A FUZZY DECISION MAKER

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Abstract

One of the most useful aspects of fuzzy set theory is its ability to represent mathematically a class of decision problems called multiple objective decisions (MODs). This class of problems often involves many vague and ambiguous (and thus fuzzy) goals and constraints. The object of the fuzzy decision methodology is to obtain a decision, optimum in the sense that some set of goals is attained while observing (i.e. not violating) a simultaneous set of constraints. This paper presents one possible fuzzy decision-making model developed by L.A. Zadeh and Richard Bellman [1] and later extended by Ronald R. Yager [5], which has proved to be useful to decision makers in many "real-world" problems. The latter fuzzy decision making method is used when the goals or objectives and the constraints are not of equal importance to the decision maker. The methodology for weighting the importances of the decision components is an exponential weighting, which are obtained using a method of paired comparisons developed by T.L. Saaty [3]. The weighting methodology is based on finding the eigenvector associated with the maximal eigenvalue of a paired comparison matrix, which is obtained from simple, binary decisions (i.e. which of two things is more important) and a fuzzyily-guessed-at scale of how much more important (on a scale of 1 to 9) one factor is than another. This last step characterizes the decision maker and properly reflects his/her biases with an easily obtained vector of exponents. A further extension of the decision making methodology was made by Yager [6] in 1981, in which partially ordered sets of fuzzy ratings could be used in the decision process as opposed to cardinal numbers. A simple BASIC program incorporating Yager's first methodology [5] was first published by Whaley [4] in 1979.

Introduction and History

In a seminal paper written in 1965 [7], Lotfi A. Zadeh described the properties of fuzzy sets, a class of objects with a continuum of grades of membership in the interval [0, 1]. This idea stands in stark contrast to conventional set theory in which objects have only membership (characteristic function) values taken from the doubleton set {0, 1}. Each object x in a fuzzy set X is assigned a grade of membership by a membership function usually denoted by $\mu(x)$ whose values range between zero and one. Many people tend to confuse the idea of a membership function $\mu(x)$ with that of a probability density function f(x), however, this is incorrect since the integral of f(x) must sum to 1. There is no such restriction on $\mu(x)$.

Many classes of objects encountered in the physical world of our experience do not have precisely defined criteria of membership, e.g. the class of expensive homes, the class of underpaid engineers, the class of integrated avionics, etc. But, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction. As an example, fuzzy set theory has been used in linear regression and its application to forecasting in uncertain (almost a universal set) environments [2] by Heshmaty & Kandel. The Japanese have been particularly adept at applying fuzzy control theory to everything from subway trains and elevators to VCR-camcorder focus, stabilization, and exposure and shifting automobile transmissions.

One of the better summaries of why fuzzy theory is useful in decisionmaking is given by Yager in [5]: " It must be kept in mind that the use of a fuzzy set does not eliminate the subjective or fuzzy nature of the concepts with which we are dealing..., but it does give us a handle for dealing with subjective concepts in a rational way, in a similar manner as the method used by theBayesian Decision Makers enables them to handle subjective probabilities and utilities."

In the next section, we will define a fuzzy set and examine some of its properties.

Fuzzy Sets

Suppose we have a set of alternatives such as a set of cities which might be destinations for some future travel plans. Let the set of four decision alternatives be denoted by;

$$X = \{ x_1, x_2, x_3, x_4 \} = \{ \text{Philadelphia, Los Angeles, Chicago, Newark} \}$$

representing our potential destination cities of choice. Note that the set X is a conventional or classical set of objects. We can define a fuzzy subset of the set X, call it A, which is characterized by a membership function $\mu_A(x_i)$ associating with each x_i X a number in the interval [0,1] which indicates the grade of membership of x_i in A. Suppose for our example A is a fuzzy subset defined as:

$$A = \{ \text{the city in X is near New York} \} = \{0.6/x_1, 0.001/x_2, 0.1/x_3, 0.9/x_4 \}$$

where the first number in each pair represents the membership value $\mu_A(x_i)$ and the second denotes the individual set member x_i. [Note that the division symbol is merely a notational separator, not an arithmetic operation.] The second number in each pair can be dropped with the understanding that the fuzzy subset A of X is characterized by the individual $\mu_A(x_i)$ and is indexed identically with the elements x_i X. As an example, we have

$$\mu_A(x_3) = \mu_A(\text{Chicago}) = 0.1.$$

Note the increasing membership values as the indexed cities come closer to New York, with Newark having the largest membership. If the set X were enlarged to include Queens and Brooklyn as x_5 and x_6, they would have membership values of 1.0, since both are not only near New York, they are suburbs within New York and thus are contained in the target city characterizing the fuzzy set A.

Note that the concept of a fuzzy set includes that of a classical set as a special case by using a step function as the fuzzy membership function. For example let the fuzzy set A defined on the set X of integers between 1 and 5, be defined as the set of integers greater than three. Thus we have:

$$X = \{ 1, 2, 3, 4, 5 \} \text{ and } A = \{ 0/1, 0/2, 0/3, 1/4, 1/5 \}.$$

The integers 1, 2, and 3 don't belong to A at all (membership value zero) and the integers 4 and 5 belong absolutely to A (membership values of one). The fuzzy subset A is actually not fuzzy at all, it is "crisp" with a membership (characteristic function) like classical sets.

Operations on Fuzzy Sets

At this point in the discussion, all subsequent fuzzy subset descriptions will drop the index following the "division" symbol and merely use ordered lists of membership values to characterize the fuzzy subset. In this "reduced" notation, fuzzy set A of the first example would become :

A = { 0.6, 0.001, 0.1, 0.9 }.

Complementation

One reasonable operation to consider on a fuzzy set would be complementation. For example, how would you describe the complementary set to A of the first example above? Probably, it would seem reasonable to consider the complement in English of the word "near", which is "far". Given that we know how to characterize the fuzzy set A, its complement sometimes denoted as A' or A, is easily computed as having membership values of 1-μ_A(). Thus given A above we would characterize its fuzzy complement A' as:

A' = { the city in X far from New York } = { 0.4, 0.999, 0.9, 0.1 }.

The membership values of the complementary set A' are just 1 - the corresponding μ_A() membership values. Los Angeles, the x_2 element, has a high membership in the fuzzy set A' defined as being far from New York.

As an example with a more or less continuous index, consider the membership function for the fuzzy set defined by tall men, whose index set is height in inches. The complementary set would be the fuzzy set of short men. In the fuzzy set of tall men, George Bush at 6'2" would have a 0.7 membership, a professional basketball player at 7'2" would have say a 0.98 membership value. In the fuzzy set of short men, these same men would have membership values of 0.3 and 0.02, respectively.

Union and Intersection

The union operation on fuzzy sets A and B defined on X, denoted as A≈ B, is the fuzzy set C with membership values that are the "MAX" (or maximum) of the component values. This operation is defined only if the index sets of A and B have the same cardinality. Notationally, we have:

C = A ≈ B

where

$$\mu_C(x) = \max[\ \mu_A(x), \mu_B(x)\ ].$$

This operation is the fuzzy analog to the normal Boolean logic operation of "OR". As an example, let A and B be fuzzy sets defined by the respective arrays of membership values defined on the set X, representing an attack aircraft's sensor suite. There are three such suites, so X, A, and B are:

X = { x1, x2, x3 } and A = { 0.6, 0.5, 0.8 } and B = { 0.7, 0.3, 0.9 }.

The fuzzy set A might represent, for example our best estimate of the relative memberships of the three sensor suites in the class of "inexpensive" electronics sensor suites. The fuzzy set B might represent our best estimate of relative membership values in the class of "efficient battlefield detectors". The fuzzy set C then represents the set of cheap "OR" efficient detectors, and has membership values of:

C = { 0.7, 0.5, 0.9 }.

The fuzzy max or "OR" operation is related to the optimism of the decision maker. In effect, it is a "best of all possible worlds" (optimistic in the sense of Leibnitz) or polyanna approach to decision making.

Suppose we wanted our sensor suite to be both inexpensive and a good detector. This would lead us very naturally to consider finding a fuzzy analog to the English and Boolean logic idea of "AND". As you might already guess, the fuzzy intersection denoted $\mathbf{A} \cap \mathbf{B}$ is a "MIN" (or minimum) operator on the membership values of A and B. As above, this intersection operation is only defined if the cardinality (count) of the index sets of A and B agree. Notationally, we have:

$$\mathbf{D} = \mathbf{A} \cap \mathbf{B}$$

where

$$\mu_D(x) = \min[\ \mu_A(x), \mu_B(x)].$$

Thus continuing our example, the set of "inexpensive and efficient" sensor suites would be characterized by the fuzzy set D whose membership values are given by:

D = { 0.6, 0.3, 0.8 }.

If we were making a decision about which sensor suite to choose, then an obvious choice would be to buy suite x_3 since it best satisfies our goals (i.e., has the highest membership value). It was exactly this type of reasoning, that lead Zadeh and Bellman [1] to their formulation of a proper approach to decision making in a fuzzy environment, which will be described and expanded upon below. Before leaving operations on fuzzy sets, we will consider how one might exponentially weight such a set with a scalar value. This operation will come in handy when we consider the relative importances of the factors entering a fuzzy decision.

Exponentiation

Let A be a fuzzy set defined on a set X and let α be a scalar value (for all the cases of interest to us the α's will be > 0). The concept of raising the fuzzy set A to the power α, denoted by A^α is a fuzzy set E defined as:

$$E = A^\alpha$$

where

$$\mu_E(x) = [\mu_A(x)]^\alpha$$

for all x X. Using the same set A of inexpensive sensor suites characterized above, if α = 2, corresponding to the English language quantifier "very", we might characterize the membership values of the three suites in the class of "very inexpensive" (real cheap) sensor suites as:

A^2 _ CON(A) = { 0.36, 0.25, 0.64 }.

This is an example of the ability of fuzzy sets to map from English language quantifiers into quantitative measures. The α of value 2, when applied in this fashion, is called a "concentrator" (sometimes referred to as CON()) when this value is used to exponentiate a fuzzy set. An α value of 0.5 (square root operation) is called a "dilator" (sometimes referred to as DIA()) and corresponds to the English quantifier "sort of" or "kind of" or "not very".

$$DIA(A) \_ A^{0.5} = \{ 0.7746, 0.7071, 0.8944 \}.$$

Thus the set DIA(A) represents the fuzzy set membership values in the set of "kind of" inexpensive sensor suites.

Note that if $\alpha > 1$, the effect of exponentiation of the fuzzy set A is to reduce the grades of memberships of all the x's since the individual fuzzy values fall between 0 and 1. This effect is less marked on large initial fuzzy membership values than on small ones. For example, $0.9^2 = 0.81$, whereas $0.1^2 = 0.01$. This fact was used by Yager [2] when he extended the work of Zadeh and Bellman [1] to consider exponentially weighting fuzzy sets to reflect the relative importances of the factors being represented by the fuzzy sets in a decision making process. Important factors are exponentially weighted heavily to force their entry into the decision (an intersection or MIN) process. This idea will be expanded below. Correspondingly, small values of the exponent $0 < \alpha < 1$, tend to make the membership values larger, which effectively takes them out of the intersection used in the decision making process. Now that we have some tools, let's explore the area of decision making in some more detail. The development will follow the historical order in which they were developed over the last two decades.

Decision Theory and the Application of Fuzzy Sets to the Process

In the last example given above, we touched on the essence of a common class of decision problems. The decision maker is often faced with the problem of selecting among a set (usually finite) of alternatives while simultaneously satisfying a set of objective criteria (goals) and observing (not violating) a set of constraints. The main contribution of Bellman and Zadeh [1] to this theory was in recognizing that a "good" decision had to satisfy both goals and constraints, and that for decision purposes, they should be treated alike. That is to say, that a "good" decision had to satisfy some conjunctive form of goals and constraints associated with the decision making environment. An optimal decision would be one that "best" satisfied all the criteria in some sense. In their 1970 [1] paper Bellman and Zadeh suggested a rule of implied conjunction with m goals and n constraints of the problem stated as an ordinary conjunctive "AND" form evaluated for a given alternative x as:

$$\mathbf{G_1}(\mathbf{x}) \cap \mathbf{G_2}(\mathbf{x}) \cap \ldots \cap \mathbf{G_m}(\mathbf{x}) \cap \mathbf{C_1}(\mathbf{x}) \cap \mathbf{C_2}(\mathbf{x}) \cap \ldots \cap \mathbf{C_n}(\mathbf{x}),$$

where $G_i(x)$ is the fuzzy value of the ith goal or objective evaluated for decision alternative x and $C_j(x)$ is the fuzzy value associated with the satisfaction of the jth constraint by decision alternative x. Note that if this form is used, the implication is that all of the goals and constraints are of equal importance We are thus impartial about satisfaction of any of the goals or constraints. The obvious thing to do now is to associate this conjunction, which in this form we know to be equivalent to a MIN operation on all the fuzzy values, with a new fuzzy set, which we will call D(x) for "decision". Note that the choice of D used in the example above defining fuzzy intersection was not an accident! What we are doing is creating the definition of a new fuzzy set whose membership values can be used to make an optimal decision. One should also note that the fact that both the objectives and constraints have been mapped into the same fuzzy [0,1] space is in stark contrast to the usual operations research approach to optimization where the objective values are used to direct the optimization process and the constraints are used to control the region searched. So in terms of fuzzy sets, our decision fuzzy set D becomes:

$$D(x) = G_1(x) \cap G_2(x) \cap \ldots \cap G_m(x) \cap C_1(x) \cap C_2(x) \cap \ldots \cap C_n(x)$$

Thus, D(x) is a fuzzy set defined over the alternatives $x \in X$. As optimal decision makers, we then select the alternative x*, that has the highest fuzzy membership in D as our decision. Thus x* satisfies the set of objectives (goals) and constraints best. Mathematically this would be stated as:

$$x^* = \arg\left\{\max_{x \in X} D(x)\right\}$$

that is x* is the argument or index value when D(x) achieves its largest value. We're now at roughly the level of decision making reached two decades ago. Let's see how it works!

Impartial Decisions for Equally Weighted Goals and Constraints

As an example of the original Bellman and Zadeh approach to fuzzy decision making, we will use an example taken from the electronics domain of high speed data bus [HSDB] design. In this design task, we are trying to choose the appropriate communications protocol from among three candidates: CSMA/CD, MAADS, and Token Passing [TP] without further explanation. [We are trying to describe fuzzy decision making, not the subtleties of electronic data bus design!] Suppose further, that these communication protocols were being evaluated within a hierarchy of system performance criteria including such factors as 1) system integrity, 2) throughput response, 3) message structure, 4) flexible network control strategy, 5) cost/complexity, 6) adaptiveness, and 7) fault tolerance. For this example, we shall consider a further breakdown (at the next level in the hierarchy) of fault tolerance into four lower level criteria (of equal importance): fault detection [FD], fault containment [FC], fault isolation [FI], and reconfiguration [RC]. We wish to choose impartially the "best" protocol among the three proposed candidates. In somewhat more detail, we have four objectives or goals to be considered in choosing among the three candidate protocols:

(1) the protocol should be able to determine the occurrence of erroneous operation (i.e., fault detection [FD]),

(2) the protocol should be able to measure the extent to which the system prohibits error and/or failures from propagating from the source throughout the system, the so-called "ripple effect" unfortunately often present in "automatically adapting" busses (i.e., fault containment [FC]),

(3) the protocol should be able to isolate a failure to the required level in order to be able to reconfigure with the failed component disabled or switched off (i.e., fault isolation [FI]),

(4) the protocol should provide mechanisms to be employed to reconfigure the system after failure detection and isolation including reassigning processing tasks from one element to another and the reassigning data flow paths (i.e., reconfiguration [RC]).

Thus the set X for our example is given by:

X = { CSMA/CD, MAAS, TP }.

The goals of the decision to be made are given by :

G_1 = FD, G_2 = FC, G_3 = FI, and G_4 = RC.

There are no constraints given explicitly, just goals present for this decision making problem. Suppose the candidate protocols were evaluated by a competent committee of bus designers with respect to the four objectives with the following (actually meaningless, except as an example of the methodology) results:

G_1 = { 0.5, 0.7, 0.3 } = fuzzy satisfaction of FD capability,
G_2 = { 0.5, 0.4, 0.8 } = fuzzy satisfaction of FC capability,
G_3 = { 0.2, 0.01, 0.6 } = fuzzy satisfaction of FI capability,
and G_4 = { 0.6, 0.4, 0.9 } = fuzzy satisfaction of RC capability.

Following the decision making methodology outlined above, we form the decision function membership function as a conjunction (a hard "AND") of the goals as:

$$D(x) = G_1 \cap G_2 \cap G_3 \cap G_4 = \min[G_1, G_2, G_3, G_4] = \{0.2, 0.01, 0.3\}$$

The now obviously best solution is given by:

$$x^* = TP = \arg\{\max_x D(x)\}.$$

The best solution is TP (token passing protocol) because it satisfies the decision with the highest value (arg max), and is in fact, the max-min solution. Remember that the procedure presented thus far is under the condition that all the objectives and constraints (if present) are equally important, which unfortunately is almost never the true state of affairs. Before proceeding, let us make some observations about fuzzy decisions.

Based upon the definitions of the intersection operation in fuzzy sets given above, we can observe the following:

An optimal decision for indifferent goals and constraints is made by:

(1) selecting for each alternative x, its smallest membership value in and of the objectives or constraints (the decision function is a fuzzy intersection), and

(2) selecting as the optimal decision, the alternative with the highest membership in the decision fuzzy set.

Decisions When Goals and Constraints are of Varying Importance

If a particular goal or constraint is of great importance, we want to be very unlikely to select an alternative as our solution that has a small membership value in this particular goal or constraint. This can be accomplished by making those alternatives that are low in important objectives have a low membership in D, the decision fuzzy set. This would minimize the chance of their being selected as the best alternative. From statement (1) in the previous paragraph above, we can conclude that the membership function for each alternative in D is determined by its lowest membership in all the objectives (from the MIN definition of fuzzy intersection). Therefore, if we make the grade of membership of alternatives that are low in some important goal or constraint satisfaction even lower, they will be even less likely to be selected as the optimal alternative, x*. This is the point in his thought process where Ronald Yager was in 1976 [5] when he came up with the idea of using exponential weighting to accomplish this differential weighting. This form of weighting helped make the fuzzy decision process more responsive to real-world needs.

Having explored previously the effects of raising a fuzzy set to a scalar power, we can see that if some method could be found to assign an appropriate value to α, indicative of a particular goal's or constraint's relative importance to the decision maker, we could obtain the desired effects on the overall decision. If we associate with each objective an $\alpha > 0$ (the more important the objective or constraint, the higher the associated α) and then consider our revised (weighted) decision function as:

$$D(x) = \{G_1^{\alpha_1}(x) \cap G_2^{\alpha_2}(x) \cap \ldots \cap G_m^{\alpha_m}(x) \cap C_{m+1}^{\alpha_{m+1}}(x) \cap C_{m+2}^{\alpha_{m+2}}(x) \cap \ldots \cap C_{m+n}^{\alpha_{m+n}}(x)\}$$

$$= \underset{\substack{i=1,m \\ j=m+1,m+n}}{\mathrm{MIN}} \left[G_1^{\alpha_1}(x), G_2^{\alpha_2}(x), \ldots, G_m^{\alpha_m}(x), C_{m+1}^{\alpha_{m+1}}(x), C_{m+2}^{\alpha_{m+2}}(x), \ldots, C_{m+n}^{\alpha_{m+n}}(x)\right].$$

Suppose that we also constrain the sum of the α's to be m+n, the total number of goals and constraints. This is to force the exponential weights to be all 1 if the unlikely case occurs that all objectives and constraints are of equal importance (the previous case of decision making). Note that with this scheme, alternatives that are weak in important goal or constraint satisfaction become even less appealing as potential optimal solutions. This is consistent with the linguistic characterization given above in the sense that the more stringent we are in enforcing our objective or constraint criteria, the more important it is. Recall the fuzzy set of "very" inexpensive or real cheap sensor suites given above. If cost was an important issue then an α value of 2 is not unreasonable as an exponential weight to reflect its importance.

We now have a hierarchical system for decision making in which each alternative is first rated on its ability to satisfy each of the objectives and constraints and then each objective or constraint is modified by the exponential weighting to reflect its "true" importance to the decision maker.

Our next problem is that of obtaining a scale upon which to measure the relative importance of each goal or constraint. Fortunately this method [3] was developed by Thomas L. Saaty, a famous operations research (the English, who named the discipline, call it "operational" research) contributor about this same time in 1976. Yager was quick to borrow the methodology for his decision theory paper [5]. Good ideas are often useful in more than one specific setting!

A Method for Comparing the Relative Importance of Fuzzy Sets

T. L. Saaty developed a procedure for obtaining a ratio scale for a group of elements based upon a complete paired comparison of the elements taken two at a time (n(n-1)/2 comparisons for n elements). This method was used by Yager [5] in his 1977 paper to obtain exponential weighting values which properly reflect the relative importance of the objective criteria and constraints entering a decision problem.

Assume we have p objects, and we want to construct a scale rating these objects with respect to each other. The objects could for example be the objective criteria and/or constraints characterizing a decision problem, in which case p=m+n. The decision maker compares the objects two-at-a-time (paired comparison). When comparing object i with object j, the decision maker is asked first to make a binary decision, which object is more important? Having made that decision, he/she is then asked to assign a value taken from the scale 1 to 9 to the more important objects domination over the less important object. These scale values are given in Table 1 below with verbal hints on how to apply them. If object i dominates object j, the assigned value is denoted as a_{ij}. The paired comparison matrix has an interesting reciprocal property given by:

$$a_{ij} = 1/a_{ji}.$$

This property allows easy generation of a paired comparison matrix of dimension p by p, B, whose elements are

$b_{ii} = 1$, $b_{ij} = a_{ij}$ and $b_{ji} = 1/ b_{ij}$ for i _ j.

Saaty showed that the eigenvector corresponding to the maximum eigenvalue associated with B is a cardinal ratio scale for the elements compared. Yager [5] multiplied the normalized eigenvector by the order of the system p to obtain exponents for weighting the fuzzy criteria in the decision. This make the exponents have value one if the factors are equally important.

Table 0-1 - Judgement Scale

Importance Value	Definition
1	Equal Importance.
3	Weak importance of one over the other.
5	Strong importance of one over the other.
7	Demonstrated importance of one over the other.
9	Absolute domination of one over the other.
2, 4, 6, 8	Intermediate values between the two adjacent judgements..

As an example of the methodology using paired comparisons, suppose three sensors, say X, Y, and Z were being rated on a scale evaluating their importance in a mission equipment package (MEP) with values taken from Table 1 above.

Y is weakly more important than X; then $a_{12} = 1/3$ and $a_{21} = 3$.

Z is somewhere between equal and weakly more important than X; then $a_{13} = 1/2$ and $a_{31} = 2$.

Y is weakly more important than Z; then $a_{23} = 3$, $a_{32} = 1/3$,

thus our matrix B of paired comparisons is given by:

$$B = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix} \begin{matrix} X \\ .Y \\ Z \end{matrix}$$

We then solve the eigen value/eigen vector problem:

$$BW = \lambda_{max} W$$

and obtain the unit (normalized) eigenvector corresponding to λ_{max} for this matrix equation. For the above example,

$$W = \begin{bmatrix} 0.16 \\ 0.59 \\ 0.25 \end{bmatrix}.$$

Using Saaty's method we have obtained a cardinal rating (or if you prefer, a fuzzy possibility estimate) of the relative importances of the three sensors of the MEP as judged by the decision maker (or team of decision makers). Note that sensor Y appears to be very important relative to Z and especially to X. Note that this is just one of many ways that fuzzy membership functions or possibility distributions can be obtained.

This same methodology can be used to compare the importance of fuzzy constraints or goals in the multi-objective decision problem. We can compare the objective criteria to obtain a measure of relative importance based on paired comparison with its peers. However, instead of using the unit eigenvector, we use the eigen vector E, a scaled version of W, because we want the average importance to be one. This insures that in the unlikely case that all the objective goals and constraints are of equal importance, the α_i's and/or α_j's will equal one, thus having no effect on the $G_i()$ and/or $C_j()$ values entering the decision function, D(). Thus, if we have m goals and n constraints to satisfy with our decision, the dimension p used for scaling the weights is equal to m plus n. The resulting scaled weight vector A' is given by:

$$A' = [\ \alpha_1\ \alpha_2\ \cdots\ \alpha_p\ ] = pW' = [\ pw_1\ pw_2\ \cdots\ pw_p\ ],$$

where W is the unit eigenvector corresponding to the largest real eigen value, λ_{max}, obtained from the paired comparison matrix B. Fortunately, from Perron's Theorem in matrix algebra, we are guaranteed that such a real, maximum eigen value exists. The theorem states that if B is a matrix with strictly positive entries, then B has a simple positive eigenvalue λ_{max} which is not exceeded in absolute value by any of its (complex) eigenvalues and that every (row or column) eigen vector corresponding to λ_{max} is a constant multiple of an eigenvector with strictly positive entries. In our case, B has 1's down the main diagonal and entries that are either positive values (1 to 9) or their reciprocals (also positive). Thus, we are guaranteed the ability (with sufficient mathematical skill) to find an appropriate weighting vector based on λ_{max}, which we may then scale for our decision making process.

A Decision Example With Varying Importance of Goals and Constraints

Assume, as in the previous example, that there are three bus protocol candidates being considered using the same four objective criteria as before. Now, however, assume that each of the objectives are of differing degrees of importance. Suppose that a qualified committee of experts met and compared each of the four objectives with each other using the judgement scale of Table 1 with the results shown in Table 2 below.

Table 0-2 - Committee Ratings of the Criteria for Bus Protocol

Pair Presented (1st or 2nd)	**Choice(1st or 2nd) &Rating(Tab.1)**
Flt. Detection[FD]--Flt Containment[FC]	1,5
Flt. Detection[FD]--Flt. Isolation[FI]	1,3
Flt. Containment[FC]--Flt. Isolation[FI]	2,7
Flt. Detection[FD]--Reconfiguration[RC]	1,1
Flt. Containment[FC]--Reconfiguration[RC]	2,7
Flt. Isolation[FI]--Reconfiguration[RC]	2,3

Thus with m=4 (the number of goals) and n=0 (the number of constraints), the order of the paired comparison matrix is p =m+n = 4. The resulting matrix B of paired comparisons is given by:

$$B = \begin{bmatrix} 1 & 5 & 3 & 1 \\ \frac{1}{5} & 1 & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{3} & 7 & 1 & \frac{1}{3} \\ 1 & 7 & 3 & 1 \end{bmatrix}, \begin{matrix} FD \\ FC \\ FI \\ RC \end{matrix}$$

where the row and column index variables are FD, FC,FI, and RC , respectively. Note the 1's on the main diagonal and the reciprocal character of the off-diagonal entries. The next task required in the decision making process is to solve for the maximum eigen value, λ_{max}, of B and for its associated column unit eigen vector, W. This can be easily by many math packages currently available on PCs. The result of the process is :

$$\lambda_{max} = 4.22.$$

The associated unit (column) eigen vector is:

$$W = \begin{bmatrix} 0.37 \\ 0.05 \\ 0.19 \\ 0.39 \end{bmatrix} \begin{matrix} FD \\ FC \\ FI \\ RC \end{matrix}.$$

The W vector can be interpreted as implying that, to the decision making committee, FD and RC were about equally important with RC slightly more important. FC was the least important factor, and FI the next least important factor. Remember that this ordering of relative importances is a reflection of the decision making entity. Depending on the competence of the decision maker or the committee of decision makers, the resulting values may, or may not, reflect reality.

The W values must now be scaled for use as exponents in exponentially weighting the goals fuzzy values by performing a scalar multiplication of W by p (=4) to produce the alpha vector of exponents. Thus the alpha vector of exponential weights is given by:

$$A = \begin{bmatrix} 1.48 \\ 0.20 \\ 0.74 \\ 1.58 \end{bmatrix} \begin{matrix} FD \\ FC \\ FI \\ RC \end{matrix}.$$

Using these weighting values, the decision function becomes:

$$\begin{aligned} D(x) &= \{G_1^{\alpha_1}(x) \cap G_2^{\alpha_2}(x) \cap G_3^{\alpha_3}(x) \cap G_4^{\alpha_4}(x)\} \\ &= \min\left[G_1^{1.48}(x), G_2^{0.20}(x), G_3^{0.74}(x), G_4^{1.58}(x)\right]. \end{aligned}$$

Substituting the fuzzy values from the previous equally-weighted decision example reflecting how well each of the three protocols satisfied the four goals, we have:

$$\begin{aligned} D(CSMA/CD) &= \{\min(0.5^{1.48}, 0.5^{0.20}, 0.2^{0.74}, 0.6^{1.58})\} \\ &= \{\min(0.36, 0.87, 0.30, 0.45)\} \\ &= 0.30. \end{aligned}$$

The limiting factor on CSMA/CD was its relatively low performance on satisfying goal 3, FI. Similarly for the other two decision candidates we can evaluate:

$$\begin{aligned} D(MAADS) &= \{\min(0.7^{1.48}, 0.4^{0.20}, 0.01^{0.74}, 0.4^{1.58})\} \\ &= \{\min(0.59, 0.83, 0.03, 0.24)\} \\ &= 0.03, \end{aligned}$$

and

$$\begin{aligned} D(TP) &= \{\min(0.3^{1.48}, 0.8^{0.20}, 0.6^{0.74}, 0.9^{1.58})\} \\ &= \{\min(0.16, 0.95, 0.69, 0.85)\} \\ &= 0.16. \end{aligned}$$

Note that the limiting factor on TP was its low rating on the relatively important goal of FD, in contrast to MAADS and CSMA/CD that were most limited on FI, a third level (in importance) criteria. Thus, the optimal decision is given by:

$$x^* = \text{CSMA/CD} = \arg\{\max_x D(x)\}.$$

With the relative importance weighting, our optimal decision has changed from the equal importance decision value of TP to the goal-weighted importance optimal decision of CSMA/CD bus protocol.

Ordinal Set Decision Making

In many decision making problems, the decision maker may not have information available to allow determination of the fuzzy satisfaction values associated with the goals and constraints. However, the decision maker might be able to order satisfaction of goals and constraints in some crude (coarse-grained) set of values such as:

{poor, barely OK, satisfactory, good, excellent }.

In a 1981 paper following the one discussed above, Yager [6] discussed just such an ordinal (as opposed to cardinal) method of decision making. The method is predicated on the fact that the truth set for boolean exponentiation of a boolean variable is the same as that for material implication at the extreme points. This allows the decision making setting to be generalized to a lattice (a mathematical domain with a partial ordering defined and containing max and min elements along with some set operations like union and intersection). One good result from this lack of precision is that the calculations can be done on the back of a convenient envelope with the relative importances taken from the partially ordered set of values rather than from an eigen vector plus normalization calculation. Similarly the fuzzy values for rating the goal and constraint satisfaction are drawn from the same partially ordered set of fuzzy ratings. With the substitution that:

$$a^b \equiv (b \Rightarrow a) = b' \vee a = \max(b', a)$$

everything described about multi-objective decision making above can be used. For partially ordered sets, complementation (as indicated by b') is particularly easy and only involves an inversion of the subscript indexing the list. For example, the complement of "barely OK" is "good", and the complement of "satisfactory" is itself. Also values are considered larger (useful in finding the max) if the subscript defining their position in the list is larger.

Aggregation

As you may have already noticed in the two examples of decision making described above, the hard "and" conjunction defined by MIN is really quite severe. This conjunction is really equivalent to a "pessimistic" or "worst-case" (survival-above-all) decision maker. Most of us fall somewhere between a "polyanna" type optimist (who would use MAX instead of MIN for a conjunction) and the "pessimal" MIN type decision maker. This brings up the possibility of aggregating the fuzzy results of goal/constraint satisfaction to somehow reflect the relative pessimism or optimism of the particular decision maker or decision making committee. One interesting proposal towards this goal was made by Zimmermann and Zysno [8,9] called the "compensatory and". The concept of compensation by appropriate aggregation perhaps needs further explanation. As noted above, the logical hard fuzzy "and" corresponding to MIN does not allow higher degree of belonging in one criteria to offset or compensate for low values of belonging in another. In [9], Zimmermann lists some eight criteria as appropriate for selecting aggregation operators including: axiomatic strength, empirical fit, adaptability, compensation, and numerical efficiency. Thus, the problem of choosing an appropriate aggregation operator can itself be viewed as a multiple objective fuzzy decision making problem. Using γ as an optimism index ($\gamma = 1$ implies total optimism), the following formula from [8, 9] can be used to aggregate decision evaluation criteria, but unfortunately it is quite sensitive to memberships near 0 or 1 due to the products and complements involved.

$$\mu_{comp}(x) = \left[\prod_{i=1}^{p} \mu_i(x)\right]^{(1-\lambda)} \left[1 - \prod_{i=1}^{p}(1-\mu_i(x))\right]^{\lambda} \quad \begin{array}{l} x \in X \\ 0 \le \lambda \le 1 \\ 0 \le \mu_i(x) \le 1 \end{array}$$

Yager in [9] had earlier proposed another aggregation operator based on a Minkowski metric. In more recent times, Yager in [10, 11] proposed an entirely new form of aggregation based on ordered weighted averages (OWAs) based on fuzzy quantifiers. His work was extended in a series of papers [12, 13, 14] by O'Hagan using the solution of a well-formed nonlinear programming problem guided by a maximum entropy objective to obtain a new type set of ordered averaging weights called maximum entropy ordered weighted averages (ME-OWAs). This methodology has proved useful in several real world problems of decision making and design including being extended to handle the propagation of uncertainty in expert systems [13] and the design of a fuzzy neuron [14].

Observations and Conclusions

Making a decision when faced with several alternatives, which initially appear equally good or desirable, can be a time consuming and often painful process. The algorithms described here and in the references overcome the (human) memory and processor limitations by allowing the decision maker to selectively evaluate small amounts of the necessary information at any one time (i.e., the fuzzy values of goal and constraint satisfaction and simple, one-at-a-time paired comparisons). Then, when it becomes necessary to evaluate all the pertinent data, the computer, calculator, or back of an envelope can be utilized to perform the decision task in a straight forward manner. The method incorporates several crucial concepts in multi-objective decision making: first, the idea of comparing objectives or constraints with regard to their relative importance to the decision maker incorporating an ability to account for trade-off effects between the criteria; second, the manner in which the relative importance of each decision criterion is included in the model corresponds to a hierarchical structure in the sense that each of the fuzzy sets corresponding to satisfaction of a criterion can be evaluated by various experts and then several criteria can be compared by a next higher level in the decision process, etc.; and third, the methodology is basically a max-min procedure operating over the fuzzy criteria evaluations, but it may be extended to more properly reflect the optimism of the decision maker by use of an appropriate aggregation operator.

References

[1] Bellman,R.E. and L.A. Zadeh.1970."Decision Making in a Fuzzy Environment," *Management Science*, **17**,141-164.

[2] Heshmaty, B. and Kandel, A. (March 1985)."Fuzzy linear regressions and its applications to forecasting in uncertain environments,"*Fuzzy Sets and Systems*, **15**:2,159-191.

[3] Saaty, T.L. 1977. "A Scaling Method for Priorities in Hierarchical Structures,"*J. Math. Psychology*, **15**,234-281.

[4]Whaley, C.P. (November 1979). "Fuzzy Decision Making,"*Interface Age*, 87-91.

[5] Yager, R.R. 1977."Multiple Objective Decision-Making Using Fuzzy Sets," *Intl. J. Man-Machine Studies*, **9**, 375-382.

[6] Yager, R.R. 1981. "A New Methodology for Ordinal Multi-Objective Decisions Based on Fuzzy Sets," *Decision Sciences*, **12**:4,589-600.

[7] Zadeh. L.A.1965."Fuzzy Sets,"*Information and Control*, **8**, 338-353.

[8] Zimmermann, H.J. and P. Zysno. 1980. "Latent Connectives in Human Decision Making,"*Fuzzy Sets and Systems*, **4**,37-51.

[9] Zimmermann, H.J., 1985. Fuzzy Set Theory and Its Applications, Kluwer•Nijhoff Publishing, Boston, MA.

[10] Yager, R.R. 1987. "On the Aggregation of Processing Units in Neural Networks,"*Proc. Intl. Conf. on Neural Networks*, 2, 327-333, San Diego, CA.

[11] Yager, R.R. 1988. "On Ordered Weighted Averaging Aggregation Operators in Multi-criteria Decisions,"*IEEE Trans. on Systems, Man, and Cybernetics*, **8**:1,183-190.

[12] O'Hagan, M. 1987. "Fuzzy Decision Aids," *Proc. 21st Ann. Asilomar Conf. on Signals, Systems, and Computers*, **2**, 624-628, IEEE and Maple Press, Pacific Grove, CA.

[13] O'Hagan, M. 1988. "Aggregating Template or Rule Antecedents in Real-Time Expert Systems with Fuzzy Logic," *Proc. 22nd Ann. Asilomar Conf. on Signals, Systems, and Computers*, **2**, 681-689, IEEE and Maple Press, Pacific Grove, CA. (Accepted for publication in *IEEE Trans. on Systems, Man, and Cybernetics*.)

[14] O'Hagan, M. 1990. "A Fuzzy Neuron Based on Maximum Entropy Ordered Weighted Averaging," *Proc. 24th Ann. Asilomar Conf. on Signals, Systems, and Computers*, **2**, 618-623, IEEE and Maple Press, Pacific Grove, CA.