N}^{\gamma}_{s,0}(\nu)$. If $\bar{N}^{\gamma}_{s,0}$ is small, almost every contending node sends, in fact all transmitters have access frequency close to p_γ, so the fairness index is close to 1. If $\bar{N}^{\gamma}_{s,0}$ is relatively small, as $\bar{N}^{\gamma}_{s,0}$ (which is mean and the variability of the number of contenders) increases, the variability of access frequency, i.e., $\frac{p_\gamma}{N^{\gamma}_{s,0}+1}$, across nodes increases resulting in a decrease in fairness. However, if $\bar{N}^{\gamma}_{s,0}$ is relatively large, the fairness index eventually increases again since, in this regime, the variability of access frequency $\frac{p_\gamma}{N^{\gamma}_{s,0}+1}$ decreases and converges to 0, which in turn increases fairness. Note that the fairness curve has its minimum value $0.73019\ldots$. Specifically, the minimizer $n^* \equiv \arg\min_{n>0} \mathrm{FI}_{ac}(n) \approx 2.9736657$ can be found by numerically solving $\frac{\mathrm{d}}{\mathrm{d}n}\mathrm{FI}_{ac}(n) = 0$. Based on this, we have following proposition.

Proposition 17. Under Assumption 1, the spatial fairness for access frequency of slotted O/QT-CSMA is worst, roughly 0.73 when the mean number of contenders of a typical transmitter is roughly 3.

D. Spatial Fairness of the Frequency of Successful Transmissions

In this section, we consider the fairness of the frequency of successful transmissions. Specifically, we will show that opportunistic CSMA schemes can, to a certain extent, remove topological unfairness. We first define the spatial fairness of the frequency of successful transmissions.

For O-CSMA, we define $\frac{p_\gamma}{N^{\gamma}_{s,0}+1}\bar{p}^{op}_{suc}(\gamma, N^{\gamma}_{s,0})$ as the frequency of successful transmissions of a typical receiver with $N^{\gamma}_{s,0}$ neighbors, where $\frac{p_\gamma}{N^{\gamma}_{s,0}+1}$ is the access frequency and $\bar{p}^{op}_{suc}(\gamma, N^{\gamma}_{s,0})$ is the conditional success probability conditioned on that its associated transmitter has $N^{\gamma}_{s,0}$ contenders, which is given by

$$\bar{p}^{op}_{suc}(\gamma, N^\gamma_{s,0}=n) = \mathbb{P}^0\left(F_\gamma > tI_{\Phi^\gamma_{M\setminus\{0\}}} l(r) | N^\gamma_{s,0}=n\right)$$
$$\approx \mathbb{E}^0\left[\int_{-\infty}^{\infty} \mathcal{L}_{I^{N^\gamma_{s,0},F_\gamma}_{\Phi^\gamma_u\setminus\{0\}}}(2i\pi l(r)ts)\frac{e^{2i\pi sF_\gamma}-1}{2i\pi s}ds\Big|N^\gamma_{s,0}=n\right], \quad (53)$$

where $I^{N^\gamma_{s,0},F_\gamma}_{\Phi^{\gamma^0}_u\setminus\{0\}}$ given $N^\gamma_{s,0}=n$ is the interference seen by a typical receiver conditioned on that it has n neighbors. Accordingly, the fairness index is given by

$$\tilde{\mathrm{FI}}^{op}_{suc}(\gamma, \bar{N}^\gamma_{s,0}) = \frac{\left(\mathbb{E}^0\left[\frac{p_\gamma}{N^\gamma_{s,0}+1}\bar{p}^{op}_{suc}(\gamma, N^\gamma_{s,0})\right]\right)^2}{\mathbb{E}^0\left[\left(\frac{p_\gamma}{N^\gamma_{s,0}+1}\bar{p}^{op}_{suc}(\gamma, N^\gamma_{s,0})\right)^2\right]}. \quad (54)$$

For QT-CSMA, we take a similar approach. We define $\frac{p_\gamma}{N^\gamma_{s,0}+1}\bar{p}^{qt}_{suc}(\gamma, N^\gamma_{s,0})$ as the frequency of successful transmission of a typical receiver with $N^\gamma_{s,0}$ neighbors, where $\frac{p_\gamma}{N^\gamma_{s,0}+1}$ is the access frequency and $\bar{p}^{qt}_{suc}(\gamma, N^\gamma_{s,0})$ is the conditional success probability conditioned on that its associated transmitter has $N^\gamma_{s,0}$ contenders, which is given by

$$\bar{p}^{qt}_{suc}(\gamma, N^\gamma_{s,0}=n) = \mathbb{P}^0\left(F^{\max}_{0,\gamma}(N^\gamma_{s,0}+1) > tI_{\Phi^\gamma_{M\setminus\{0\}}} l(r) | N^\gamma_{s,0}=n\right).$$
$$\approx \mathbb{E}^0\left[\int_{-\infty}^{\infty} \mathcal{L}_{I^{N^\gamma_{s,0},F^{\max}_{0,\gamma}(N^\gamma_{s,0}+1)}_{\Phi^\gamma_u\setminus\{0\}}}(2i\pi l(r)ts)\frac{e^{2i\pi sF^{\max}_{0,\gamma}(N^\gamma_{s,0}+1)}-1}{2i\pi s}ds\Big|N^\gamma_{s,0}=n\right]. \quad (55)$$

The fairness metric of interest in this section corresponds to the second type (51) only, and the corresponding fairness index of successful transmission is given by

$$\tilde{\mathrm{FI}}^{qt}_{suc}(\gamma, \bar{N}^\gamma_{s,0}) = \frac{\left(\mathbb{E}^0\left[\frac{p_\gamma}{N^\gamma_{s,0}+1}\bar{p}^{qt}_{suc}(\gamma, N^\gamma_{s,0})\right]\right)^2}{\mathbb{E}^0\left[\left(\frac{p_\gamma}{N^\gamma_{s,0}+1}\bar{p}^{qt}_{suc}(\gamma, N^\gamma_{s,0})\right)^2\right]}. \quad (56)$$

Using the expression of $\mathcal{L}_{I^{N^\gamma_{s,0},F^{\max}_{0,\gamma}(N^\gamma_{s,0}+1)}_{\Phi^{\gamma^0}_u\setminus\{0\}}}$ in (37) and $N^\gamma_{s,0} \sim \mathrm{Poisson}(\bar{N}^\gamma_{s,0})$, $\tilde{\mathrm{FI}}^{qt}_{suc}$ can be numerically computed.

$\tilde{\mathrm{FI}}^{op}_{suc}$ and $\tilde{\mathrm{FI}}^{qt}_{suc}$are plotted in Fig.6b for $\gamma=0$. The figure shows that the fairness on the frequency of successful transmissions achieved by QT$_0$-CSMA is improved versus that of O-CSMA. The gain is significant in the regime where $\bar{N}^\gamma_{s,0}$ is less than or equal to roughly 10. In this regime, QT$_0$-CSMA increases the success probability of receivers a lot. This reduces the performance differences among nodes caused by different access frequency (or topology) since nodes with a large number of neighbors who get low access frequency have higher success probability. In other words, the higher success probability compensates the low access frequency, which decreases the variability in performance. In the regime where $\bar{N}^\gamma_{s,0}$ is large (or ν is small), the density of concurrent transmitters becomes small, which generates weak interference. Thus, most nodes succeed in their transmissions with high probability irrespective of the number of neighbors, so in this regime there is no much gain from opportunism increasing the success probability. Thus, QT$_0$-CSMA and O-CSMA have almost the same performance. As γ increases, fairness decreases and eventually converges to the fairness curve of O-CSMA where $\gamma \to \infty$ since there will be little difference between $\bar{p}^{qt}_{suc}$ and $\bar{p}^{op}_{suc}$.

So far, it has been shown that opportunistic CSMA can improve fairness. However, with this results only, it is not clear how these protocols tradeoff the density of successful transmissions versus fairness. We consider this next.

E. Tradeoff between Spatial Fairness and Spatial Reuse

It has been noted that there exists a limited set of network topologies (e.g., line or circle networks) where both high fairness and "spatial reuse" or "throughput" can be both achieved [41], [14]. However, in the random networks we consider, there will be tradeoffs due to the randomness of node locations, contentions and protocols. To explore these we introduce the following notations.

Definition 18. (FD-pair) We call (a,b) an achievable FD-pair if a fairness index a and density of successful transmissions b can be achieved under a given protocol parameter choice.

Definition 19. (Dominance) For FD-pairs (a,b) and $(c,d) \in \mathbb{R}^2_+$, we say that the (a,b) *dominates* (c,d) if $a \geq c$ and $b \geq d$. We denote this relation with $(c,d) \preceq (a,b)$.

We consider the set of FD-pairs that are not dominated by any other pairs, i.e., the Pareto-frontier. For a given FD-pair (a,b), we define the set of FD-pairs dominated by (a,b) as follows.

Definition 20. (Dominated set) For a FD-pair $(a,b) \in \mathbb{R}^2_+$, we call the set

$$\Lambda(a,b) = \big\{(x,y) \in \mathbf{R}^2_+ \text{ s.t. } (x,y) \preceq (a,b)\big\} \tag{57}$$

as the dominated set of the FD-pair (a,b). Note that $(a,b) \in \Lambda(a,b)$.

In particular, we define the dominated set for O-CSMA, for a given t and λ, by

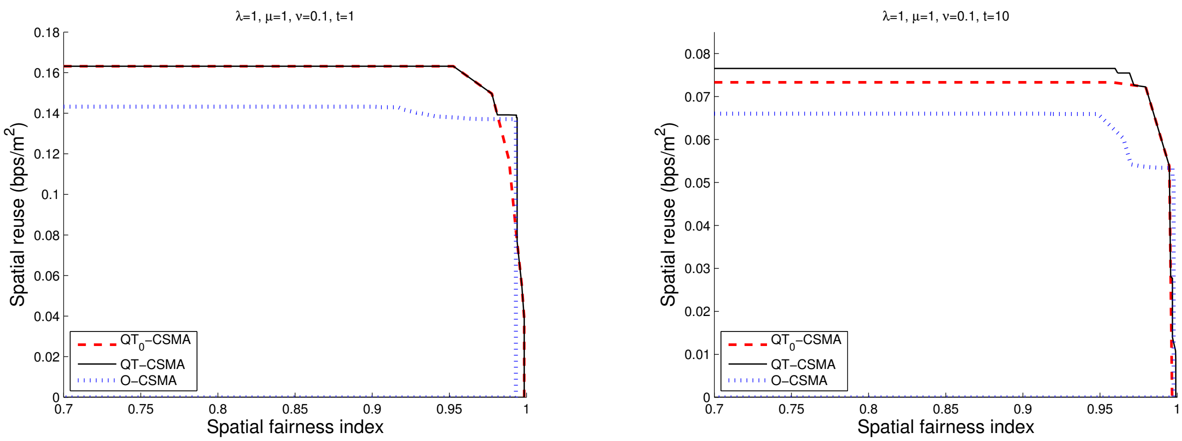
$$\Omega^{op}(\lambda,t) = \bigcup_{\gamma \geq 0, \nu \geq 0} \Lambda(\tilde{\text{FI}}^{op}_{suc}(\lambda,\gamma,\nu,t), d^{op}_{suc}(\lambda,\gamma,\nu,t)). \tag{58}$$

The dominated set for QT-CSMA is similarly defined. The dominated set QT_0-CSMA for a given t and λ is defined as

$$\Omega^{qt}_0(\lambda,t) = \bigcup_{\nu \geq 0} \Lambda(\tilde{\text{FI}}^{qt}_{suc}(\lambda,0,\nu,t), d^{qt}_{suc}(\lambda,0,\nu,t)). \tag{59}$$

Definition 21. (Pareto-Frontier) For a given set of FD-pairs, the subset of FD-pairs which are not dominated by any other FD-pairs is called as Pareto-frontier.

We plotted three pareto-frontiers for O-CSMA, QT-CSMA, and QT_0-CSMA or their dominated sets $\Omega^{op}(\lambda,t)$,$\Omega^{qt}(\lambda,t)$ and $\Omega^{qt}_0(\lambda,t)$ for two decoding SIR $t=1$ and $t=10$ in Fig. 7a and 7b respectively. We note that the dominant set of QT-CSMA is the super set of that of O-CSMA. This gain comes from the joint improvement of spatial reuse and fairness performance. One notable thing is that $\Omega^{qt}_0(\lambda,t)$ is very close to $\Omega^{qt}(\lambda,t)$, which shows again the effectiveness of quantile-based approach in taking advantage of dynamic channel variations and mult-user diversity.



(a) Case $t = 1$: low decoding SIR regime

(b) Case $t = 10$: high decoding SIR regime

Figure 7: Comparison of the dominated sets of O-CSMA, QT-CSMA and QT_0-CSMA.

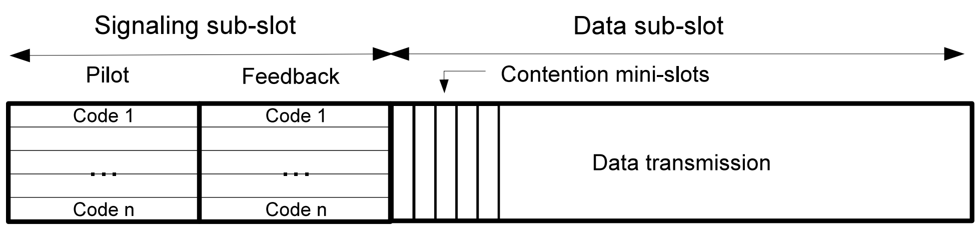


Figure 8: Slot structure for CDMA case.

VI. System implementation

In this section, we briefly describe a possible implementation of the O/QT-CSMA protocols. We will focus on the aspect of slot structure together with the operation of protocols but not give full fledged detailed protocols - this is beyond the scope of this paper. We will make several key assumptions in our development. First, we assume that nodes are perfectly synchronized with external and/or internal aids, e.g., each node might use pilot signals from GPS-synchronized cellular base stations and might run a distributed synchronization algorithm to further synchronize as done in [46]. For a discussion of the impact of imperfect sync, see [35]. Second, we assume that fading is symmetric and changing each slot. Third, the node density λ is assumed to be in a moderate range, i.e., not unrealistically dense.

In what follows, we describe a possible frame structure for transmissions. It is composed of two sub-slots : one for signaling and the other for data transmission, see Fig. 8.

A. Signaling sub-slot

In the signaling sub-slot, each transmitter sends a pilot signal to its associated receiver and the receiver feeds back the measured channel status between them. We consider the method suggested in [2], where the receiver estimates its channel status based on the received pilot power. However, the received pilot at the receiver may include both desired and undesired signals (interference) which makes the estimate inaccurate. To mitigate the impact of interference, we can assign each transmitter a CDMA code (CDMA approach) or a time slot (TDMA approach) for sending its pilot signal. However, since the number of codes or time slots are limited, two or more nodes can use the same resource, so the protocol should be designed to work in such a way that two nodes using the same resource are sufficiently separated to mitigate the impact of interference, e.g., as done in [25], [30]. One can also use an OFDM approach like [46], where each node picks a single sub-carrier to send its pilot signal.

Once the receiver estimates its channel status, it feeds back the estimate, say $\hat{F}_{ii}$. If the transmitter uses O-CSMA and $\hat{F}_{ii} > \gamma$, then it contends otherwise it defers. While, if it uses QT-CSMA and $\hat{F}_{ii} > \gamma$, then it updates its channel status statistics $\hat{G}_\gamma$(i.e. empirical cdf of $\hat{F}_{ii}$) and determines its current discretized version of quantile as $\frac{k}{w}$ if $\frac{k}{w} \leq \hat{G}_\gamma(\hat{F}_{ii}) < \frac{k+1}{w}$ for $k = 0, \cdots, w-1$, where w is the maximum number of mini-slots (similar to the size of the contention window) used for contention.

B. Data sub-slot

In the data transmission sub-slot, qualified nodes contend and only winners transmit. In the contention phase composed of w mini-slots, each qualified node sets its timer value to a random value uniformly distributed in $\{1, 2, \cdots, w\}$ if it is O-CSMA node, or sets it to k defined above if it is QT-CSMA node. All qualified nodes start to decrement their timer values by one every mini-slot. All nodes whose timer values reach 0 start transmitting from the mini-slot until the end of the data sub-slot. Other nodes which sense the transmissions prior to their timeout defer in the slot.

Since w is finite, a collision can occur if two or more nodes select the same minislot. The transmission probability (considering collision) with w mini-slots is computed as

$$p_{tx}^{col}(w) = p_\gamma \sum_{i=0} \left[\sum_{t=1}^{w} \frac{1}{w} \left(1 - \frac{t}{w}\right)^i \right] \frac{p_\gamma \bar{N}_0^i}{i!} e^{-p_\gamma \bar{N}_0} = \frac{p_\gamma \left(1 - \exp\left\{-p_\gamma \bar{N}_0\right\}\right)}{w \left(\exp\left(p_\gamma \bar{N}_0 / w\right) - 1\right)}. \tag{60}$$

Note that we can easily show $p_{tx}^{col}(w)$ converges as $w \to \infty$ to the transmission probabilities without collision given (18) and (28). Note that to see overall impact of collision, p_{suc} also needs to be recomputed considering collision. Studying the impact of collision is left to future work.

VII. Conclusion

In this paper, we considered the spatial reuse and fairness for wireless ad-hoc networks using two different channel-aware CSMA protocols. We used an analytical framework based on stochastic geometry to derive the transmission probability and success probability for a typical node, and from there two spatial performance metrics, the density of successful transmissions and spatial fairness index were computed. By capturing the delicate interactions among system parameters (qualification threshold γ, carrier sense threshold ν), the density of transmitters, aggregate interference, and spatial reuse and fairness, we showed that QT-CSMA achieves higher spatial reuse and fairness than O-CSMA, and more interestingly the simple version of QT-CSMA with one less parameter achieves robust spatial reuse in a wide range of node densities. To better understand the interactions between joint spatial reuse and fairness performance, we characterized the dominated sets of spatial fairness-reuse pair under Q/O-CSMA. Although O-CSMA has one more parameter to adjust, its dominated set was smaller than that of QT-CSMA. Suprisingly, the simple version of QT-CSMA (QT$_0$-CSMA) has almost the same dominated set as QT-CSMA. Thus, we conclude that the joint carrier sense and quantile scheduling approach of QT-CSMA is not only an effective way in improving spatial reuse/fairness performance by compromizing interference and transmitter density but also a practical low complexity approach with one less parameter and robustness.

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Appendix A
Derivation of $u'(n, t_0, \tau, \lambda)$

In this appendix, we derive u' for $n \in \{0, 1, 2, \cdots\}$, $t_0 \in [0, 1)$, $\tau > 0$, and $\lambda > 0$, which is defined as

$$u'(n, t_0, \tau, \lambda) = \mathbb{P}(E_1 = 1 \mid E_0 = 1, N_0 = n, T_0 = t_0, \{y_0, y_1\} \subset \Phi, |y_0 - y_1| = \tau).$$

Let $\Phi = \{X_i\}$ be an homogeneous PPP with density λ. N_0 is the number of neighbors of y_0 and T_0 is the timer value of y_0. In the sequel, we omit the conditioning events $\{y_0, y_1\} \subset \Phi$ and $|y_0 - y_1| = \tau$ for simplicity. By applying Bayes' rule, we have

$$\mathbb{P}(E_1 = 1 \mid E_0 = 1, N_0 = n, T_0 = t_0) = \frac{\mathbb{P}(E_1 = 1, E_0 = 1 \mid N_0 = n, T_0 = t_0)}{\mathbb{P}(E_0 = 1 \mid N_0 = n, T_0 = t_0)}. \tag{61}$$

The denominator is simply given by

$$\mathbb{P}(E_0 = 1 \mid N_0 = n, T_0 = t_0) = (1 - t_0)^n \tag{62}$$

since for y_0 to transmit, all its n neighbors should have timer values larger than t_0 independently. To compute the numerator, we condition on the event that y_0 and y_1 are neighbors, i.e., $\{y_1 \in \mathcal{N}_0\} = \{F'_{01} > \nu l(\|y_0 - y_1\|)\}$, then by the law of total probability we have

$$\begin{aligned}\mathbb{P}(E_1 = E_0 = 1 \,|N_0 = n, T_0 = t_0) &= \underbrace{\mathbb{P}(E_1 = E_0 = 1, y_1 \in \mathcal{N}_0 \mid N_0 = n, T_0 = t_0)}_{=0} \\ &+ \underbrace{\mathbb{P}(E_1 = E_0 = 1 \mid N_0 = n, T_0 = t_0, y_1 \notin \mathcal{N}_0)}_{C}\underbrace{\mathbb{P}(y_1 \notin \mathcal{N}_0 \mid N_0 = n, T_0 = t_0)}_{D}. \end{aligned} \tag{63}$$

If y_0 and y_1 can see each other, it is impossible for both to transmit at the same time, so the first term is equal to 0. We denote the second and third terms by C and D respectively.

A. Computing C

The term C is a conditional probability conditioned on that y_0 and y_1 do not see each other. However, they may share some neighbors; so to compute this term we need to further condition on the event that some of y_0's contenders are also seen by y_1. Let $\mathcal{K} = \mathcal{N}_0 \cap \mathcal{N}_1$ be the set of shared neighbors between y_0 and y_1 and $K(\tau) \equiv |\mathcal{K}|$ be the number of them. Note that $K(\tau) \sim \text{Poisson}(\bar{K}(\tau))$ where the mean $\bar{K}(\tau)$ is computed as

$$\begin{aligned}\bar{K}(\tau) &= \mathbb{E}\left[\sum_{X_i \in \Phi} \mathbf{1}\left\{F'_{i0} > \nu l(\|X_i - y_0\|)\right\} \mathbf{1}\left\{F'_{i1} > \nu l(\|X_i - y_1\|)\right\}\right] \\ &= \lambda \int_{\mathbb{R}^2} \mathbb{P}(F_{i0} > \nu l(\|x\|))\mathbb{P}(F_{i1} > \nu l(\|x - y_1\|))dx \\ &= \lambda \int_0^{2\pi} \int_0^{\infty} e^{-\mu\nu\left(l(r) + l(\sqrt{\tau^2 + r^2 - 2\tau r \cos\theta})\right)} r dr d\theta. \end{aligned} \tag{64}$$

By conditioning on the number of shared neighbors, $K = K(\tau)$, we rewrite the C as

$$C = \sum_{k=0}^{n} \underbrace{\mathbb{P}(E_0 = E_1 = 1 \mid N_0 = n, T_0 = t_0, K = k, y_1 \notin \mathcal{N}_0)}_{\equiv A_k} \underbrace{\mathbb{P}(K = k \mid N_0 = n, T_0 = t_0, y_1 \notin \mathcal{N}_0)}_{\equiv B_k}. \tag{65}$$

We let the first and second terms in the summation be A_k and B_k respectively.

1) Computing B_k: B_k is the probability that two nodes y_0 and y_1 share k common contenders conditioned on that y_0 has n contenders. Since each contender of y_0 is independently seen by y_1, the number of shared contenders K for a given N_0 is a Binomial random variable with parameters n and p_s, where p_s is the probability that one of y_0's neighbors is seen by y_1. Then, we have

$$\begin{aligned} B_k &= \mathbb{P}(K = k \mid N_0 = n, T_0 = t_0, y_1 \notin \mathcal{N}_0) \\ &\stackrel{a}{=} \mathbb{P}(K = k \mid N_0 = n, y_1 \notin \mathcal{N}_0) = \binom{n}{k} p_s^k (1-p_s)^{n-k} \end{aligned} \tag{66}$$

where in $\stackrel{a}{=}$, we used the fact that $\{T_0 = t_0\}$ is independent of $\{K = k\}$, and $p_s(\tau)$ is computed as

$$\begin{aligned} p_s(\tau) &= \mathbb{P}(X \in \mathcal{N}_1 \mid X \in \mathcal{N}_0) \\ &\stackrel{a}{=} \frac{\mathbb{E}\left[\mathbb{P}(X \in \mathcal{N}_1, X \in \mathcal{N}_0 \mid X)\right]}{\mathbb{E}\left[\mathbb{P}(X \in \mathcal{N}_0 \mid X)\right]} \\ &= \frac{\lambda \int_{\mathbf{R}^2} \mathbb{P}(F_0' > \nu l(||x||)) \mathbb{P}(F_1' > \nu l(||x - y_1||)) dx}{\lambda \int_{\mathbf{R}^2} \mathbb{P}(F > \nu l(||x||)) dx} \\ &= \frac{\bar{K}(\tau)}{\bar{N}_0}. \end{aligned} \tag{67}$$

In $\stackrel{a}{=}$, we conditioned on the location of X in $\mathbb{R}^2$. Using (12), (67) can be rewritten as $p_s = p_s(\tau, \lambda) = 2 - \frac{b(\tau,\lambda)}{\bar{N}_0}$.

2) Computing A_k: A_k can be rewritten as

$$A_k = \int_0^1 \mathbb{P}(E_1 = E_0 = 1 \mid N_0 = n, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0) dt_1$$

by conditioning on that the timer value of y_1 equal to t_1, i.e., $\{T_1 = t_1\}$. Note that either all the shared neighbors in $\mathcal{K}$ have timer values larger than t_1, i.e., $\{T_j^c \geq t_1, \forall C_j \in \mathcal{K}\}$ where T_j^c is the timer value of contender $C_j \in \mathcal{K}$, or there exist one or more neighbors with timer value(s) smaller than t_1, i.e., $\{\exists C_j \ s.t.\ T_j^c < t_1\}$. Using the law of total probability, the probability inside the integral can be written as follows:

$$\begin{aligned} \mathbb{P}(E_1 = & E_0 = 1 \mid N_0 = n, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0) \\ &= \underbrace{\mathbb{P}(E_1 = E_0 = 1 \mid N_0 = n, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0, T_j^c \geq t_1 \forall C_j \in \mathcal{K})}_{\equiv A_{k1}} && (68) \\ &\quad \times \underbrace{\mathbb{P}\left(T_j^c \geq t_1 \forall C_j \in \mathcal{K} \mid N_0 = n, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0\right)}_{\equiv A_{k2} = (1-t_1)^k} && (69) \\ &+ \underbrace{\mathbb{P}(E_1 = 1, E_0 = 1 \mid N_0 = n, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0, \exists C_j \ s.t.\ T_j^c < t_1)}_{\equiv A_{k3} = 0} && (70) \\ &\quad \times \mathbb{P}(\exists C_j \ s.t.\ T_j^c < t_1 \mid N_0 = 1, T_0 = t_0, K = k, T_1 = t_1, y_1 \notin \mathcal{N}_0). \end{aligned}$$

Let (68), (69), and (70) be A_{k1}, A_{k2}, and A_{k3} respectively. We have $A_{k3} = 0$ since if there exist any neighbor with timer value strictly smaller than t_1, it prevents y_1 from transmitting, so E_1 can not be 1. A_{k2} is the probability that all shared neighbors in $\mathcal{K}$ have timer values larger than t_1, which is simply given as $A_{k2} = (1 - t_1)^k$ since each timer is independent and uniform in $[0, 1]$. Before we compute A_{k1}, we need to define several random variables.

- Let N_1 ∼Poisson($\bar{N}_0$) be a random variable denoting the number of contenders of y_1.
- Let N_{1x} ∼Poisson($\bar{N}_{1x}$) be a random variable denoting the number of contenders of y_1 which are not shared by y_0. Note that $N_{1x} + K = N_1$ and $\bar{N}_{1x} = \bar{N}_0(1 - p_s)$.
- Let $N_{1x}^{<t_1}$ ∼Poisson($\bar{N}_{1x}^{<t_1}$) be a random variable denoting the number of contenders of y_1 which are not shared by y_0 and with timer values smaller than t_1. Note that $\bar{N}_{1x}^{<t_1} = t_1\bar{N}_{1x} = t_1\bar{N}_0(1 - p_s)$.

To compute A_{k1}, we consider the following two sub-cases $t_1 \leq t_0$ and $t_0 < t_1$. If $t_1 \leq t_0$, then

- y_1 transmits (or $E_1 = 1$) only if it finds no additional neighbors who have timer values smaller than t_1 and are not seen by y_0, i.e, if $N_{1x}^{<t_1} = 0$, and
- y_0 transmits (or $E_0 = 1$) only if all $T_j^c \sim$ Uniform$[t_1, 1]$ $\forall C_j \in \mathcal{K}$ is larger than t_0, which happens with probability $\left(\frac{1-t_0}{1-t_1}\right)^k$ and remaining $n - k$ contenders have timer values larger than t_0, which happens with probability $(1 - t_0)^{n-k}$.

Note that, as in the previous case, $\{E_0 = 1\}$ and $\{E_1 = 0\}$ are conditionally independent; so we have that

$$A_{k1} = e^{-\bar{N}_{1x}^{<t_1}} \frac{(1 - t_0)^n}{(1 - t_1)^k} \quad \text{if } t_0 \geq t_1. \tag{71}$$

In the other case where $t_0 < t_1$,

- y_0 transmits (or $E_0 = 1$) if $n - k$ neighbors have timer values larger than t_0, which happens with probability $(1 - t_0)^{n-k}$, and
- y_1 transmits (or $E_1 = 1$) only when it finds no additional neighbors who have timer values smaller than t_1 and do not see y_0, i.e., $N_{1x}^{<t_1} = 0$.

Note that, as previous case, $\{E_0 = 1\}$ and $\{E_1 = 0\}$ are conditionally independent; so we have

$$A_{k1} = e^{-\bar{N}_{1x}^{<t_1}} (1 - t_0)^{n-k} \quad \text{if } t_0 < t_1. \tag{72}$$

A_{k1} in the above two cases can be written as follows using indicator functions:

$$A_{k1} = e^{-t_1\bar{N}_0(1-p_s)} \left(\frac{(1 - t_0)^n}{(1 - t_1)^k} \mathbf{1}\{t_1 \leq t_0\} + (1 - t_0)^{n-k} \mathbf{1}\{t_0 < t_1\} \right). \tag{73}$$

Unconditioning with respect to the event $\{T_1 = t_1\}$ in $A_{k1}A_{k2}$ gives

$$\begin{aligned} A_k &= \int_0^1 e^{-t_1\bar{N}_0(1-p_s)} \left((1 - t_0)^n \mathbf{1}\{t_1 \leq t_0\} + (1 - t_0)^{n-k}(1 - t_1)^k \mathbf{1}\{t_0 < t_1\} \right) dt_1 \\ &= (1 - t_0)^n \int_0^{t_0} e^{-t_1\bar{N}_0(1-p_s)} dt_1 + (1 - t_0)^{n-k} \int_{t_0}^1 (1 - t_1)^k e^{-t_1\bar{N}_0(1-p_s)} dt_1 \\ &\overset{a}{=} (1 - t_0)^n \left(\frac{1 - e^{-t_0\bar{N}_0(1-p_s)}}{\bar{N}_0(1 - p_s)} + \frac{(t_0 - 1)e^{-\bar{N}_0(1-p_s)} \left(\Gamma(k + 1, \eta) - \Gamma(k + 1) \right)}{\eta^{k+1}} \right) \\ &\overset{b}{=} (1 - t_0)^n \left(\frac{1 - e^{-t_0\bar{N}_0(1-p_s)}}{\bar{N}_0(1 - p_s)} + \frac{(1 - t_0)e^{-\bar{N}_0(1-p_s)} k!}{\eta^{k+1}} \left(1 - e^{-\eta} \sum_{j=0}^k \frac{\eta^j}{j!} \right) \right). \end{aligned} \tag{74}$$

where in $\overset{a}{=}$, $\Gamma(a, x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function with $\Gamma(a) \equiv \Gamma(a, 0)$ and $\eta = \bar{N}_0(1 - p_s)(t_0 - 1)$, and in $\overset{b}{=}$ we used the fact that $\frac{\Gamma(k+1,\eta)}{\Gamma(k+1)} = \sum_{j=0}^k \frac{\eta^j}{j!} e^{-\eta}$. Replacing (74) and

(66) in (65) gives

$$C = \mathbb{P}(E_1 = E_0 = 1 | N_0 = n, T_0 = t_0, y_1 \notin \mathcal{N}_0) = \sum_{k=0}^{n} A_k B_k. \tag{75}$$

B. Computing D

We now compute $D = \mathbb{P}(y_1 \notin \mathcal{N}_0 \mid N_0 = n, T_0 = t_0)$ in (63). Note that $\{N_0 = n\}$ and $\{y_1 \notin \mathcal{N}_0\}$ are not independent since it is likely that y_1 is the neighbor of y_0 if $N_0 = n$ is large, but $\{y_1 \notin \mathcal{N}_0\}$ and $\{T_0 = t_0\}$ are independent since being neighbors of a node does not depend on timer values. Thus, we have $D = \mathbb{P}(y_1 \notin \mathcal{N}_0 | N_0 = n)$. Applying Bayes' rule, we get

$$D = \frac{\mathbb{P}(y_1 \notin \mathcal{N}_0, N_0 = n)}{\mathbb{P}(N_0 = n)} = \frac{\mathbb{P}(N_0 = n \mid y_1 \notin \mathcal{N}_0)\mathbb{P}(y_1 \notin \mathcal{N}_0)}{\mathbb{P}(N_0 = n)}, \tag{76}$$

where we have

$$\mathbb{P}(N_0 = n \mid y_1 \notin \mathcal{N}_0) = \frac{\bar{N}_0^n}{n!} e^{-\bar{N}_0} \tag{77}$$

and

$$\mathbb{P}(y_1 \notin \mathcal{N}_0) = \mathbb{P}(F'_{10} < vl(\tau)) = G(\nu l(\tau)) \tag{78}$$

for numerator. To compute $\mathbb{P}(N_0 = n)$, the denominator[10] in (76), we need to consider whether y_1 is seen by y_0 or not. Using the law of total probability, we have

$$\begin{aligned}
\mathbb{P}(N_0 = n) =& \mathbb{P}(N_0 = n \mid y_1 \in \mathcal{N}_0)\mathbb{P}(y_1 \in \mathcal{N}_0) + \mathbb{P}(N_0 = n \mid y_1 \notin \mathcal{N}_0)\mathbb{P}(y_1 \notin \mathcal{N}_0) \\
=& \frac{\bar{N}_0^{n-1}}{(n-1)!} e^{-\bar{N}_0} \left(1 - G(\nu l(\tau))\right) + \frac{\bar{N}_0^n}{n!} e^{-\bar{N}_0} G(\nu l(\tau)) \\
=& \frac{\bar{N}_0^{n-1}}{(n-1)!} e^{-\bar{N}_0} \left(1 + \left(\frac{\bar{N}_0}{n} - 1\right) G(\nu l(\tau))\right)
\end{aligned} \tag{79}$$

Note that as expected $\mathbb{P}(N_0 = n) \to \mathbb{P}(N_0 = n \mid y_1 \notin \mathcal{N}_0)$ as $\tau \to \infty$ (or $G(\tau) \to 1$), and $\mathbb{P}(N_0 = n) \to \mathbb{P}(N_0 = n \mid y_1 \in \mathcal{N}_0)$ as $\tau \to 0$ (or $G(\tau) \to 0$). Then, replacing (77), (78), and (79) in (76) gives the following for $n \geq 0$,

$$D = \frac{\bar{N}_0 G(\nu l(\tau))}{n + (\bar{N}_0 - n) G(\nu l(\tau))}. \tag{80}$$

Recall that D is the probability that y_1 is not the neighbor of y_0 given $N_0 = n$. Thus, $D \to 1 (D \to 0)$ as $\tau \to \infty (\tau \to 0)$ makes sense.

C. Computing u'

Now, replacing term C in (75) and D in (80) to (63) gives

$$\mathbb{P}(E_1 = E_0 = 1 | N_0 = n, T_0 = t_0) = \frac{\bar{N}_0 G(\nu l(\tau))}{n + (\bar{N}_0 - n) G(\nu l(\tau))} \sum_{k=0}^{n} A_k B_k. \tag{81}$$

[10] Recall that $\mathbb{P}(N_0 = n)$ is indeed $\mathbb{P}\left(N_0 = n \mid ||y_0 - y_1|| = \tau\right)$.

Finally, (61) is given as

$$u'(n,t_0,\tau,\lambda)=\frac{\bar{N}_0G(\nu l(\tau))}{n+(\bar{N}_0-n)G(\nu l(\tau))}\Bigg(\frac{1-e^{-t_0\bar{N}_0(1-p_s)}}{\bar{N}_0(1-p_s)}+ \\ +(1-t_0)e^{-\bar{N}_0(1-p_s)}\sum_{k=0}^{n}\frac{k!}{\eta^{k+1}}\left(1-e^{-\eta}\sum_{j=0}^{k}\frac{\eta^j}{j!}\right)\binom{n}{k}p_s^k(1-p_s)^{n-k}\Bigg), \tag{82}$$

where $p_s=p_s(\tau,\lambda)=2-\frac{b(\tau,\lambda)}{\bar{N}_0}$ and $\eta=\bar{N}_0(1-p_s)(t_0-1)$.

D. Impact of n and t_0

Fig.9a gives plots for $u'(n,t_0,\tau,\lambda)$ for $t_0=0.5$, $\lambda=1$ and for $n=0,\cdots,20$. Observe how u' changes as the distance τ between y_0 and y_1 changes. As τ gets large, y_1 behaves like a typical node in space which is not affected by the existence of y_0. The latter case is verified by the fact that all curves u' converge to the value $\frac{1-e^{-\bar{N}_0}}{\bar{N}_0}$ as $\tau\to\infty$, which is indeed the transmission probability of a typical CSMA node. As τ gets small, there is a strong correlation between y_1 and y_0 which are likely to be neighbors. The behavior of u' in this case depends on the value of n. In particular, if $n=0$, u' increases as $\tau\to 0$; since y_1 will see no contenders as is the case for y_0, while if $n>0$, as $\tau\to 0$, y_1 will see one or more contenders as seen by y_0, and it will be more likely that y_1 is a neighbor of y_0. If y_1 is a neighbor of y_0, then due to the condition $\{E_0=1\}$, y_1 must have a timer value larger than t_0, so the conditional transmission probability u' approaches 0. As n increases, y_1 is more likely to be preempted by y_0 and its neighbors, thus u' decreases.

Fig.9b shows the impact of y_0's timer value, t_0, on u' for $n=5$. Note that the condition $\{E_0=1\}$ implies that n neighbors of y_0 have timer values between t_0 and 1. Thus, if t_0 gets large, y_1 will transmit with high probability since the neighbors of y_0 will have timer values larger than t_0, which can be easily preempted by y_1's timer. While if t_0 gets small, y_1 is more likely to be preempted by y_0's neighbors, so u' decreases in this case.

APPENDIX B
CONVERGENCE OF $I_{\Phi_M^{\gamma}}$

Consider a Matérn CSMA process which is induced by a CSMA mechanism from a PPP with density λ. In this CSMA network, the density of active transmitters λp_{tx} converges to a value as $\lambda\to\infty$ due to the CSMA protocol. Accordingly, the amount of interference *seen at a typical receiver* also converges to a random variable, say $I_{\Phi_M^{dens}}$. The objective of this section is to prove that $I_{\Phi_M^{dens}}$ is almost surely finite, i.e., $\mathbb{P}(I_{\Phi_M^{dens}}<\infty)$=1.

To that end, we first show that for a given monotonically increasing sequence of node densities $\lambda^{[1]}\leq\lambda^{[2]}\leq\lambda^{[3]}\leq\cdots$ and associated *marked* PPPs $\Psi^{[i]}$, $\Psi^{[2]}$, $\Psi^{[3]}$, $\cdots$ denoting transmitters, the sequence of aggregate interference seen by a typical receiver $I_{\Phi_M^{[1]}},I_{\Phi_M^{[2]}},I_{\Phi_M^{[3]}},\cdots$,[11] stochastically and monotonically

[11] $\Phi_M^{[i]}$ is a Matérn CSMA point process associated with the original marked process $\Psi^{[i]}$. The relation is explained in detail later.

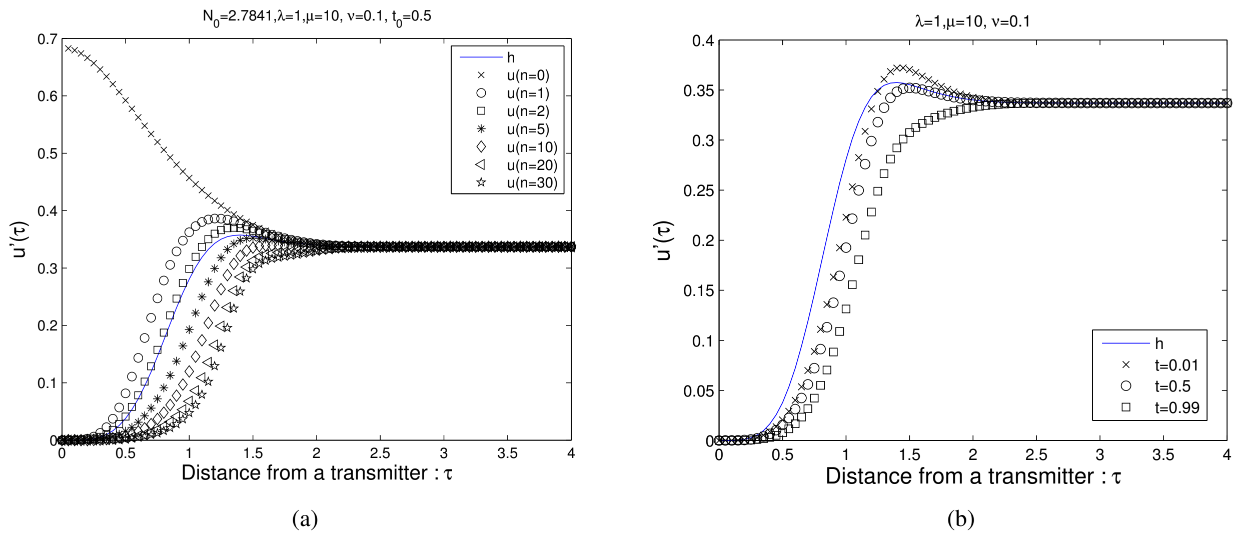

Figure 9: $h(\cdot)$ is a function computed in (11). $u'(\cdot)$ is a function we computed above for $N_0 = n \geq 0$ and $t_0 \in [0, 1)$.

increases, i.e.,

$$I_{\Phi_M^{[1]}} \leq^{st} I_{\Phi_M^{[2]}} \leq^{st} I_{\Phi_M^{[3]}} \leq^{st} \cdots . \tag{83}$$

We will use coupling argument to show this. To that end, we will construct the copy $\Psi^{(n)}$ of each process $\Psi^{[n]}$ and couple them such that following strict inequalities hold with probability 1:

$$I_{\Phi_M^{(1)}} \leq I_{\Phi_M^{(2)}} \leq I_{\Phi_M^{(3)}} \leq \cdots , \tag{84}$$

where $I_{\Phi_M^{(n)}}$ is the associated aggregate interference of $\Psi^{(n)}$ seen at a typical receiver for $n = 1, 2, \cdots$[12]. Note that we use square bracket $[n]$ to denote n-th original process and curly bracket (n) to denote the copy of it. Clearly, (84) implies convergence of $I_{\Phi_M^{(n)}}$ to a random variable, say $I_{\Phi_M^{dens}}$ (possibly infinite). We will then use the results in [45] to show that $I_{\Phi_M^{dens}}$ is almost surely finite.

A. Constructing n-th network $\Psi^{(n)}$

We consider

$$\Psi^{(n)} = \left\{ \left(X_i^{(n)}, E_i^{(n)}, T_i^{(n)} \right) \right\} \tag{85}$$

given $(X_0^{(n)} = 0, E_0^{(n)} = 1, T_0^{(n)}, \mathbf{F}_0^{(n)}, \mathbf{F'}_0^{(n)}) \in \Psi^{(n)}$, with a node density $\lambda^{[n]}$, where $T_i^{(n)} \sim$ Uniform$[0, 1]$ are marks corresponding timer values and

$$E_i^{(n)} = \mathbf{1} \left\{ T_i^{(n)} < \min_{X_j^{(n)} \in \mathcal{N}_i^{(n)}} T_j^{(n)} \right\}$$

is the transmission indicator of node $X_i^{(n)}$. Let $\Phi^{(n)} = \{X_i^{(n)}\}$ and $\Phi_M^{(n)} = \{X_i^{(n)} \in \Phi^{(n)} \mid E_i^{(n)} = 1\}$. Then, the aggregate interference seen by the receiver of a transmitter X_0 is given as $I_{\Phi_M^{(n)}} =$

[12] $\Phi_M^{(i)}$ is a Matérn CSMA process associated with the copied process $\Psi^{(i)}$.

$\sum_{X_j \in \Phi_M^{(n)} \setminus \{0\}} F_{ji}^{(n)} / l(\|X_j - (0,r)\|)$.

B. Constructing coupled timers

We scale the timer values $T_i^{(n)}$ for the n-th network such that it is uniformly distributed in $[0, \frac{\lambda^{[n]}}{\lambda^{[n+1]}}]$, i.e., let $T_{ci}^{(n)} = \frac{\lambda^{[n]}}{\lambda^{[n+1]}} T_i^{(n)}$. Let a PPP with scaled timer values as

$$\Psi_c^{(n)} = \left\{ \left(X_i^{(n)}, E_i^{(n)}, T_{ci}^{(n)} \right) \mid (X_i^{(n)}, E_i^{(n)}, T_i^{(n)}) \in \Psi^{(n)} \right\}. \tag{86}$$

Note that the timer value scaling maintains the order of timer values, so there is no change in E_i. Let $\Phi_c^{(n)} = \{X_i^{(n)}\}$ and $\Phi_{cM}^{(n)} = \{X_i^{(n)} \in \Phi_c^{(n)} \mid E_i^{(n)} = 1\}$, then, the aggregate interference from $\Phi_{cM}^{(n)}$ is the same as $I_{\Phi_M^{(n)}}$, i.e.,

$$I_{\Phi_{cM}^{(n)}} = I_{\Phi_M^{(n)}}. \tag{87}$$

C. Differential PPP

Consider an another marked PPP

$$\hat{\Psi}^{(n)} = \left\{ \left(\hat{X}_i^{(n)}, \hat{E}_i^{(n)}, \hat{T}_i^{(n)} \right) \right\} \tag{88}$$

with density $\hat{\lambda}^{[n]} = \lambda^{[n+1]} - \lambda^{[n]}$ for the given $\lambda^{[n+1]} \geq \lambda^{[n]}$ and $\hat{\Phi}^{(n)} = \{\hat{X}_i^{(n)}\}$ and $\hat{\Phi}_M^{(n)} = \{\hat{X}_i^{(n)} \in \hat{\Phi}^{(n)} \mid \hat{E}_i^{(n)} = 1\}$[13]. By construction we will ensure that $\hat{\Psi}^{(n)}$ is independent of $\Psi^{(n)}$. We let the timer values $\hat{T}_i^{(n)}$ be uniformly distributed on $[\frac{\lambda^{[n]}}{\lambda^{[n+1]}}, 1]$ and have *nodes* $\hat{X}_i^{(n)}$ *to contend nodes with those in* $\hat{\Psi}^{(n)}$ *as well as in* $\Psi^{(n)}$. Let $\hat{F}_{ji}'^{(n)}$ be the fading channel between $\hat{X}_j^{(n)} \in \hat{\Phi}^{(n)}$ and $\hat{X}_i^{(n)} \in \hat{\Phi}^{(n)}$ and $H_{ji}'^{(n)}$ as the fading channel between a transmitter $X_j^{(n)} \in \Phi^{(n)}$ and a transmitter $\hat{X}_i^{(n)} \in \hat{\Phi}^{(n)}$. Then, we can define two different neighborhoods for $\hat{X}_i^{(n)}$: one in $\hat{\Phi}^{(n)}$ given as

$$\hat{\mathcal{N}}_i^{(n)} = \left\{ \hat{X}_j^{(n)} \in \hat{\Phi}^{(n)} \mid \hat{F}_{ji}'^{(n)} > \nu l(\left\| \hat{X}_j^{(n)} - \hat{X}_i^{(n)} \right\|), j \neq i \right\} \tag{89}$$

and the other in $\Phi^{(n)}$ given as

$$\mathcal{M}_i^{(n)} = \left\{ X_j^{(n)} \in \Phi^{(n)} \mid H_{ji}'^{(n)} > \nu l(\left\| X_j^{(n)} - \hat{X}_i^{(n)} \right\|) \right\}. \tag{90}$$

Using above definitions, we can define the transmission indicator as

$$\hat{E}_i^{(n)} = \mathbf{1} \left\{ \hat{T}_i^{(n)} < \min \left\{ \min_{\hat{X}_j^{(n)} \in \hat{\mathcal{N}}_i^{(n)}} \hat{T}_j^{(n)}, \min_{X_k^{(n)} \in \mathcal{M}_i^{(n)}} T_k^{(n)} \right\} \right\}. \tag{91}$$

Note that every node in $\hat{\Phi}^{(n)}$ which contends with at least one node in $\Phi^{(n)}$ defers its transmission since its timer value is always larger than $\frac{\lambda^{[n]}}{\lambda^{[n+1]}}$. Let

$$\Delta I_{\hat{\Phi}_M^{(n)}} = \sum_{\hat{X}_j^{(n)} \in \hat{\Phi}_M^{(n)}} H_{j0}^{(n)} / l(\left\| \hat{X}_j^{(n)} - (0,r) \right\|)$$

be the interference seen by the receiver of the transmitter X_0 in $\Phi_M^{(n)}$ from transmitters only in $\hat{\Phi}_M^{(n)}$, where $H_{j0}^{(n)}$ is the fading channel gain between a transmitter $\hat{X}_j^{(n)} \in \hat{\Phi}^{(n)}$ and the receiver of $X_0 \in \Phi^{(n)}$.

[13] Note that we do *not* condition on $0 \in \hat{\Phi}_M^{(n)}$.

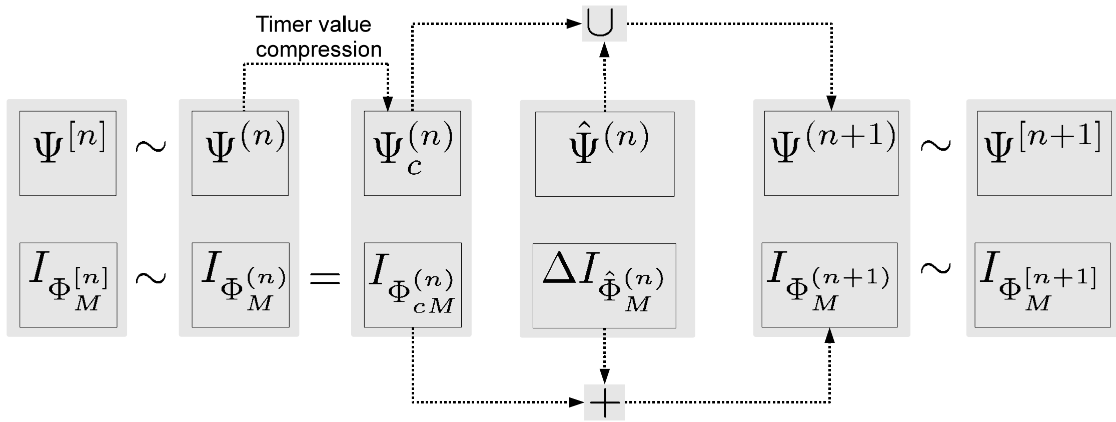

Figure 10

D. *Constructing* $n+1th$ *network* $\Psi^{(n+1)}$

Now, we construct $\Psi^{(n+1)}$ by the union of the the timer scaled point process and the differential point process:

$$\Psi^{(n+1)} = \Psi_c^{(n)} \cup \hat{\Psi}^{(n)}. \tag{92}$$

Note that $\Psi_c^{(n)}$ and $\Psi^{\{n\}}$ contribute $\frac{\lambda^{[n]}}{\lambda^{[n+1]}}$ and $1-\frac{\lambda^{[n]}}{\lambda^{[n+1]}}$ fraction of nodes to $\Psi^{(n+1)}$ respectively. This makes a randomly chosen node in $\Psi^{(n+1)}$ has a timer value uniformly distributed on [0,1]. *Thus,* $\Psi^{(n+1)} = \{(X_i^{(n+1)}, E_i^{(n+1)}, T_i^{(n+1)})\}$ *is indeed a marked PPP with density* $\lambda^{[n+1]}$*, where* $T_i^{(n+1)}$ *is uniformly distributed on [0,1].* Let $I_{\Phi_M^{(n+1)}}$ be the aggregate interference seen by a typical receiver, then, it is given as the sum of the interferences

$$I_{\Phi_M^{(n+1)}} = I_{\Phi_{cM}^{(n)}} + \Delta I_{\hat{\Phi}_M^{(n)}}, \tag{93}$$

since $I_{\Phi_{cM}^{(n)}}$ is independent of nodes in $\Psi^{\{n\}}$, which is the direct result of the timer value separation. Now clearly the two point processes $\Psi_c^{(n)}$ and $\Psi^{(n+1)}$ are coupled such that their aggregate interference satisfy :

$$I_{\Phi_{cM}^{(n)}} \le I_{\Phi_M^{(n+1)}}. \tag{94}$$

Using (87), this can be rewritten as $I_{\Phi_M^{(n)}} \le I_{\Phi_M^{(n+1)}}$, which implies the stochastic dominance relation $I_{\Phi_M^{(n)}} \le^{st} I_{\Phi_M^{(n+1)}}$. Since $I_{\Phi_M^{(n)}} \sim I_{\Phi_M^{[n]}}$ and $I_{\Phi_M^{(n+1)}} \sim I_{\Phi_M^{[n+1]}}$, we have

$$I_{\Phi_M^{[n]}} \le^{st} I_{\Phi_M^{[n+1]}}. \tag{95}$$

Fig.10 summarizes our coupling argument.

E. *Convergence of* $I_{\Phi_M^{[n]}}$

Since increasing random variables converge (possibly to ∞), $I_{\Phi_M^{(n)}}$ (or equivalently $I_{\Phi_M^{[n]}}$) converges to a random variable as $n \to \infty$. Let the converging value be denoted by $I_{\Phi_M^{dens}} \equiv \lim_{n\to\infty} I_{\Phi_M^{[n]}}$. We rewrite the results from [45] in our context as follows.

Proposition 22. *(Existence of shot noise) [45] For a given random variable I, if $\lim_{s\to 0}\mathbb{E}[e^{-sI}] = 1$, then, I has a well-behaved distribution or finite almost surely.*

Note that Φ_M^{dens} is a Matérn CSMA point process with density λ_{dens}, which is stationary and ergodic. Then, we can apply following results from [45].

Proposition 23. *(Necessary and sufficient conditions, Corollary 1.2 in [45]). The necessary and sufficient conditions for the existence of $I_{\Phi_M^{dens}}$ are*

$$\int_0^\infty \int_0^\delta xw(x;y)ydxdy < \infty, \text{ and} \tag{96}$$

$$\left| \int_0^\infty \int_0^\delta w(x;y)ydxdy \right| < \infty \tag{97}$$

for $\delta > 0$, where $w(x;y) = P(F > xy^\alpha) = \exp\{-\mu xy^\alpha\}$.

Using the fact that $x \le \delta$ and $w(x;y) > 0$, it is sufficient to show (97) only, i.e., we have

$$\begin{aligned}\int_0^\infty \int_0^\delta xw(x;y)ydxdy &\le \delta \int_0^\infty \int_0^\delta w(x;y)ydxdy \\ &= \frac{\delta}{\mu}\int_0^\infty y^{1-\alpha}(1-\exp\{-\mu\delta y^\alpha\})dy \qquad (98) \\ &\overset{a}{=} -\frac{\delta^{2-\frac{2}{\alpha}}}{\alpha\mu^{2/\alpha}}\Gamma\left(\frac{2-\alpha}{\alpha}\right) < \infty. \qquad (99)\end{aligned}$$

In $\overset{a}{=}$, we used the results in 370p of [19], which holds for $\alpha > 2$. Using Theorem 2 of [45] and bounding technique, one can also show the existence of the *approximate* of $I_{\Phi_M^{dens}}$, which is a non-homogeneous Poisson shot noise.