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Network Pricing: How to Induce Optimal Flows Under Strategic Link Operators

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Network pricing games provide a framework for modeling real-world settings with two types of strategic agents: operators of a network and users of the network. Operators of the network post a price so as to attract users and maximize profit; users of the network select routes based on these prices and congestion from other users.

Motivated by the fact that an equilibrium in these games may not exist, may not be unique and may induce an inefficient network performance, our main result is to observe that a simple regulation on the network owners market solves all these three issues. Specifically, if an authority could set appropriate caps (upper bounds) on the tolls (prices) operators can charge, then: the game among the link operators has a unique and strong Nash equilibrium and the users' game results in a Wardrop equilibrium that achieves the optimal total delay. We call any price vector with these properties a great set of tolls and investigate the efficiency of great tolls with respect to the users' surplus. We derive a bicriteria bound that compares the users' surplus under great tolls with the users' surplus under optimal tolls.

Lastly, we consider two different extensions of the model. First, we assume that operators face operating costs that depend on the amount of flow on the link, for which we prove existence of great tolls. Second, we allow operators to own more than one link. In this case, we prove that when operators own complementary links (i.e., links for which an increase in toll value may only increase the flow on the other owned links), any toll vector that induces the optimal flow and that is upper bounded by the marginal tolls is a great set of tolls, and furthermore show that when all links in the network are complementary, then the aforementioned toll vector is also a strong cap equilibrium.

Key words: network pricing games, competition regulation, price caps, selfish routing

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1. Introduction

Network pricing games provide a framework for modeling real-world settings with two types of strategic agents: operators of a network and users of the network. Operators of the network post a price for usage of the links they own so as to attract users and maximize profit, while users of the network select routes based on these prices and congestion from other users. A landmark example of this type of interaction, that sets the ground in which we describe our work, occurs in transportation networks. Here, link owners are road operators and may toll the usage of their road. Users are travelers that seek to minimize their travel time plus payments. The challenge in these games is that there are two levels of competition: one, among the owners to attract users to their link so as to maximize profit; and second, among users of the network to select routes that are cheap yet not too congested.

In the absence of self-interested link owners, these games reduce to the well-studied network congestion games—a widely accepted and practically useful model for selfish resource allocation in transportation and communication networks (see, e.g., Beckmann et al. 1956, Roughgarden 2005, Correa and Stier-Moses 2011, and references therein). In congestion games, self-interested users travel in a network from their origin to their destination with the goal of minimizing their own travel cost. The natural solution concept of the game is the so-called Wardrop equilibrium flow, under which all users route along shortest paths, given the strategic choices of other users. We assume that the total amount of traffic is dependent on the disutility the users experience, a model also known as congestion games with elastic demand.

Since selfish behavior usually drives systems to socially inefficient situations, a central authority is typically interested in optimizing the social welfare for the network users—a task that can be implemented by setting appropriate *marginal tolls* (Beckmann et al. 1956), which simply charge each user the negative externality she imposes on the system. However, the goal of the link owners is to set tolls so as to attract users and maximize their own profit. Imagine a link owner increasing the price. Clearly, some of the users will move to alternative routes, thereby increasing the congestion on these routes and making them less attractive. This implies that link owners have an incentive to set a price that is higher than what is socially desired and thereby introduce new inefficiencies. For instance, under marginal tolls, some operators may want to increase their toll in order to make a higher profit. In this regard, an equilibrium for the link owners is a toll vector such that a change in a single toll does not increase the profit of the corresponding toll operator (under the implied user Wardrop equilibrium flow).

In this more complex game-theoretic environment, (i) an equilibrium may fail to exist (the only case where an equilibrium is proved to exist is in parallel link networks with affine latency functions, see, e.g., Acemoglu and Ozdaglar 2007a, Hayrapetyan et al. 2007, Harks et al. 2019); (ii) an equilibrium might not be unique (see, e.g., Acemoglu and Ozdaglar 2007b, Harks et al. 2019) and; (iii) the total delay of the equilibrium flow can be arbitrarily higher than the optimal delay, implying that the network can behave arbitrarily worse than the case where tolls are completely absent (see, e.g., Acemoglu and Ozdaglar 2007b, Harks et al. 2019).

In the face of these challenges, we set out to find ways to mitigate the effect of selfish toll operator behavior. We introduce competition regulation by allowing a regulator to set specific price caps on the toll values that each toll operator can set on her link. Different price caps for different operators is consistent with the practice in some privately operated networks of highways. For example, in Santiago de Chile there are currently 12 different operators who set tolls on different urban highways, as shown in Fig. 1. The current regulation sets a price cap that is unique to each highway and the toll operators are allowed to set tolls upper bounded by the caps (Gonzalez 2016). As it turns out, introducing such regulation can resolve all of the above issues, as there are caps for which the game has a unique (strong) Nash equilibrium, under which the Wardrop equilibrium is optimal.

Figure 1 Privately operated highways in Santiago de Chile. Each highway has its own toll cap.

Contribution

Since the unregulated network pricing game behaves so poorly, in Section 3 we set out to study a simple mechanism that improves it. In particular, we investigate the regulatory policy of setting upper bounds (caps) on the toll values that each operator is allowed to set. Note that there is a tension between the toll charged by the operator and the amount of flow she will get. It is plausible for a toll operator to gain from decreasing her toll since her link will attract more flow, which may result in an overall higher profit. For large enough caps it indeed happens that it is optimal for the operator to set a toll upper bounded by the cap. Our main result, Theorem 1, shows that when the central planner chooses the marginal tolls as caps, then the unique Nash equilibrium for the operators is to set the tolls equal to the caps, which is known to induce the optimal flow as a Wardrop equilibrium. In what follows, any toll vector that induces the optimal flow as Wardop equilibrium will be called an optimal toll. Under this definition, the theorem shows something stronger: any optimal cap vector upper bounded by the marginal tolls when chosen as toll caps leads to a unique Nash equilibrium in which every operator charges precisely the cap. Moreover, we show this equilibrium is robust to coalitions, a concept known as *strong Nash equilibrium*. We show that even though the flow on a given link is a decreasing function of the toll on that link (as we show in Lemma 1), the profit of the toll operator as a function of the toll she charges is an increasing function up to a certain point (Lemma 2), which is a key tool to prove Theorem 1.

Inspired by Theorem 1 we study *great tolls* in more detail in Section 4. These are optimal toll vectors that when they are set as caps are themselves the unique Nash equilibrium. It is easy to observe that given an optimal flow, if for every commodity there are some users that do not travel, then great tolls are unique. However, there are also simple examples in which great tolls are not unique. A basic question to ask is how efficient great tolls are with respect to the users' surplus. Note that the users' total delay under great tolls is fixed, since all great tolls are optimal; thus, our question is equivalent to that of finding great tolls that minimize the total payments. By the results in Section 3, any optimal toll vector that is upper bounded by the marginal toll vector, is a great set of tolls. As a benchmark we use the *minimum payment tolls*, defined as those tolls that among the optimal ones, minimize the total payments. Minimum payment tolls, just as great tolls upper bounded by the marginal tolls, can be computed by a linear program. On the negative side, we give an example that shows that the users' surplus under great tolls can be arbitrary worse than the optimal users' surplus. On the positive side, we prove that the users' surplus under great tolls is at least as high as the optimal surplus if each user had no less than half the original valuation (Theorem 2).

In Section 5, we study two different extensions of our model. First, we consider the setting in which each of the operators faces operating costs for maintaining the link. We prove that the main result, the existence of great tolls, is still valid. However, if toll caps are too low or operating costs are too high, there might be links for which the profit in the Nash equilibrium is negative.

Second, we study the setting in which operators are allowed to own more than one link. We start by showing that for instances where players own complementary links, i.e., links for which an increase of the toll value in one of them may only increase the flow on the other (complementary) links, any optimal tolls upper bounded by the marginal tolls are great tolls (Theorem 5). Intuitively, this holds because an operator that operates a single link has an incentive to use the upper bound as toll and this incentive remains if by doing so she only gains more flow on her other links and thus only gains more profit. Then we show that when all the links are complementary, e.g., in a parallel link network, such tolls are additionally a strong Nash equilibrium for the operators (Theorem 6). Lastly, we show that Theorems 5 and 6 are essentially tight by providing two examples.

Related Work

Acemoglu and Ozdaglar (2007a) introduced a model of price competition between link operators where each user has some fixed reservation value for travel. They show that increasing competition among operators from a monopoly to an oligopoly may cause a reduction in efficiency, measured as the difference between the users' willingness to pay and the delay, and provide a (tight) bound on efficiency in pure strategy equilibria. In a follow up work, Acemoglu and Ozdaglar (2007b) generalized the above study to slightly more general topologies in which parallel paths with multiple links may replace the parallel links. They showed that even this slight generalization can make the game arbitrarily inefficient, where the efficiency is measured as mentioned above.

Hayrapetyan et al. (2007) considered instances on parallel links where the demand to be routed is elastic and decreases in a concave way as the cost for using the network increases. The social cost in that work is the sum of the players' profits plus a term that represents the utility gathered by the traffic that gets routed, with a nice tradeoff occurring between these two terms. For that game they showed that in a network with parallel links and linear latencies, there is always a pure Nash equilibrium with the price of anarchy, i.e., the measure for inefficiency, being bounded by a constant factor even when the latency functions are relaxed to be convex. For the case where latencies have zero value under zero flow, they improve the constant. Following that work, Ozdaglar (2008) studied the same model and managed to prove tight bounds on the efficiency of that game. Musacchio (2009) and Musacchio and Wu (2007) rederived and generalized those (upper) bounds for the case of series-parallel networks via a connection to electrical circuits; see the survey by Ozdaglar and Srikant (2007) for further discussion. Johari et al. (2010) study an extension of network pricing games in which operators compete in prices and investments.

Harks et al. (2019) use a very similar policy to regulate competition between link operators. There, a regulator is able to set a unique price cap for all link operators. As it turns out this restricts the regulator so that the induced network performance is not always optimal. For two-link parallel networks this reduction in performance is characterized for different classes of latency function.

Our price competition model corresponds to Bertrand competition in a network setting (Dixon 2001, Chapter 6). Under this setting Chawla et al. (2008) addressed questions regarding the price of anarchy and price of stability with respect to two objectives: the social welfare of all the players (users and sellers), and the total profit obtained by all the sellers. Their work only considers capacity-based (capacity-based congestion corresponds to latency functions which are identically zero until capacity is reached, and then jump to infinity) and no regulation is imposed on the game.

Their results show a dependence of the price of anarchy/stability on the number of monopolistic links, namely the links whose removal disconnects an origin-destination pair. In contrast, our model does not suffer from monopolies: setting caps on prices prevents monopolistic links from charging arbitrarily large prices. Following the same model and focusing on the social welfare of the consumers as the objective, Chawla and Niu (2009) extended the results of Chawla et al. (2008).

An interesting and related model is raised by Anshelevich and Sekar (2015). They consider the edges of a network as goods, and each edge is owned by a different profit maximizing seller. In the first stage of the game, sellers set prices for the use of their edges and have production costs depending on the level of use. In the second stage, the users of the network, i.e., the buyers, choose origin to destination paths so as each of them maximizes her utility minus the payments to the sellers. The main difference with the model presented here is that users impose an externality on the sellers, via the production costs, and not on the other users, as is the case in our model. This crucially affects the equilibrium pricing and the social welfare (e.g., for single commodity networks, if no monopolies are present, there is always an optimal equilibrium, which is not the case in our model).

In other related work, Papadimitriou and Valiant (2010) consider the case where the routing is no longer selfish, but is controlled by the edges of the network, and each edge either minimizes its average latency, or announces a suitable price to its neighbors in order to maximize its profit. Caragiannis et al. (2017) consider a model of buyers and sellers of a similar product, that under some reformulation can be seen as a variant of the parallel links model of Acemoglu and Ozdaglar (2007a) with heterogeneous buyers but constant latencies. Instead of minimizing traffic costs, maximizing the profit from tolls, has been considered in the past (Kuiteing et al. 2016, Castelli et al. 2017, Briest et al. 2012). There, a central authority/unique owner has control of all the toll-able edges, yet, more importantly, the edge costs are constants rather than flow dependent. Recently, Schmand et al. (2019) consider a two-stage game in which operators compete in investments so as to increase the bandwith of a link to attract users.

The study of network congestion games where a central operator is allowed to charge tolls in order to improve efficiency has a long history, starting with Beckmann et al. (1956). Cole et al. (2003) and Fleischer (2005) provided upper bounds on tolls that induce the optimal flow as an equilibrium, and Dial (1999) considered the objective of minimizing users' payments among optimal flow-inducing tolls. The study of network users games where for each link a (potentially adversarially chosen) upper bound on the toll is present was first considered by Bonifaci et al. (2011) and later by Jelinek et al. (2014) and Fotakis et al. (2015). Results in these papers show that when upper bounds are present, optimality cannot in general be achieved, yet, on the positive side, algorithms are proposed under rather restrictive settings with provable guarantees regarding the efficiency of the network.

The transportation literature also addresses questions regarding the pricing of privately operated roads in transportation networks. A recent stream of papers focusses on the so-called build-operate-transfer (BOT) scheme in which private investors build and operate roads at their own expenses, and in return receive the revenue from tolls charged within some years, after which the roads are transferred to the government. See, for example, Yang and Meng (2000), Yang et al. (2002), and Meng and Lu (2017).

2. Preliminaries

We study a network pricing game, where nonatomic players, which we call *users*, selfishly minimize their cost (delays plus tolls) across a network; on top of this, each network link is operated by a different selfish agent which maximizes profit by charging tolls on users traversing her link.

The Network Users' Game: Selfish Routing

Let $G=(V,E)$ be a network, with V the set of nodes, and E the set of directed edges/links of the network. We consider a multi-commodity flow instance, described by origin-destination node pairs $\{(o^k,d^k)\}_{k\in K}$, for a finite set of commodities K. For each commodity k, we assume that the total amount of traffic is dependent on the costs the traffic experiences; the higher the costs, the lower the traffic. We analyze the elastic traffic demand model as introduced by Beckmann et al.

(1956) and later used for network pricing games by Hayrapetyan et al. (2007) and Ozdaglar (2008). We model elastic demand with a *utility function* $u^k : [0, r^k] \to \mathbb{R}_+$ for each $k \in K$, where $u^k(x)$ captures the reservation value for travel of the particle of the demand (i.e., the infinitesimally small user) at level x and $r^k \in \mathbb{R}_+$ is the maximum demand of commodity k. We assume that $u^k(\cdot)$ is nonincreasing and continuous for each $k \in K$, so that, in a sense, the users are ordered decreasingly according to their utility for traveling. Let $\boldsymbol{u} = (u^k)_{k\in K}$ be the vector of all utility functions. Define the aggregate utility function $U^k : [0, r^k] \to \mathbb{R}_+$ by $U^k(x) = \int_0^x u^k(y)\, dy$. By definition, this function is nondecreasing, concave and continuously differentiable.

For each link $e \in E$, there is a latency function $\ell_e : \mathbb{R}_+ \to \mathbb{R}_+$, that represents the delay experienced by users traversing this link, as a function of the total flow on the link. We will assume this function to be nondecreasing, convex and smooth.

Paths and Flows. For each commodity $k \in K$, let $\mathcal{P}^k$ denote the set of $o^k - d^k$ paths and let $\mathcal{P} = \cup_k \mathcal{P}^k$ be the union of all these paths. A *flow* for commodity k is a nonnegative vector $\boldsymbol{x}^k = (x^k_P)_{P\in\mathcal{P}^k}$ such that $\sum_{P\in\mathcal{P}^k} x^k_P \le r^k$. For each commodity $k \in K$, let $r^k_{\boldsymbol{x}} = \sum_{P\in\mathcal{P}^k} x^k_P$ be the amount of flow that is routed on the network by $\boldsymbol{x}^k$. A *flow* $\boldsymbol{x}$ is a vector $(\boldsymbol{x}^k)_{k\in K}$, where each $\boldsymbol{x}^k$ is a flow for commodity k. Let X denote the set of all flows. For a flow $\boldsymbol{x} \in X$ and $e \in E$, let $x^k_e = \sum_{P\in\mathcal{P}^k : e\in P} x^k_P$ be the amount of flow that $\boldsymbol{x}^k$ routes on each link e and let $x_e = \sum_{k\in K} x^k_e$ be the amount of flow that $\boldsymbol{x}$ routes on e. With a slight abuse of notation, we will also denote $\boldsymbol{x} = (x_e)_{e\in E}$, the link-wise description of a flow.

Wardrop Equilibria and Optimal Flows. Given a flow $\boldsymbol{x} \in X$, the delay experienced on e is $\ell_e(x_e)$. In the case a toll $t_e \ge 0$ is charged for link usage, the combined cost of traversing e is $[\ell_e(x_e) + \alpha \cdot t_e]$, where $\alpha > 0$ represents the trade-off factor between delay and tolls and is assumed to be identical for all users. Without loss of generality, we can assume α to be equal to 1 because we can always divide all tolls by α. A flow $\boldsymbol{x} \in X$ is a *Wardrop equilibrium* if all the routed traffic goes through shortest paths for the respective commodity, the utility for each traveling user is at least equal to the common shortest path cost of her commodity, and the utility of each user not traveling is at most equal to the common shortest path cost of her commodity. Formally, for every k, for every path $P \in \mathcal{P}^k$ with $x_P^k > 0$, and every path $P' \in \mathcal{P}^k$, $\sum_{e \in P}[\ell_e(x_e) + t_e] \le \sum_{e \in P'}[\ell_e(x_e) + t_e]$ and $\sum_{e \in P}[\ell_e(x_e) + t_e] \le u^k(r_{\boldsymbol{x}}^k)$ with $r_{\boldsymbol{x}}^k = r^k$ if $\sum_{e \in P}[\ell_e(x_e) + t_e] < u^k(r_{\boldsymbol{x}}^k)$. Given such a flow, any $o^k - d^k$ path achieving the minimum end-to-end cost will be called an *active path*, and any link that belongs to an active path will be called an *active link*.

For any toll vector $\boldsymbol{t} \ge 0$, a Wardrop equilibrium exists, moreover it minimizes the convex Wardrop potential $\Phi_{\boldsymbol{t}}(\boldsymbol{x})$ (Beckmann et al. 1956), and thus is given by:

$$\boldsymbol{x}(\boldsymbol{t}) \triangleq \arg\max_{\boldsymbol{x} \in X} \left\{ \sum_{k \in K} U^k(r_{\boldsymbol{x}}^k) - \sum_{e \in E} \int_0^{x_e} (\ell_e(y) + t_e)\, dy \right\}. \tag{1}$$

We will restrict attention to instances in which the Wardrop equilibrium is unique for all $\boldsymbol{t} \ge 0$. In particular, this can be achieved by assuming that $\ell_e(x)$ is strictly increasing for all $e \in E$, or by assuming that there is no pair of paths with common endpoints and constant latency. In particular, $(x_e(\boldsymbol{t}))_{e \in E}$ is a well-defined function, which is moreover continuous by Berge's theorem (Berge 1963).

Given a flow $\boldsymbol{x} \in X$ and toll vector $\boldsymbol{t}$, define the *total users' cost* by $C(\boldsymbol{x}, \boldsymbol{t}) = \sum_{e \in E}[\ell_e(x_e) + t_e]x_e$ and the *users' surplus* (or *consumer surplus*) by

$$CS(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{u}) = \sum_{k \in K} U^k(r_{\boldsymbol{x}}^k) - C(\boldsymbol{x}, \boldsymbol{t}).$$

We define the *social welfare* by

$$SW(\boldsymbol{x}, \boldsymbol{u}) = \sum_{k \in K} U^k(r_{\boldsymbol{x}}^k) - \sum_{e \in E} \ell_e(x_e) \cdot x_e.$$

Notice that the social welfare is simply the users' surplus plus the toll operators' profits, yet the tolls do not appear in the definition as they are transfers from users to toll operators.

An *optimal flow* $\boldsymbol{x}^*(\boldsymbol{u})$ is a flow that maximizes the social welfare w.r.t. to $\boldsymbol{u}$. The classical result by Beckmann et al. (1956) shows that a flow $\boldsymbol{x}^*(\boldsymbol{u})$ is an optimal flow if and only if $\boldsymbol{x}^*(\boldsymbol{u}) = \boldsymbol{x}(\hat{\boldsymbol{t}}(\boldsymbol{u}))$, where $\hat{t}_e(\boldsymbol{u}) = x_e^*(\boldsymbol{u})\ell_e'(x_e^*(\boldsymbol{u}))$: the toll vector $\hat{t}$ is known as the *marginal tolls*. By our assumptions on the latency functions, optimal flows are unique. Any toll vector that induces the optimal flow will be called *optimal*.

The Network Operators' Game: Price Competition on Tolls

In our model, additionally, every link $e \in E$ is operated by a different operator: these are the players of the price competition game. Each player e is allowed to charge a nonnegative toll t_e for the usage of her link. Under the resulting toll vector $\boldsymbol{t}$, each link gets flow $x_e(\boldsymbol{t})$ according to the induced Wardrop equilibrium, and thus the profit of player e is given by $\pi_e(\boldsymbol{t}) \triangleq t_e x_e(\boldsymbol{t})$. We are interested in the equilibrium outcomes of this game.

Profit functions. For each player $e \in E$, her strategy is given by toll t_e, and her profit is given by $\pi_e(t_e, \boldsymbol{t}_{-e}) = t_e x_e(t_e, \boldsymbol{t}_{-e})$, where we use the standard game-theoretic notation $\boldsymbol{t} = (t_e, \boldsymbol{t}_{-e})$.

Regulated Network Pricing Game and Nash Equilibria. The regulated network pricing game we consider is the following. A central planner may choose a cap vector $\bar{\boldsymbol{t}} \geq 0$ for tolls, and each player wants to maximize her own profit under this constraint. We study two solution concepts, (pure) Nash equilibrium and strong (pure) Nash equilibrium. Tolls $\boldsymbol{t}$ are a Nash equilibrium for the network pricing game if for every $e \in E$, t_e is the best response of player e to $\boldsymbol{t}_{-e}$, i.e., we have $t_e \in \mathcal{BR}_e(\boldsymbol{t}_{-e})$, where the best response mapping $\mathcal{BR}_e(\boldsymbol{t}_{-e})$ is defined as

$$\mathcal{BR}_e(\boldsymbol{t}_{-e}) \triangleq \arg\max\{\pi_e(s_e, \boldsymbol{t}_{-e}) : \ 0 \leq s_e \leq \bar{t}_e\}.$$

Tolls $\boldsymbol{t}$ are a strong Nash equilibrium if there is no coalition that jointly decides for a deviation so that all operators in the coalition increase their profit (Aumann 1959). Formally, tolls $\boldsymbol{t}$ are a strong Nash equilibrium if for any set $E^{\Delta} \subseteq E$, there exists no $\boldsymbol{t}'$ with $\{e \in E : t'_e \neq t_e\} \subseteq E^{\Delta}$ such that $\pi_e(\boldsymbol{t}') > \pi_e(\boldsymbol{t})$ for all $e \in E^{\Delta}$. Notice that any strong Nash equilibrium is a Nash equilibrium.

Definition 1 (Cap equilibrium and great tolls). Given an instance of the profit maximization game, we say that a nonnegative vector $\bar{\boldsymbol{t}} = (\bar{t}_e)_{e \in E}$ is

(a) a (strong) cap equilibrium if when restricting the strategy space for every player e to tolls $s_e \in [0, \bar{t}_e]$, then $(s_e)_{e \in E} = (\bar{t}_e)_{e \in E}$ is the unique (strong) Nash equilibrium;

(b) a great set of tolls if it is optimal (i.e., induces the optimal flow) and a cap equilibrium.

3. Regulated Network Pricing Game

In this section, we study the regulated profit maximization game and some of its structural properties. We prove that the marginal tolls, when used as caps, are always a Nash equilibrium for the profit maximization game, thus resolving the issues of equilibrium existence, uniqueness and inefficiency raised in the literature. Our main result, presented in Theorem 1, strengthens the above by showing that a potentially larger set of optimal tolls, when used as caps, leads to a unique Nash equilibrium, which is furthermore robust to coalitions.

We start the section by proving two monotonicity properties of the Wardrop flow as a function of tolls. These properties are related to the lower level game.

LEMMA 1. *Let $\boldsymbol{t}, \boldsymbol{t}' \geq 0$ be two toll vectors such that $\boldsymbol{t} \leq \boldsymbol{t}'$ and $E^{<} \triangleq \{e \in E : t_e < t'_e\}$ is nonempty. Then, there exist $e_1, e_2 \in E^{<}$ such that:*

(i) $x_{e_1}(\boldsymbol{t}') \leq x_{e_1}(\boldsymbol{t})$,

(ii) $[x_{e_2}(\boldsymbol{t}') - x_{e_2}(\boldsymbol{t})][\ell_{e_2}(x_{e_2}(\boldsymbol{t}')) + t'_{e_2} - \ell_{e_2}(x_{e_2}(\boldsymbol{t})) - t_{e_2}] \leq 0.$

Proof. To prove (i), we compare flows $\boldsymbol{x}(\boldsymbol{t})$ and $\boldsymbol{x}(\boldsymbol{t}')$ with respect to the Wardrop potentials. By the optimality of the Wardrop flow on its respective potential, we get the following inequalities:

$$\sum_{k \in K} U^k(r^k_{\boldsymbol{x}(\boldsymbol{t})}) - \sum_{e \in E} \int_0^{x_e(\boldsymbol{t})} (\ell_e(x) + t_e)\, dx \geq \sum_{k \in K} U^k(r^k_{\boldsymbol{x}(\boldsymbol{t}')}) - \sum_{e \in E} \int_0^{x_e(\boldsymbol{t}')} (\ell_e(x) + t_e)\, dx$$
$$\sum_{k \in K} U^k(r^k_{\boldsymbol{x}(\boldsymbol{t}')}) - \sum_{e \in E} \int_0^{x_e(\boldsymbol{t}')} (\ell_e(x) + t'_e)\, dx \geq \sum_{k \in K} U^k(r^k_{\boldsymbol{x}(\boldsymbol{t})}) - \sum_{e \in E} \int_0^{x_e(\boldsymbol{t})} (\ell_e(x) + t'_e)\, dx.$$

Combining these inequalities we get

$$\sum_{e \in E^{<}} [t'_e - t_e][x_e(\boldsymbol{t}) - x_e(\boldsymbol{t}')] \geq 0,$$

and thus since $E^{<}$ is nonempty, there must exist an $e_1 \in E^{<}$ such that $x_{e_1}(\boldsymbol{t}') \leq x_{e_1}(\boldsymbol{t})$, proving (i).

Let us now prove (ii). Notice that since $\boldsymbol{x}(\boldsymbol{t}) + (\boldsymbol{x}(\boldsymbol{t}') - \boldsymbol{x}(\boldsymbol{t})) = \boldsymbol{x}(\boldsymbol{t}')$ is a feasible flow, then $\boldsymbol{x}(\boldsymbol{t}') - \boldsymbol{x}(\boldsymbol{t})$ is a feasible direction for $\boldsymbol{x}(\boldsymbol{t})$ in the space of feasible flows. By the first-order optimality conditions of the Wardrop potential,

$$\sum_{k \in K} \sum_{P \in \mathcal{P}^k} \left[u^k(r^k_{\boldsymbol{x}(\boldsymbol{t})}) - \sum_{e \in P} [\ell_e(x_e(\boldsymbol{t})) + t_e] \right] [x^k_P(\boldsymbol{t}') - x^k_P(\boldsymbol{t}))] \leq 0.$$

Analogously, $\boldsymbol{x}(\boldsymbol{t})-\boldsymbol{x}(\boldsymbol{t}')$ is a feasible direction for $x(\boldsymbol{t}')$, and thus

$$\sum_{k\in K}\sum_{P\in\mathcal{P}^k}\Big[u^k(r^k_{\boldsymbol{x}(\boldsymbol{t}')})-\sum_{e\in P}[\ell_e(x_e(\boldsymbol{t}'))+t'_e]\Big][x^k_P(\boldsymbol{t})-x^k_P(\boldsymbol{t}')]\le 0.$$

Adding up these inequalities and using that for any two flows $\boldsymbol{y},\boldsymbol{z}$, we have

$$\sum_{k\in K}\sum_{P\in\mathcal{P}^k}\big[y^k_P\cdot\sum_{e\in P}(\ell_e(z_e)+t_e)\big]=\sum_{e\in E}[y_e\cdot(\ell_e(z_e)+t_e)],$$

we obtain

$$\sum_{k\in K}\big[r^k_{\boldsymbol{x}(\boldsymbol{t}')}-r^k_{\boldsymbol{x}(\boldsymbol{t})}\big]\big[u^k(r^k_{\boldsymbol{x}(\boldsymbol{t})})-u^k(r^k_{\boldsymbol{x}(\boldsymbol{t}')})\big]+\sum_{e\in E^=}[x_e(\boldsymbol{t}')-x_e(\boldsymbol{t})][\ell_e(x_e(\boldsymbol{t}'))-\ell_e(x_e(\boldsymbol{t}))]$$
$$+\sum_{e\in E^<}[x_e(\boldsymbol{t}')-x_e(\boldsymbol{t})][\ell_e(x_e(\boldsymbol{t}'))+t'_e-\ell_e(x_e(\boldsymbol{t}))-t_e]\le 0,$$

where $E^=\triangleq\{e\in E: t_e=t'_e\}$. Observe now that the first and second summation terms are non-negative, as u^k is decreasing for all k and ℓ_e is noncreasing for all e. Thus, $\sum_{e\in E^<}[x_e(\boldsymbol{t}')-x_e(\boldsymbol{t})][\ell_e(x_e(\boldsymbol{t}'))+t'_e-\ell_e(x_e(\boldsymbol{t}))-t_e]\le 0$, implying that there exists $e_2\in E^<$ such that $[x_{e_2}(\boldsymbol{t}')-x_{e_2}(\boldsymbol{t})][\ell_{e_2}(x_{e_2}(\boldsymbol{t}'))+t'_{e_2}-\ell_{e_2}(x_{e_2}(\boldsymbol{t}))-t_{e_2}]\le 0$, proving *(ii)*. □

The following result gives an intriguing inequality satisfied by any profit maximizing toll. It says that in a (local) maximum, the toll is at least as high as the induced marginal costs of the users. The interpretation of this result is that firms have an incentive to increase prices above what is socially desired due to the congestion effects of the users. This property is a consequence of the first-order optimality conditions, in combination with the monotonicity properties stated above. We will later see this lemma is crucial for our main result.

LEMMA 2. *Let $\boldsymbol{t}\ge 0$ be a toll vector. If t_e is a local optimum for the profit maximization problem (that is, for objective $\pi_e(\cdot,\boldsymbol{t}_{-e})$) then $x_e(\boldsymbol{t})\cdot\ell'_e(x_e(\boldsymbol{t}))\le t_e$.*

Proof. Notice first that in the case $x_e(\boldsymbol{t})=0$ the result obviously holds and thus we may restrict ourselves to the case $x_e(\boldsymbol{t})>0$. By continuity of the induced Wardrop flow, $t_e=0$ is not a local maximizer. Also by continuity and t_e being a local maximizer, there exists a $\delta>0$ so that $0<x_e(s_e,\boldsymbol{t}_{-e})<x_e(\boldsymbol{t})$ for any $s_e\in(t_e,t_e+\delta)$.

Now, since t_e is a local maximizer, we can use the first-order optimality conditions, $D^+[\pi_e(t_e,\boldsymbol{t}_{-e})] \le 0$, where $D^+[f(x)] \triangleq \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$ is the upper-right Dini derivative. Using the linearity of Dini derivatives, we get

$$x_e(\boldsymbol{t}) + t_e D^+[x_e(\boldsymbol{t})] = D^+[\pi_e(t_e,\boldsymbol{t}_{-e})] \le 0. \tag{2}$$

On the other hand, since $x_e(s_e,\boldsymbol{t}_{-e}) < x_e(\boldsymbol{t})$ for any $s_e \in (t_e, t_e+\delta)$, by Lemma 1 (ii), we have that $[\ell_e(x_e(s_e,\boldsymbol{t}_{-e})) + s_e - \ell_e(x_e(\boldsymbol{t})) - t_e]/[x_e(s_e,\boldsymbol{t}_{-e}) - x_e(\boldsymbol{t})] \le 0$. After re-arranging terms, we get

$$\frac{\ell_e(x_e(s_e,\boldsymbol{t}_{-e})) - \ell_e(x_e(\boldsymbol{t}))}{x_e(s_e,\boldsymbol{t}_{-e}) - x_e(\boldsymbol{t})} \le -\frac{s_e - t_e}{x_e(s_e,\boldsymbol{t}_{-e}) - x_e(\boldsymbol{t})}.$$

Taking $\limsup_{s_e \to t_e^+}$ in the expression above, we get that the left hand side converges to $\ell_e'(x_e(\boldsymbol{t}))$, whereas the right hand side converges to $-\left(D^+[x_e(\boldsymbol{t})]\right)^{-1}$. Since $D^+[x_e(\boldsymbol{t})] < 0$, we conclude that $\ell_e'(x_e(\boldsymbol{t})) \le -\frac{1}{D^+[x_e(\boldsymbol{t})]}$. This, in combination with (2), gives

$$x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) \le -\frac{x_e(\boldsymbol{t})}{D^+[x_e(\boldsymbol{t})]} \le t_e,$$

which proves the result. $\square$

Our main result below shows a strong consequence of the lemma above. All optimal tolls upper bounded by the marginal tolls, when used as caps, lead to a unique (strong) Nash equilibrium and thus are great tolls.

Theorem 1. *Let $\bar{\boldsymbol{t}}$ be an optimal toll vector with $\bar{\boldsymbol{t}} \le \hat{\boldsymbol{t}}$, where $\hat{t}_e(\boldsymbol{u}) = x_e^*(\boldsymbol{u})\ell_e'(x_e^*(\boldsymbol{u}))$. Then $\bar{\boldsymbol{t}}$ is a (strong) cap equilibrium.*

Remark 1. Our proof below shows a property that is stronger than strong Nash Equilibrium. In fact, what we prove is that for any $E^\Delta \subseteq E$, there exists no $\boldsymbol{t}'$ with $\{e \in E : t_e' \ne t_e\} \subseteq E^\Delta$ such that $\pi_e(\boldsymbol{t}') \ge \pi_e(\boldsymbol{t})$ for all $e \in E^\Delta$ with at least one strict inequality. This property clearly implies the classical notion of strong Nash equilibrium stated in Section 2, but we believe it might be of independent interest. The same stronger property is proved in Theorem 4 and Theorem 6.

Proof. First, we prove that $\bar{\boldsymbol{t}}$ is a strong Nash equilibrium. By way of contradiction, let E^{Δ} be a set of links and $\boldsymbol{t} \leq \bar{\boldsymbol{t}}$ with $e \in E^{<} \triangleq \{e \in E : t_e < \bar{t}_e\} \subseteq E^{\Delta}$ such that $\pi_e(\boldsymbol{t}) \geq \pi_e(\bar{\boldsymbol{t}})$ for all $e \in E^{\Delta}$ with at least one strict inequality. Then for all $e \in E^{<}$, we have $t_e x_e(\boldsymbol{t}) \geq \bar{t}_e x_e(\bar{\boldsymbol{t}})$, which implies $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ and

$$t_e \geq \frac{\bar{t}_e x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}. \tag{3}$$

By Lemma 1 (ii), there exists $e \in E^{<}$ such that $[x_e(\bar{\boldsymbol{t}}) - x_e(\boldsymbol{t})][\ell_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e - \ell_e(x_e(\boldsymbol{t})) - t_e] \leq 0$, which in combination with $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ gives

$$\ell_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e \geq \ell_e(x_e(\boldsymbol{t})) + t_e. \tag{4}$$

Let $e \in E^{<}$ be a link satisfying (4). We have the following inequalities

$$\ell_e'(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \leq \ell_e(x_e(\boldsymbol{t})) - \ell_e(x_e(\bar{\boldsymbol{t}})) \leq \bar{t}_e - t_e \leq \bar{t}_e \left(1 - \frac{x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right), \tag{5}$$

where the first inequality follows from convexity of ℓ_e, the second from (4), and the third from (3). Since $\bar{t}_e \leq \hat{t}_e = \ell_e'(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}})$ as $\bar{t}$ is optimal, we obtain from (5) that

$$\ell_e'(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \leq \ell_e'(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}}) \left(\frac{x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right).$$

Since $\ell_e'(x_e(\bar{\boldsymbol{t}})) > 0$ and $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$, we conclude that $x_e(\boldsymbol{t}) \leq x_e(\bar{\boldsymbol{t}})$, a contradiction.

Now we show that there is a unique Nash equilibrium. By way of contradiction, suppose there exists another Nash equilibrium $\boldsymbol{t} \leq \bar{\boldsymbol{t}}$ (with at least one strict inequality) for the profit maximization game with caps $\bar{\boldsymbol{t}}$. Since all players are playing their best response, we can use Lemma 2, concluding that $x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) \leq t_e$ for all $e \in E^{<}$ (which is nonempty by assumption), which in turn gives

$$x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) \leq t_e < \bar{t}_e \leq \hat{t}_e = x_e^* \ell_e'(x_e^*),$$

concluding that $x_e(\boldsymbol{t}) < x_e^* = x_e(\bar{\boldsymbol{t}})$ for all $e \in E^{<}$. But from Lemma 1 (i) there exists $e_1 \in E^{<}$ such that $x_{e_1}(\boldsymbol{t}) \geq x_{e_1}(\bar{\boldsymbol{t}})$, a contradiction. $\square$

REMARK 2. If for an optimal flow we have that for every commodity there are some users that travel and some users that do not travel, then marginal tolls are the unique optimal toll vector and thus the unique toll vector that satisfies the condition of Theorem 1. Otherwise, there might be multiple optimal toll vectors. We refer to Appendix A for a related discussion on the linear program MPT, whose feasible set is all optimal toll vectors and its solution minimizes the users' surplus at equilibrium.

4. Maximizing Consumer Surplus

We now consider the question how efficient great tolls are with respect to the consumer surplus. The main result in this subsection, Theorem 2, only applies to single-source, single-sink networks, and thus we will restrict ourselves to these instances and omit any dependence on $k \in K$.

In what follows, we compare the maximum consumer surplus of an optimal toll vector upper bounded by the marginal tolls (this is a great set of tolls by Theorem 1) to the maximum consumer surplus of any optimal toll vector. Among all optimal toll vectors, let $\boldsymbol{t}^M$ be the one that maximizes the consumer surplus, and let $\boldsymbol{t}^B$ the one that maximizes the consumer surplus, restricted to being upper bounded by the marginal tolls. Both these toll vectors can be calculated by means of a linear program, i.e., linear programs BMT and MPT in Appendix A for $\boldsymbol{t}^B$ and $\boldsymbol{t}^M$, respectively, and in general they may differ. The following example shows that the consumer surplus under $\boldsymbol{t}^B$ can be arbitrarily lower than the consumer surplus under $\boldsymbol{t}^M$.

EXAMPLE 1. Consider the Braess network of Figure 2 with r units of flow to be routed and $u(x) = 2r$ for $x \in [0, r]$.

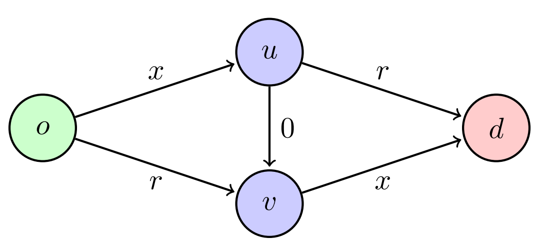

Figure 2 The network of Example 1.

An optimal flow in this case is to split the demand into half on the upper and half on the lower path; in particular, no flow traverses the middle uv link. This way, marginal tolls are given by $(\hat{t}_{ou}, \hat{t}_{uv}, \hat{t}_{vd}) = (r/2, 0, r/2)$, and notice that this is the only feasible toll vector that satisfies the conditions of Theorem 1, achieving a value of $2r^2 - 2r^2 = 0$. On the other hand, it is easy to see that consumer surplus maximizing optimal toll vector just needs to assign a sufficiently large value for t_{uv} (more precisely, $t_{uv} \geq r/2$) and the rest of the tolls can be zero, therefore the optimal consumer surplus is $2r^2 - 3r^2/2 = r^2/2$.

The main result of this subsection is a comparison in the spirit of the bicriteria bound of Roughgarden and Tardos (2002): by how much should we decrease the utility function in order to induce the same level of consumer surplus when comparing $\boldsymbol{t}^B$ to $\boldsymbol{t}^M$. A consequence of this result is that the consumers' surplus under $\boldsymbol{t}^B$ is at least as much as the optimal surplus if each of them had half the valuation.

For particular values of the parameter $\beta \in [1, \infty)$ in the next theorem, we refer to Theorem 7 and Theorem 8 in Appendix A. For the proof of Theorem 2 we will need the following lemma. The proof of Lemma 3 can be found in Appendix A.

LEMMA 3. *Let $\tilde{u}$ and u be utility functions so that $\tilde{u}(x) = \alpha \cdot u(x)$, where $0 < \alpha \leq 1$. Then*

(i) $r_{\boldsymbol{x}^*(u)} \geq r_{\boldsymbol{x}^*(\tilde{u})}$,

(ii) $CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) \geq CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u})$,

(iii) $CS(\boldsymbol{x}^*(u), \boldsymbol{t}^M, u) \geq CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M, \tilde{u})$.

THEOREM 2. *Let $C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \leq \beta \cdot C(\boldsymbol{x}^*(u), \boldsymbol{t}^M)$ for some $\beta \in [1, \infty)$. Then*

$$CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) \geq CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M, \tilde{u}),$$

where $\tilde{u}(x) = \frac{\beta}{2\beta - 1} \cdot u(x)$.

Proof. By the optimal flow characterization of Beckmann et al. (1956), we have that $r_{\boldsymbol{x}^*(u)} < r$ for all u with $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}) > r \cdot u(r)$, and $r_{\boldsymbol{x}^*(u)} = r$ for all u with $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}) \leq r \cdot u(r)$.

We consider the following three cases: (1) $u(r) < C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$, (2) $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r \le u(r) < \frac{2\beta-1}{\beta} \cdot C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$ and (3) $u(r) \ge \frac{2\beta-1}{\beta} \cdot C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$.

Case (1). Assume that $u(r) < C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$. Then $r_{\boldsymbol{x}^*(u)} < r$ and thus

$$CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) = CS(\boldsymbol{x}^*(u), \boldsymbol{t}^M, u) \ge CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M, \tilde{u}),$$

where the equality follows since $CS(\boldsymbol{x}^*(u), \boldsymbol{t}, u) = U(r_{\boldsymbol{x}^*(u)}) - r_{\boldsymbol{x}^*(u)} \cdot u(r_{\boldsymbol{x}^*(u)})$ for all optimal $\boldsymbol{t}$ and the inequality by Lemma 3.

Case (2). Assume that $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r \le u(r) < \frac{2\beta-1}{\beta} \cdot C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$. Then $r_{\boldsymbol{x}^*(u)} = r$ and since $\tilde{u}(r) = \frac{\beta}{2\beta-1} \cdot u(r) < C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$, we have $r_{\boldsymbol{x}^*(\tilde{u})} < r$. Thus

$$CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) \ge CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}) = CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M, \tilde{u}),$$

where the inequality follows by Lemma 3 and the equality since $CS(\boldsymbol{x}^*(u), \boldsymbol{t}, u) = U(r_{\boldsymbol{x}^*(u)}) - r_{\boldsymbol{x}^*(u)} \cdot u(r_{\boldsymbol{x}^*(u)})$ for all optimal $\boldsymbol{t}$.

Case (3). Assume that $u(r) \ge \frac{2\beta-1}{\beta} \cdot C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$. Then $r_{\boldsymbol{x}^*(u)} = r$ and since $\tilde{u}(r) = \frac{\beta}{2\beta-1} \cdot u(r) \ge C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$, we have $r_{\boldsymbol{x}^*(\tilde{u})} = r$. Thus,

$$\begin{aligned} C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) - C(\boldsymbol{x}^*(u), \boldsymbol{t}^M) &\le \left(1 - \frac{1}{\beta}\right) \cdot C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \\ &\le \left(1 - \frac{1}{\beta}\right) \cdot C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}) \\ &\le r \cdot \frac{\beta - 1}{2\beta - 1} \cdot u(r) - \int_0^r u(x)dx + \int_0^r u(x)dx \\ &\le -\int_0^r \frac{\beta}{2\beta - 1} \cdot u(x)dx + \int_0^r u(x)dx \\ &= -\int_0^r \tilde{u}(x)dx + \int_0^r u(x)dx = U(r) - \tilde{U}(r), \end{aligned}$$

where the first inequality follows from $C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \le \beta \cdot C(\boldsymbol{x}^*(u), \boldsymbol{t}^M)$, which is true by assumption, the second from $C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \le C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})$, the third from $\frac{\beta}{2\beta-1} \cdot u(r) \ge C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}})/r$, and the fourth from $r \cdot u(r) \le \int_0^r u(x)dx$ and the definition of $\tilde{u}(x)$. Rearranging terms yields

$$CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) \ge \tilde{U}(r) - C(\boldsymbol{x}^*(u), \boldsymbol{t}^M) = \tilde{U}(r) - C(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M) = CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^M, \tilde{u}),$$

as needed. $\square$

REMARK 3. Theorem 2 cannot be extended to multicommodity networks. Consider, for example, the network of Figure 3. Commodities $o-d_1$, $o-d_2$ have a maximum demand of 1. If $u^1(x)=8$ for $x\in[0,1]$ and $u^2(x)=4$ for $x\in[0,1]$, then the optimal flow is 3/4 for commodity 1 and 1 for commodity 2. The unique optimal toll vector is $(3/4,7/4)$; inducing a consumer surplus of 1/2.

If $u^1(x)=4$ for $x\in[0,1]$ and $u^2(x)=2$ for $x\in[0,1]$, then the optimal flow is 0 for commodity 1 and 1 for commodity 2. The unique optimal toll vector is $(0,0)$; inducing a consumer surplus of 1.

$$o \xrightarrow{2+x} d_1 \xrightarrow{x} d_2$$

Figure 3 The network of Remark 3.

5. Extensions

In this last section, we consider two different extensions of our model. First, we study an extension in which operators face operating costs. The main result states that any optimal toll vector upper bounded by the marginal tolls is a great set of tolls with a unique equilibrium. Second, we investigate the setting in which operators own multiple links. We show that marginal tolls induce a unique Nash equilibrium as long as the links owned by each operator are complementary.

5.1. When Links Have Operating Costs

Extra Preliminaries

In this case, each operator $e\in E$ is allowed to charge a nonnegative toll t_e for its usage while she additionally faces operating costs as a function of the amount of flow on the link. We assume that this (cost) function $c_e:\mathbb{R}_+\to\mathbb{R}_+$ is nondecreasing, convex and smooth. Convexity of operating costs reflects that with higher traffic, there is a higher chance for accidents and correspondingly higher costs for clean-up and maintenance that the toll operator is responsible for. Under the resulting toll vector $\boldsymbol{t}$, each link gets flow $x_e(\boldsymbol{t})$ according to the induced Wardrop equilibrium, and thus the profit of player e is given by $\pi_e(\boldsymbol{t})\triangleq t_e x_e(\boldsymbol{t})-c_e(x_e(\boldsymbol{t}))$.

Profit functions. For each player $e \in E$, her strategy is given by toll $t_e \geq 0$, and her profit is given by $\pi_e(t_e, \boldsymbol{t}_{-e}) = t_e x_e(t_e, \boldsymbol{t}_{-e}) - c_e(x_e(\boldsymbol{t}))$.

Optimal Flows. Given a flow $\boldsymbol{x}$, the *social welfare* is defined by

$$SW(\boldsymbol{x}, \boldsymbol{u}) = \sum_{k \in K} U^k(r^k_{\boldsymbol{x}}) - \sum_{e \in E} \ell_e(x_e) \cdot x_e - \sum_{e \in E} c_e(x_e).$$

An *optimal flow* $\boldsymbol{x}^*(\boldsymbol{u})$ is a flow that maximizes the social welfare w.r.t. to $\boldsymbol{u}$. The vector of marginal tolls $\hat{\boldsymbol{t}}$, now defined as $\hat{t}_e = x^*_e(\boldsymbol{u})\ell'_e(x^*_e(\boldsymbol{u})) + c'_e(x^*_e(\boldsymbol{u}))$ *induces* the optimal flow, that is $\boldsymbol{x}(\hat{\boldsymbol{t}}) = \boldsymbol{x}^*(\boldsymbol{u})$.

Results

By the next theorem, even with operating costs the existence of great tolls remains guaranteed. The proofs of the two theorems are along the lines of the proof of Theorem 1 (their analogue in the basic model) and can be found in Appendix B.

THEOREM 3. *Let $\bar{\boldsymbol{t}}$ be an optimal toll vector with $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}$. Then $\bar{\boldsymbol{t}}$ is a cap equilibrium.*

REMARK 4. Theorem 3 does not guarantee that each provider earns a positive profit. If the cap is too low or the operating costs are too high, the profit of a provider might be negative. Notice that it is easy to check whether a given cap induces a negative profit, as tolls and flows are known in the equilibrium.

THEOREM 4. *Let $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}$ be an optimal toll vector. If $\pi_e(\bar{\boldsymbol{t}}) \geq 0$ for all $e \in E$ and operating costs are affine, i.e., $c_e(x) = a_e \cdot x_e + b_e$ with $a_e \geq 0$ for all $e \in E$, then $\bar{\boldsymbol{t}}$ is a strong cap equilibrium.*

REMARK 5. If operating costs are linear, i.e., $c_e(x) = a_e \cdot x_e$ with $a_e \geq 0$ for all $e \in E$, we have by definition of $\hat{\boldsymbol{t}}$ that $\pi_e(\hat{\boldsymbol{t}}) \geq 0$ for all $e \in E$. Thus, marginal tolls are a strong cap equilibrium if operating costs are linear.

5.2. Allowing Multiple Links per Operator

Extra Preliminaries

In the case of multiple links per operator the network users' game remains the same, but we have to redefine some notions for the the network operators' game. For every player $i \in [n]$, where n is the number of players, player i owns a subset $E_i \subseteq E$ and is allowed to charge a nonnegative toll t_e to every $e \in E_i$. We assume that E_i and E_j are disjoint for all $i, j \in [n]$ with $i \neq j$. Under the resulting toll vector $\boldsymbol{t}$, each link gets flow $x_e(\boldsymbol{t})$ according to the induced Wardrop equilibrium and the profit from link e is $\pi_e(\boldsymbol{t}) := t_e x_e(\boldsymbol{t})$. The profit of player i is given by $\pi_i(\boldsymbol{t}) := \sum_{e \in E_i} \pi_e(\boldsymbol{t})$.

Profit functions. For each player $i \in [n]$, her strategy is given by toll vector $\boldsymbol{t}_i = (t_e)_{e \in E_i}$, and her profit as a function of her strategy is given by $\pi_i(\boldsymbol{t}_i, \boldsymbol{t}_{-i}) = \sum_{e \in E_i} \pi_e(\boldsymbol{t}_i, \boldsymbol{t}_{-i}) = \sum_{e \in E_i} t_e x_e(\boldsymbol{t}_i, \boldsymbol{t}_{-i})$. We will also need the profit function of a link e when tolls on other links are fixed according to some vector $\boldsymbol{t}_{-e}$, defined as $\pi_e(t_e, \boldsymbol{t}_{-e})$. Whenever $\boldsymbol{t}_{-e}$ is clear from the context, we will simply write $x_e(t_e)$ and $\pi_e(t_e)$. Note that all these profit functions are continuous since for every toll vector $\boldsymbol{t}$, the flow function $\boldsymbol{x}(\boldsymbol{t})$ is continuous.

The following definition will be important for the analysis when operators own multiple links.

Definition 2. A set of links $E' \subseteq E$ is called complementary if for all $e \in E'$, all $0 \leq t_e \leq t'_e$ and all $\boldsymbol{t}_{-e} \geq 0$, it is $x_{e'}(t_e, \boldsymbol{t}_{-e}) \leq x_{e'}(t'_e, \boldsymbol{t}_{-e})$ for all $e' \in E'$ with $e' \neq e$.

Series-Parallel Graphs. A directed $o-d$ multi-graph is *series-parallel* if it consists of a single link (o, d) or from two series-parallel graphs with terminals (o_1, d_1) and (o_2, d_2) composed either in series or in parallel. In a *series composition*, d_1 is identified with o_2, o_1 becomes o, and d_2 becomes d. In a *parallel composition*, o_1 is identified with o_2 and becomes o, and d_1 is identified with d_2 and becomes d. Any series-parallel graph has a decomposition tree that reveals all the "building blocks" of the graph, i.e., every parent is a series or a parallel combination of its children.

The following proposition shows that a set of links whose pairs cannot belong to the same path in a series-paralllel graph, is complementary. The proof lies in Appendix B

Proposition 1. *Let G be series-parallel and $E' \subseteq E$. If for all $e, e' \in E'$ and all paths $P : e \in P$, we have $e' \notin P$, then E' is complementary.*

Note that the above proposition implies that in parallel-link networks any set $E' \subseteq E$ is complementary.

The following example shows that two links of different paths need not be complementary.

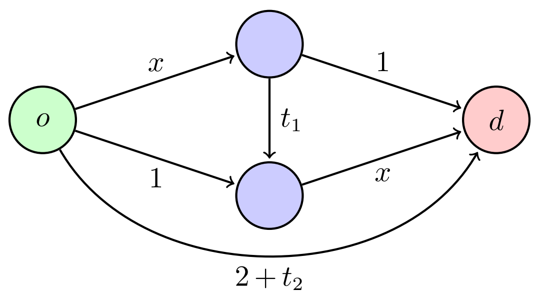

Figure 4 Noncomplementary links.

Example 2. Consider the graph of Figure 4 with 2 units of flow to be routed from o to d and assume that the utilities are sufficiently large so that all users travel. If $t_1 = 0$ and $t_2 = 0$, then 1 unit of flow is routed on the zig-zag path, and 1 unit of flow is routed on the direct path from o to d. If $t_1 = 1$ and $t_2 = 0$, then 3/5 units of flow are routed on the upper and lower path, and 4/5 units of flow are routed on the direct path from o to d. Hence by increasing t_1, the flow on the direct path is reduced.

Results

The existence of great tolls is guaranteed if all players own complementary links.

Theorem 5. *Let $\mathcal{G}$ be an instance in which each player owns a set of complementary links and let $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}$ be an optimal toll vector. Then $\bar{\boldsymbol{t}}$ is a cap equilibrium.*

The proof can be found in Appendix B and is quite technical. The main idea of the proof is that an operator that operates a single link has an incentive to use the upper bound as toll and this incentive remains if by doing so she only gains more flow on her other links and thus only gains more profit.

For series-parallel graphs we get the following corollary. For its proof, one simply has to combine Proposition 1, that yields that players own complementary links, with Theorem 5.

Corollary 1. *Let $\mathcal{G}$ be an instance in which each player owns a set of links that in pairs cannot belong to the same path and let $\bar{\boldsymbol{t}} \leq \hat{\boldsymbol{t}}$ be an optimal toll vector. Then $\bar{\boldsymbol{t}}$ is a cap equilibrium.*

Theorem 5 is tight in the sense that for instances where some player operates noncomplementary links, there exist optimal tolls upper bounded by the marginal ones, and such that when they are set as caps they are not a Nash equilibrium. An example follows. Note that the network is series-parallel, which makes also Corollary 1 tight.

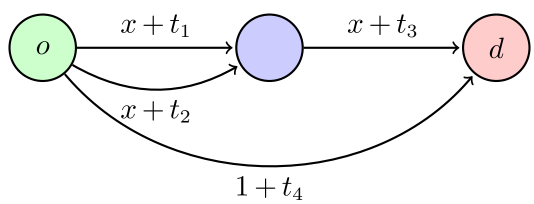

Figure 5 The single-commodity series-parallel network used in Example 3.

Example 3. Consider the series-parallel graph of Figure 5 with 1 unit of flow to be routed and assume that the utilities are sufficiently large so that all users travel. Let player 1 operate links e_1 and e_3 and the other links being operated by other players, one link per player. It is easy to check that links e_1 and e_3 are noncomplementary, e.g., increasing the toll on e_1 may decrease the flow on e_3. In this instance, the optimal flow $\boldsymbol{x}^*$ routes $x_1^* = 1/6$ units through e_1, $x_2^* = 1/6$ units through e_2, $x_3^* = 1/3$ units through e_3 and $x_4^* = 2/3$ units through e_4. Thus, the vector of marginal tolls is $\hat{\boldsymbol{t}} = (1/6, 1/6, 1/3, 0)$. If all players play the marginal tolls then player 1 has profit equal to $(1/6)^2 + (1/3)^2 = 5/36$. But playing marginal tolls is not a best response for player 1 as she can play $t_1 = 1/8 < \hat{t}_1$ and $t_3 = 1/3 = \hat{t}_3$ getting a profit of $121/864 > 5/36$ (the equilibrium flow under $\boldsymbol{t} = (1/8, 1/6, 1/3, 0)$ is $\boldsymbol{x}(\boldsymbol{t}) = (7/36, 11/72, 25/72, 47/72)$). Thus, when $\hat{\boldsymbol{t}}$ are set as caps, they are not a Nash equilibrium.

It is worth noting that on instances in which players operate complementary links, $\bar{\boldsymbol{t}}$ is not a strong Nash equilibrium (despite being a Nash equilibrium). Example 4 has such an instance.

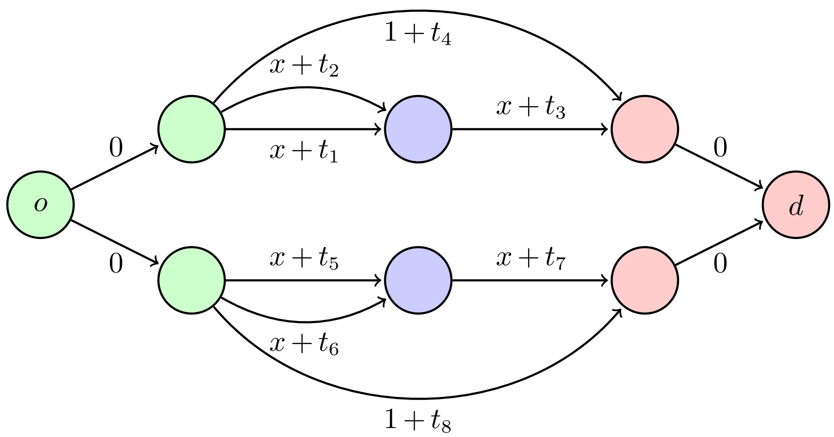

Figure 6 The single-commodity series-parallel network used in Example 4.

Example 4. Consider the series-parallel graph of Figure 6 with 2 units of flow to be routed and assume that utilities are high enough so that all users travel. Let player 1 operate links e_1 and e_7, player 2 operate links e_3 and e_5 and the other links being operated by other players, one link per player. It is easy to check that links e_1 and e_7 are complementary, and the same holds for links e_3 and e_5. Note that Figure 6 is essentially two copies of Figure 5 connected in parallel. Based on that one can derive that the optimal flow $\boldsymbol{x}^*$ is such that $(x_1^*, x_2^*, x_3^*, x_4^*) = (1/6, 1/6, 1/3, 0)$ and $(x_5^*, x_6^*, x_7^*, x_8^*) = (1/6, 1/6, 1/3, 0)$, and the marginal tolls for links e_1 through e_8 are $(\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4) = (1/6, 1/6, 1/3, 0)$ and $(\hat{t}_5, \hat{t}_6, \hat{t}_7, \hat{t}_8) = (1/6, 1/6, 1/3, 0)$. If all players play the marginal tolls then player 1 and player 2 both have profit equal to $(1/6)^2 + (1/3)^2 = 5/36$. But player 1 and player 3 can form a coalition and together deviate and play $t_1 = 1/8 < \hat{t}_1$ and $t_3 = 1/3 = \hat{t}_3$, and $t_5 = 1/8 < \hat{t}_1$ and $t_7 = 1/3 = \hat{t}_3$ getting this way a profit of $121/864 > 5/36$ each. Thus, when $\hat{\boldsymbol{t}}$ are set as caps, they are not a strong Nash equilibrium.

Yet, we have the following theorem. Its proof can be found in Appendix B.

Theorem 6. *Let $\mathcal{G}$ be an instance where all links are complementary links and let $\bar{\boldsymbol{t}} \le \hat{\boldsymbol{t}}$ be an optimal toll vector. Then $\bar{\boldsymbol{t}}$ is a strong cap equilibrium.*

6. Conclusion

In this work, we propose a simple regulation policy for network owners in network pricing games. Network owners have an incentive to set prices that are higher than their marginal externality, yielding inefficiencies for the network users. Moreover, Nash equilibria in the network owners market need not even exist. We have showed that if a central authority regulates the market by setting the appropriate upper bounds on the prices, all of the above issues get resolved. In particular, the well-known marginal tolls as caps have the property that the network owners market has a unique (strong) Nash equilibrium and induces a socially optimal flow. In general, every toll vector that is upper bounded by marginal tolls and induces the optimal flow will result as the unique Nash equilibrium when being imposed as caps.

Given that the optimal social welfare can be achieved by imposing great tolls as caps, we asked the question on how to fairly distribute the social welfare among the network owners and network users. We have studied the efficiency of great tolls that are upper bounded by marginal tolls from the perspective of the network users. Another approach could be to study the efficiency of great tolls from the perspective of the network owners. In order to do so, a complete characterization of great tolls would be needed.

The main result, that marginal tolls are great tolls, applies to the setting in which each link owners owns only one link in the network. The result extends to the setting in which link owners own complementary links (i.e., links for which an increase in toll value may only increase the flow on the other owned links). In general, the result that marginal tolls are great tolls breaks down if network owners own non-complementary links, e.g., multiple links on the same path. This implies that market regulation works well in markets with a high degree of competition, whereas it is less effective in markets with less competition.

Appendix A: Missing proofs of Section 4

We start with the proof of Lemma 3 that we used to the prove Theorem 2.

Proof of Lemma 3. We first note that, by the definition of the total users' cost $C(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{t})$, for an equilibrium $\boldsymbol{x}(\boldsymbol{t})$ under tolls $\boldsymbol{t}$, the common users' path cost equals $C(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{t})/r_x$.

To prove (i), we can assume that $r_{\boldsymbol{x}^*(u)} < r$ as otherwise the result follows trivially, since $r \geq r_{\boldsymbol{x}^*(\tilde{u})}$. By recalling that the optimal flow is a Wardrop equilibrium w.r.t. the marginal latency functions, $r_{\boldsymbol{x}^*(u)} < r$ implies that the common users' path cost at that equilibrium is equal to $u(r_{\boldsymbol{x}^*(u)})$, i.e., $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}(u))/r_{\boldsymbol{x}^*(u)} = u(r_{\boldsymbol{x}^*(u)})$.

To reach a contradiction, let $r_{\boldsymbol{x}^*(u)} < r_{\boldsymbol{x}^*(\tilde{u})}$. Hall (1978) and Lin et al. (2004) show that the common path cost of a Wardrop equilibrium is nondecreasing in r, and additionally, the common path cost is strictly increasing in r when the latencies are strictly increasing. Since an optimal flow is a Wardrop equilibrium w.r.t. the marginal latency functions, and the latency functions are strictly increasing, we have $C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}(u))/r_{\boldsymbol{x}^*(u)} < C(\boldsymbol{x}^*(\tilde{u}), \hat{\boldsymbol{t}}(\tilde{u}))/r_{\boldsymbol{x}^*(\tilde{u})}$. Additionally, by the definition of equilibrium, $C(\boldsymbol{x}^*(\tilde{u}), \hat{\boldsymbol{t}}(\tilde{u}))/r_{\boldsymbol{x}^*(\tilde{u})} \leq \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})$. Putting it all together and using the hypothesis for $u(x)$ and $\tilde{u}(x)$ for the first inequality, we get

$$\tilde{u}(r_{\boldsymbol{x}^*(u)}) \leq u(r_{\boldsymbol{x}^*(u)}) = C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}(u))/r_{\boldsymbol{x}^*(u)} < C(\boldsymbol{x}^*(\tilde{u}), \hat{\boldsymbol{t}}(\tilde{u}))/r_{\boldsymbol{x}^*(\tilde{u})} \leq \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})}),$$

a contradiction, since $\tilde{u}(x)$ is nonincreasing.

To prove (ii) and (iii), we consider the following three cases: (1) $r_{\boldsymbol{x}^*(u)} < r$ and $r_{\boldsymbol{x}^*(\tilde{u})} < r$, (2) $r_{\boldsymbol{x}^*(u)} = r$ and $r_{\boldsymbol{x}^*(\tilde{u})} < r$ and (3) $r_{\boldsymbol{x}^*(u)} = r$ and $r_{\boldsymbol{x}^*(\tilde{u})} = r$. We only give the proof of (ii), the proof of (iii) follows analogously, simply by changing $\boldsymbol{t}^B$ with $\boldsymbol{t}^M$.

Case (1). Assume that $r_{\boldsymbol{x}^*(u)} < r$ and $r_{\boldsymbol{x}^*(\tilde{u})} < r$. Then, by definition of a Wardrop equilibrium, $CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) = U(r_{\boldsymbol{x}^*(u)}) - r_{\boldsymbol{x}^*(u)} \cdot u(r_{\boldsymbol{x}^*(u)})$ and $CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}) = \tilde{U}(r_{\boldsymbol{x}^*(\tilde{u})}) - r_{\boldsymbol{x}^*(\tilde{u})} \cdot \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})$. Thus, we have

$$\begin{aligned} CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) &= \int_{x=0}^{r_{\boldsymbol{x}^*(u)}} [u(x) - u(r_{\boldsymbol{x}^*(u)})]\, dx = \int_{x=0}^{r_{\boldsymbol{x}^*(u)}} \frac{1}{\alpha} [\tilde{u}(x) - \tilde{u}(r_{\boldsymbol{x}^*(u)})]\, dx \\ &\geq \int_{x=0}^{r_{\boldsymbol{x}^*(\tilde{u})}} [\tilde{u}(x) - \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})]\, dx = CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}), \end{aligned}$$

where for the second equality we used the definition of $\tilde{u}(x)$ and for the inequality we used $\alpha \in (0,1]$, $r_{\boldsymbol{x}^*(u)} \geq r_{\boldsymbol{x}^*(\tilde{u})}$ and $\tilde{u}(r_{\boldsymbol{x}^*(u)}) \leq \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})$.

Case (2). Assume that $r_{\boldsymbol{x}^*(u)} = r$ and $r_{\boldsymbol{x}^*(\tilde{u})} < r$. Then, by definition of a Wardrop equilibrium, $CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}) = \tilde{U}(r_{\boldsymbol{x}^*(\tilde{u})}) - r_{\boldsymbol{x}^*(\tilde{u})} \cdot \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})$. Thus, we have

$$\begin{aligned} CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) &= U(r) - C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \\ &\geq \int_{x=0}^{r} [u(x) - u(r)]\, dx = \int_{x=0}^{r} \frac{1}{\alpha} [\tilde{u}(x) - \tilde{u}(r)]\, dx \\ &\geq \int_{x=0}^{r_{\boldsymbol{x}^*(\tilde{u})}} [\tilde{u}(x) - \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})]\, dx = CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}), \end{aligned}$$

where the first inequality follows from $C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \leq C(\boldsymbol{x}^*(u), \hat{\boldsymbol{t}}(u)) \leq r \cdot u(r)$, because of all the demand being routed under $\boldsymbol{x}^*(u)$ and the constraint $\nu_t - \nu_s \leq u(r)$ in (BMT2), and the second inequality follows from $\alpha \in (0,1]$, $r > r_{\boldsymbol{x}^*(\tilde{u})}$ and $\tilde{u}(r) \leq \tilde{u}(r_{\boldsymbol{x}^*(\tilde{u})})$.

Case (3). Assume that $r_{\boldsymbol{x}^*(u)} = r$ and $r_{\boldsymbol{x}^*(\tilde{u})} = r$. Then we have

$$\begin{aligned} CS(\boldsymbol{x}^*(u), \boldsymbol{t}^B, u) &= U(r) - C(\boldsymbol{x}^*(u), \boldsymbol{t}^B) \\ &\geq \tilde{U}(r) - C(x^*(\tilde{u}), \boldsymbol{t}^B) = CS(\boldsymbol{x}^*(\tilde{u}), \boldsymbol{t}^B, \tilde{u}), \end{aligned}$$

where the inequality follows from $u(x) \geq \tilde{u}(x)$ for all $x \in [0, r]$, which implies $U(r) \geq \tilde{U}(r)$, and $\boldsymbol{x}^*(u) = \boldsymbol{x}^*(\tilde{u}) = \boldsymbol{x}^*$, since all the flow is routed by the optimal solution under either utility function. □

Theorem 7, Corollary 2 and Theorem 8, presented next, can be used to get an upper bound on β in Theorem 2. The family of Braess graphs (see, e.g., Roughgarden 2006, Kleer and Schäfer 2016) can be used to show that these bounds are tight.

Theorem 7 considers an approximation bound based on properties of the latency functions. The result can be interpreted as follows: if the sensitivity of links to changes in the flows is bounded by a factor, then the performance of the proposed solution is also bounded by that factor (plus one) times the value of the optimal solution.

Theorem 7. *Suppose all latency functions ℓ in the profit maximization game satisfy* $\sup_{x \geq 0} \frac{x\ell'(x)}{\ell(x)} \leq \gamma$, *then* $C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^B) \leq \beta \cdot C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^M)$, *where* $\beta = \gamma + 1$.

Proof. By the assumption on latency functions,

$$C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^B) \leq \sum_e [\ell_e(x_e^*(\boldsymbol{u})) + x_e^*(\boldsymbol{u})\ell_e'(x_e^*(\boldsymbol{u}))] x_e^*(\boldsymbol{u}) \leq \sum_e (1+\gamma)\ell_e(x_e^*(\boldsymbol{u})) x_e^*(\boldsymbol{u}) \leq (\gamma+1)\, C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^M),$$

proving the result. □

The following result is a direct corollary of Theorem 7.

Corollary 2. *For polynomial latency functions of degree at most d and nonnegative coefficients,* $C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^B) \leq (d+1)\, C(\boldsymbol{x}^*(\boldsymbol{u}), \boldsymbol{t}^M)$.

For Theorem 8, we will need Lemma 4, but first we define programs BMT and MPT that compute tolls $\boldsymbol{t}^B$ and $\boldsymbol{t}^M$, respectively. Both programs are in variables $(\boldsymbol{t}, \boldsymbol{\nu})$ and define a potential ν^k for each commodity k, in such a way that any flow-carrying path is indeed a shortest path. The following LP encodes all possible great set of tolls upper bounded by marginals

$$\text{(BMT)} \quad \left\{ \begin{array}{lccl} \max \sum_{k \in K} U^k(r^k_{\boldsymbol{x}^*}) & - & \sum_{e \in E} [\ell_e(x_e^*(\boldsymbol{u})) + t_e] x_e^*(\boldsymbol{u}) & \\ \quad \nu_u^k - \nu_v^k + t_e & = & -\ell_e(x_e^*(\boldsymbol{u})) & \forall k,\, e=(u,v) : x_e^{*k} > 0 \\ \quad \nu_u^k - \nu_v^k + t_e & \geq & -\ell_e(x_e^*(\boldsymbol{u})) & \forall k,\, e=(u,v) : x_e^{*k} = 0 \\ \quad \nu_{d^k}^k - \nu_{o^k}^k & = & u^k(r^k_{\boldsymbol{x}^*(\boldsymbol{u})}) & \forall k : 0 < r^k_{\boldsymbol{x}^*} < r^k \\ \quad \nu_{d^k}^k - \nu_{o^k}^k & \leq & u^k(r^k) & \forall k : r^k_{\boldsymbol{x}^*} = r^k \\ \quad \nu_{d^k}^k - \nu_{o^k}^k & \geq & u^k(0) & \forall k : r^k_{\boldsymbol{x}^*} = 0 \\ \quad t_e & \leq & \hat{t}_e & \forall e \in E \\ \quad t_e & \geq & 0 & \forall e \in E. \end{array} \right.$$

The first two set of constraints ensure that the flow is routed on shortest paths for the respective commodity. The third, fourth and fifth sets of constraints ensure that the utilities of the users traveling (or not) have the correct relation with the cost of the flow carrying paths of the respective commodity.

With a similar linear program, by dropping the constraints that upper bound the tolls, we encode all possible optimal tolls (MPT stands for *Minimum Payment Tolls*):

$$
\text{(MPT)}\quad\left\{\begin{array}{lccl}
\max \sum_{k\in K} U^k(r^k_{\boldsymbol{x}^*}) - \sum_{e\in E}[\ell_e(x^*_e(\boldsymbol{u}))+t_e]x^*_e(\boldsymbol{u}) & & & \\
\nu^k_u - \nu^k_v + t_e & = & -\ell_e(x^*_e(\boldsymbol{u})) & \forall k, e=(u,v): x^{*k}_e > 0 \\
\nu^k_u - \nu^k_v + t_e & \geq & -\ell_e(x^*_e(\boldsymbol{u})) & \forall k, e=(u,v): x^{*k}_e = 0 \\
\nu^k_{d^k} - \nu^k_{o^k} & = & u^k(r^k_{\boldsymbol{x}^*(\boldsymbol{u})}) & \forall k: 0 < r^k_{\boldsymbol{x}^*} < r^k \\
\nu^k_{d^k} - \nu^k_{o^k} & \leq & u^k(r^k) & \forall k: r^k_{\boldsymbol{x}^*} = r^k \\
\nu^k_{d^k} - \nu^k_{o^k} & \geq & u^k(0) & \forall k: r^k_{\boldsymbol{x}^*} = 0 \\
t_e & \geq & 0 & \forall e \in E.
\end{array}\right.
$$

Since Theorem 8 considers single-commodity instances and great tolls other than the marginal tolls exist only when all the demand is routed, we may simplify BMT to BMT', which considers a single commodity of unit demand (this assumption is w.l.o.g. as latency functions can always be adjusted appropriately) in which all the flow is routed (so we omit the dependence on u) and minimizes the users' cost (since the aggregate utility of the optimal flow is still fixed):

$$
\text{(BMT')}\quad\left\{\begin{array}{lccl}
\min \sum_{e\in E}[\ell_e(x^*_e)+t_e]x^*_e & & & \\
\nu_u - \nu_v + t_e & = & -\ell_e(x^*_e) & \forall e=(u,v): x^*_e > 0 \\
\nu_u - \nu_v + t_e & \geq & -\ell_e(x^*_e) & \forall e=(u,v): x^*_e = 0 \\
t_e & \leq & \hat{t}_e & \forall e \in E \\
t_e & \geq & 0 & \forall e \in E.
\end{array}\right.
$$

Lemma 4 is a structural lemma which allows us to upper bound the value of (BMT'). This is naturally important in order to derive an approximation bound. We will first need some definitions. Consider a directed network G and the undirected network G^u that comes from G if we drop the directions of its links. Any path in G^u is called an undirected path in G. For an undirected path P, the links that are traversed in their actual direction are called *forward* links, denoted by P^+, and the ones traversed in their reversed direction are called *backward* links, denoted by P^-. Finally, an undirected path has J alternations if, when traversing it, there are exactly J times where a forward link is followed by a backward link. For more on such alternating paths, see, e.g., Lin et al. (2011), Nikolova and Stier-Moses (2015), Kleer and Schäfer (2016).

Lemma 4. *There exists an undirected $o-d$ path P such that the first and the last link of P belong in P^+, all $e \in P^+$ are flow carrying, and*

$$C(\boldsymbol{x}^*, \boldsymbol{t}^B) = \sum_{e\in P^+} \ell_e(x^*_e) - \sum_{e\in P^-} [\ell_e(x^*_e) + \hat{t}_e].$$

Proof. We start by taking the dual of (BMT'), where variables $\boldsymbol{y}'$ are associated with shortest path constraints, and variables $\boldsymbol{z}$ with the upper bounds on tolls.

$$
(\text{DBMT}') \quad \begin{cases} \max \sum_{e\in E}[-y'_e\ell_e(x^*_e) - z_e\hat{t}_e + \ell_e(x^*_e)x^*_e] & \\ \sum_{e\in\delta^+(u)} y'_e - \sum_{e\in\delta^-(u)} y'_e & = \ 0 \ \ \forall u \in V \\ y'_e - z_e & \le \ x^*_e \ \forall e \in E \\ y'_e & \ge \ 0 \ \ \forall e : x^*_e = 0 \\ z_e & \ge \ 0 \ \ \forall e \in E. \end{cases}
$$

Let now $y_e \triangleq y'_e - x^*_e$, and notice that the nonnegativity constraints for y'_e where $x^*_e = 0$ can be re-written as $y_e \ge 0$. Let us observe that $-\boldsymbol{y}$ is a unit demand undirected flow (i.e., a flow without sign constraints): By our unit demand assumption,

$$
\begin{aligned}
\sum_{e\in\delta^+(o)} (-y_e) - \sum_{e\in\delta^-(o)} (-y_e) &= \sum_{e\in\delta^+(o)} x^*_e - \sum_{e\in\delta^-(o)} x^*_e = 1 \\
\sum_{e\in\delta^+(d)} (-y_e) - \sum_{e\in\delta^-(d)} (-y_e) &= \sum_{e\in\delta^+(d)} x^*_e - \sum_{e\in\delta^-(d)} x^*_e = -1 \\
\sum_{e\in\delta^+(u)} (-y_e) - \sum_{e\in\delta^-(u)} (-y_e) &= \sum_{e\in\delta^+(u)} x^*_e - \sum_{e\in\delta^-(u)} x^*_e = 0,
\end{aligned}
$$

where in the last equation $u \ne o, d$. This way, we can re-formulate (DBMT′) as

$$
(\text{DBMT}) \quad \begin{cases} -\min \ \ \sum_{e\in E}[y_e\ell_e(x^*_e) + z_e\hat{t}_e] & \\ -\boldsymbol{y} \text{ undirected unit flow} & \\ z_e - y_e & \ge 0 \ \forall e \in E \\ y_e & \ge 0 \ \forall e : x^*_e = 0 \\ z_e & \ge 0 \ \forall e \in E. \end{cases}
$$

Now we make the following observation: Given any $\boldsymbol{y}$ satisfying the constraints above, there is a unique best choice of $\boldsymbol{z} = \boldsymbol{z}(\boldsymbol{y})$; namely, if $y_e \ge 0$ then $z_e = y_e$, and if $y_e < 0$ then $z_e = 0$. Furthermore, observe that $\boldsymbol{z}(\boldsymbol{y})$ is piece-wise affine: As long as no $\boldsymbol{y}$ nonnegativity constraint becomes active, $\boldsymbol{z}$ changes linearly as a function of $\boldsymbol{y}$. Now let $(\boldsymbol{y}, \boldsymbol{z}(\boldsymbol{y}))$ be an optimal solution for (DBMT): We show that we can choose this optimal solution in such a way that it does not support any cycles. Suppose there exists an undirected cycle $C = C^+ \cup C^-$ (where C^+ and C^- are defined so that the links in C^+ traverse the cycle in the opposite direction than the links in C^-), with variables $y_e \ne 0$ for all $e \in C$. Given $\varepsilon \in \mathbb{R}$, consider the perturbation $\boldsymbol{y}^\varepsilon$ such that for all $e \in C^+$, $y^\varepsilon_e = y_e + \varepsilon$ and for all $e \in C^-$, $y^\varepsilon_e = y_e - \varepsilon$; notice that up to the point where, for the first time, some y_e becomes zero, the perturbations $(\boldsymbol{y}^\varepsilon, \boldsymbol{z}(\boldsymbol{y}^\varepsilon))$ are feasible for (DBMT), and the objective function changes linearly with ε. Since neither of these perturbations can improve the objective, the objective has to be constant for these perturbations. This way, we choose either a positive or negative ε until one of the y variables reaches zero. The resulting perturbation $(\boldsymbol{y}^\varepsilon, \boldsymbol{z}(\boldsymbol{y}^\varepsilon))$ is thus optimal and it does not contain C in its support. We can continue this procedure until all cycles are eliminated.

As a conclusion, there exists an optimal solution $(\boldsymbol{y}, \boldsymbol{z}(\boldsymbol{y}))$ whose support is an undirected simple $o-d$ path, that we will call P. By following this path and using the conservation constraints we have that each $e \in P$ satisfies that either $y_e = -1$ (and thus $z_e = 0$) or $y_e = 1$ (and thus $z_e = 1$). Letting $P^+ = \{e \in P : y_e = -1\}$ and $P^- = \{e \in P : y_e = 1\}$ we have that the optimal value of (DBMT) equals

$$\sum_{e \in P^+} \ell_e(x_e^*) - \sum_{e \in P^-} [\ell_e(x_e^*) + \hat{t}_e];$$

furthermore, due to the nonnegativity constraints for $\boldsymbol{y}$, all links $e \in P^+$ are necessarily flow carrying.

Finally, we prove that the first and last links of P are forward, which, additionally, by flow conservation, implies that links in P^+ are forward and links in P^- are backward. We just prove it for the first link, as the argument for the last is analogous. Let e_1 be the first link of P: By flow conservation, $y_{e_1} = -1$ and thus $x_{e_1}^* > 0$. Since x^* is optimal then it is acyclic, so e_1 has to be forward. This completes the proof. $\square$

We make the following assumption on the instance: there exists a $J \geq 0$ such that any simple $o-d$ undirected path has at most J alternations. The smallest constant J satisfying this condition will be called the *alternation number*, and our approximation bound will only depend on this number.

THEOREM 8. *Consider a single-commodity and unit demand instance of the network pricing game whose underlying network has alternation number J. We have, $C(\boldsymbol{x}^*, \boldsymbol{t}^B) \leq (J+1) \cdot C(\boldsymbol{x}^*, \boldsymbol{t}^M)$.*

Proof. By Lemma 4, we have $C(\boldsymbol{x}^*, \boldsymbol{t}^B) = \sum_{e \in P_+} \ell_e(x_e^*) - \sum_{e \in P_-} [\ell_e(x_e^*) + \hat{t}_e]$. Since the alternation number of G is J, we can decompose P in at most J segments of consecutive forward and backward links, $P = A_1 - B_1 - A_2 - B_2 - \ldots - B_J - A_{J+1}$, from which $C(\boldsymbol{x}^*, \boldsymbol{t}^B) \leq \sum_{j=1}^{J+1} \left[\sum_{e \in A_j} \ell_e(x_e^*)\right]$. Let now $(\boldsymbol{t}^*, \boldsymbol{\nu}^*)$ be an optimal solution for (MPT); since all links $e \in A_j$, $j = 1, \ldots, J+1$ are flow carrying, for each $j = 1, \ldots, J+1$, we have $\sum_{e \in A_j} \ell_e(x_e^*) \leq \nu_d^* - \nu_o^*$. Combining the inequalities, we obtain $C(\boldsymbol{x}^*, \boldsymbol{t}^B) \leq \sum_{j=1}^{J+1} \left[\sum_{e \in A_j} \ell_e(x_e^*)\right] \leq (J+1)[\nu_d^* - \nu_o^*] = (J+1) \cdot C(\boldsymbol{x}^*, \boldsymbol{t}^M)$. $\square$

Appendix B: Missing proofs of Section 5

B.1. Missing proofs from Section 5.1

In order to prove Theorem 3 we first restate the profit maximization property that yields the main result for the model with operating costs.

LEMMA 5. *Let $\boldsymbol{t} \geq 0$ be a toll vector and $e \in E$ with $x_e(\boldsymbol{t}) > 0$. If t_e is a local optimum for the profit maximization problem (i.e., for objective $\pi_e(\cdot, \boldsymbol{t}_{-e})$) then $x_e(\boldsymbol{t}) \cdot \ell_e'(x_e(\boldsymbol{t})) + c_e'(x_e(\boldsymbol{t})) \leq t_e$.*

Proof. In order to use the first-order optimality conditions, we need to ensure that the flow on e neither suddenly drops to zero nor it remains constant. By continuity, there exists an interval

$[t_e, t_e + \delta']$ where $x_e(\cdot) > 0$, with $\delta' > 0$; furthermore, by local optimality, we may choose $0 < \delta < \delta'$ so that t_e is a profit maximizing toll on the interval. This in particular implies that $0 < x_e(s_e) < x_e(\boldsymbol{t})$ for any $s_e \in (t_e, t_e + \delta)$.

Now, since t_e is a local maximizer, we can use the first-order optimality conditions, $D^+[\pi_e(t_e)] \le 0$. Using the linearity of Dini derivatives, we get

$$x_e(\boldsymbol{t}) + t_e D^+[x_e(\boldsymbol{t})] - c_e'(x_e(\boldsymbol{t})) \cdot D^+[x_e(\boldsymbol{t})] = D^+[\pi_e(t_e)] \le 0. \tag{6}$$

On the other hand, since $x_e(s_e) < x_e(\boldsymbol{t})$ for any $s_e \in (t_e, t_e + \delta)$, by Lemma 1 (ii), we have that $[\ell_e(x_e(s_e)) + s_e - \ell_e(x_e(\boldsymbol{t})) - t_e]/[x_e(s_e) - x_e(\boldsymbol{t})] \le 0$. After re-arranging terms, we get

$$\frac{\ell_e(x_e(s_e)) - \ell_e(x_e(\boldsymbol{t}))}{x_e(s_e) - x_e(\boldsymbol{t})} \le -\frac{s_e - t_e}{x_e(s_e) - x_e(\boldsymbol{t})}.$$

Taking $\limsup_{s_e \to t_e^+}$ in the expression above, we get that the left hand side converges to $\ell_e'(x_e(\boldsymbol{t}))$, whereas the right hand side converges to $-(D^+[x_e(\boldsymbol{t})])^{-1}$. Since $D^+[x_e(\boldsymbol{t})] < 0$, we conclude that $\ell_e'(x_e(\boldsymbol{t})) \le -\frac{1}{D^+[x_e(\boldsymbol{t})]}$. This, in combination with (6), gives

$$x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) + c_e'(x_e(\boldsymbol{t})) \le -\frac{x_e(\boldsymbol{t})}{D^+[x_e(\boldsymbol{t})]} + c_e'(x_e(\boldsymbol{t})) \le t_e,$$

which proves the result. $\square$

Proof of Theorem 3. First, we prove that $\bar{\boldsymbol{t}}$ is a Nash equilibrium. By way of contradiction, suppose that $t_e < \bar{t}_e$ maximizes profit given $t_f = \bar{t}_f$ for all $f \neq e$. Since $\bar{t}_e > 0$, we must have $x_e(\boldsymbol{t}) > 0$, and thus by Lemma 1 (i), $x_e(t_e, \bar{\boldsymbol{t}}_{-e}) > 0$. By Lemma 5, we have $t_e \ge x_e(t_e, \bar{\boldsymbol{t}}_{-e})\ell_e'(x_e(t_e, \bar{\boldsymbol{t}}_{-e})) + c_e'(t_e, \bar{\boldsymbol{t}}_{-e})$, implying

$$x_e(t_e, \bar{\boldsymbol{t}}_{-e})\ell_e'(x_e(t_e, \bar{\boldsymbol{t}}_{-e})) + c_e'(t_e, \bar{\boldsymbol{t}}_{-e}) \le t_e < \bar{t}_e \le \hat{t}_e = x_e^* \ell_e'(x_e^*) + c_e'(x_e^*),$$

and thus $x_e(t_e, \bar{\boldsymbol{t}}_{-e}) < x_e^*$, contradicting Lemma 1 (i).

Now that we have proved existence, we show that there is a unique Nash equilibrium. By way of contradiction, suppose there exists another Nash equilibrium $\boldsymbol{t} \neq \bar{\boldsymbol{t}}$ for the profit maximization game with caps $\bar{\boldsymbol{t}}$ and let $E^< = \{e \in E : t_e < \bar{t}_e\}$. Observe that for a deviation to be possible $\hat{t}_e \ge \bar{t}_e > 0$ for all $e \in E^<$, and also there is at least one player with $t_e < \bar{t}_e$ and $x_e(\boldsymbol{t}) > 0$. If not, then for all players $e \in E^<$, we have $x_e(\boldsymbol{t}) = 0$, but this contradicts Lemma 1 (i) as there should exist an $e_1 \in E^<$ such that $x_{e_1}(\boldsymbol{t}) \ge x_{e_1}(\bar{\boldsymbol{t}})$ and $x_{e_1}(\hat{\boldsymbol{t}}) = x_{e_1}(\bar{\boldsymbol{t}}) > 0$ since $\hat{t}_{e_1} > 0$.

So we can assume that there are some players with $t_e < \bar{t}_e$ and $x_e(\boldsymbol{t}) > 0$. Since all players are playing their best response we can use Lemma 5, concluding that $x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) + c_e'(x_e(\boldsymbol{t})) \le t_e$ for all $e \in E^<$ with $x_e(\boldsymbol{t}) > 0$, which in turn gives

$$x_e(\boldsymbol{t})\ell_e'(x_e(\boldsymbol{t})) + c_e'(x_e(\boldsymbol{t})) \le t_e < \bar{t}_e \le \hat{t}_e = x_e^* \ell_e'(x_e^*) + c_e'(x_e^*)$$

concluding that $x_e(\boldsymbol{t}) < x_e^* = x_e(\bar{\boldsymbol{t}})$ for all $e \in E^<$ with $x_e(\boldsymbol{t}) > 0$. But from Lemma 1 (i) there exists $e_1 \in E^<$ such that $x_{e_1}(\boldsymbol{t}) \ge x_{e_1}(\bar{\boldsymbol{t}})$ and by $\hat{t}_{e_1} > 0$, $x_{e_1}(\bar{\boldsymbol{t}}) = x_{e_1}(\hat{\boldsymbol{t}}) > 0$, a contradiction. $\square$

Proof of Theorem 4. By Theorem 3, it is sufficient to prove that $\bar{\boldsymbol{t}}$ is a strong Nash equilibrium. By way of contradiction, let E^{Δ} be a set of links and $\boldsymbol{t} \le \bar{\boldsymbol{t}}$ with $\{e \in E : t_e \neq \bar{t}_e\} \subseteq E^{\Delta}$ such that $\pi_e(\boldsymbol{t}) \ge \pi_e(\bar{\boldsymbol{t}})$ for all $e \in E^{\Delta}$ with at least one strict inequality. Then for all $e \in E^{<} \subseteq E^{\Delta}$, we have $(t_e - a_e) \cdot x_e(\boldsymbol{t}) - b_e \ge (\bar{t}_e - a_e) \cdot x_e(\bar{\boldsymbol{t}}) - b_e \ge 0$, where the second inequality holds by assumption, which implies $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ and

$$t_e - a_e \ge \frac{(\bar{t}_e - a_e) \cdot x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}. \tag{7}$$

By Lemma 1 *(ii)*, there exists $e \in E^{<}$ such that $[x_e(\bar{\boldsymbol{t}}) - x_e(\boldsymbol{t})][\ell_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e - \ell_e(x_e(\boldsymbol{t})) - t_e] \le 0$, which in combination with $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$ gives

$$\ell_e(x_e(\bar{\boldsymbol{t}})) + \bar{t}_e \ge \ell_e(x_e(\boldsymbol{t})) + t_e. \tag{8}$$

Let $e \in E^{<}$ be a link satisfying (8). We have the following inequalities

$$\ell_e'(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \le \ell_e(x_e(\boldsymbol{t})) - \ell_e(x_e(\bar{\boldsymbol{t}})) \le \bar{t}_e - t_e \le (\bar{t}_e - a_e) \cdot \left(1 - \frac{x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right), \tag{9}$$

where the first inequality follows from convexity of ℓ_e, the second from (8), and the third from (7). Since $\bar{t}_e \le \hat{t}_e = \ell_e'(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}}) + a_e$ as $\bar{t}$ is optimal, we obtain from (9) that

$$\ell_e'(x_e(\bar{\boldsymbol{t}}))(x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})) \le \ell_e'(x_e(\bar{\boldsymbol{t}}))x_e(\bar{\boldsymbol{t}}) \left(\frac{x_e(\boldsymbol{t}) - x_e(\bar{\boldsymbol{t}})}{x_e(\boldsymbol{t})}\right).$$

Since $\ell_e'(x_e(\bar{\boldsymbol{t}})) > 0$ and $x_e(\boldsymbol{t}) > x_e(\bar{\boldsymbol{t}})$, we conclude that $x_e(\boldsymbol{t}) \le x_e(\bar{\boldsymbol{t}})$, a contradiction.

B.2. Missing proofs from Section 5.2

Proof of Proposition 1. We will first show that for series-parallel graphs under any fixed demand r and for any link e, setting $t_e' > t_e$ only increases the common travel cost at equilibrium. The proof is by induction on the decomposition of the series-parallel graph. In the base case of a single link, this is true by Lemma 1.

Let G be a series composition of series-parallel graphs G_1 and G_2, and w.l.o.g. let e belong to G_1. Under toll t_e and demand r let the travel costs through G_1 and G_2 be $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. By induction, under toll t_e', the travel cost through G_1 only increases to $\mathcal{L}_1' \ge \mathcal{L}_1$ while the travel cost through G_2 remains the same. Thus the common cost at equilibrium only increases.

For the other case, let now G be a parallel composition of series-parallel graphs G_1 and G_2, and w.l.o.g. let e belong in G_1. Under toll t_e and demand r let the common travel cost through G_1 and G_2 be $\mathcal{L}$ and let the traffic routed through G_1 and G_2 be r_1 and r_2 respectively. By induction, under toll t_e' if we let r_1 units go through G_1, the travel cost through G_1 only increases to some $\mathcal{L}' \ge \mathcal{L}$. Thus at equilibrium $r_1' \le r_1$ are routed through G_1 and $r_2' \ge r_2$ units are routed through G_2 in order to equalize the travel costs through G_1 and G_2, which implies that the common cost at equilibrium only increases.

Given the above we go on to prove that for elastic demand for any link e, setting $t'_e > t_e$ only increases the common travel cost at equilibrium and only decreases the demand being routed. Let r be the demand routed at equilibrium under toll t_e and $\mathcal{L}$ be the common cost at equilibrium. If r units are routed when t'_e is set, then, by above, the travel cost only increases to some $\mathcal{L}' \geq \mathcal{L}$. Yet this may not be the equilibrium since some users may incur travel cost greater than their utility. By removing flow (while in equilibrium) we reduce the travel cost and increase the minimum utility among the users that travel, until these two get equal at which point we have an equilibrium with the traffic demand routed equal to some $r' \leq r$ and the travel cost equal to some $\mathcal{L}'' : \mathcal{L} \leq \mathcal{L}'' \leq \mathcal{L}'$, proving the claim.

Now we go back to prove the proposition. Let E' be a set of links such that for all $e, e' \in E'$ and all paths $P : e \in P$, we have $e' \notin P$. We will prove that E' is complementary. Consider an arbitrary edge $e \in E'$. In order for e not to be in the same path with some other arbitrary edge $e' \in E'$ it should be that in the decomposition of G, e lies in some graph G_1, e' lies in some graph G_2 and G_1 and G_2 are connected in parallel. To prove that $x_{e'}(t_e, \boldsymbol{t}_{-e}) \leq x_{e'}(t'_e, \boldsymbol{t}_{-e})$ it suffices to show that G_1 gets at most the same flow and G_2 gets at least the same flow at equilibrium under t'_e compared to under t_e, since, if this is the case, all of G_2's edges may only gain flow.

We prove this by induction on the decomposition of the series-parallel graph starting from the full graph. For the base case, let G_1 and G_2's composition happen the first time that a parallel composition between a graph containing e_1 and some other graph occurs (some series compositions may have occurred earlier). By what we have proved earlier we know that under t'_e only less flow may go through G_1 and G_2, yet their common travel cost will only increase. This implies that the flow through G_2 only increases, since its tolls remained unchanged, implying at the same time that the flow through G_1 only decreases.

For the induction, consider all compositions of a graph containing e_1 and some other graph that happens before the composition of G_1 and G_2, and let the induction hypothesis hold for them. Using the induction hypothesis for the last time that such a composition occurrs, we get that the graph that contains e_1 gets at most the same flow under t'_e and has at least the same cost. This further implies the flow through G_1 and G_2 only decreases and the cost only increases under t'_e. Similar to above, the latter implies the flow through G_2 only increases, since its tolls remained unchanged, implying at the same time that the flow through G_1 only decreases, completing the induction and the proof. □

Proof of Theorem 5. First, we prove that if $\mathcal{G}$ admits a Nash equilibrium then it must be $\bar{\boldsymbol{t}}$. To reach a contradiction, assume there exists a Nash equilibrium $\boldsymbol{t} \neq \bar{\boldsymbol{t}}$. Since $\bar{\boldsymbol{t}}$ are set as caps, $\boldsymbol{t} \neq \bar{\boldsymbol{t}}$ implies that $\boldsymbol{t} \leq \bar{\boldsymbol{t}}$ and the set $E^< = \{e \in E : t_e < \bar{t}_e\}$ is nonempty. By Lemma 1 (i) there exists an

$e_1 \in E^<$ such that $x_{e_1}(\boldsymbol{t}) \geq x_{e_1}(\bar{\boldsymbol{t}})$. Let player k be the player operating e_1. We will show that $\boldsymbol{t}$ is not a Nash equilibrium, by showing that player k is not on her best response.

It is $x_{e_1}(\bar{\boldsymbol{t}}) > 0$ or else $0 \leq t_{e_1} \leq \bar{t}_{e_1} \leq \hat{t}_{e_1} = x_{e_1}(\bar{\boldsymbol{t}})\ell'_{e_1}(x_{e_1}(\bar{\boldsymbol{t}})) = 0$ implying $t_{e_1} = \bar{t}_{e_1}$ contradicting $e_1 \in E^<$. This further implies that $t_{e_1} > 0$, otherwise player k is not on her best response since by continuity she can increase the toll to $t_{e_1} = \delta$ for some small enough $\delta > 0$ so that e_1 still gets positive flow and thus she gains strictly more profit: positive profit from e_1 (instead of zero) and at least the same profit from her other links since they don't lose flow, by the complementarity condition.

Consider the profit function $\pi_{e_1}(\cdot)$ of link e_1 when all other tolls are kept fixed. Toll t_{e_1} cannot be a local optimum of $\pi_{e_1}(\cdot)$ since by Lemma 2 (recall $t_{e_1} > 0$) and $x\ell'(x)$ being increasing, it would be $t_{e_1} \geq x_{e_1}(\boldsymbol{t}) \cdot \ell'_{e_1}(x_{e_1}(\boldsymbol{t})) \geq x_{e_1}(\bar{\boldsymbol{t}}) \cdot \ell'_{e_1}(x_{e_1}(\bar{\boldsymbol{t}})) = \hat{t}_{e_1} \geq \bar{t}_{e_1}$, contradicting $e_1 \in E^<$. Additionally, any local optimum $\pi_{e_1}(\cdot)$ must be above t_{e_1} as otherwise, if there was a local optimum $t'_{e_1} < t_{e_1}$, then, by Lemma 1(i) it would be $x_{e_1}(t'_{e_1}) \geq x_{e_1}(t_{e_1})$ and by a similar reasoning as above it would be $t'_{e_1} \geq \bar{t}_{e_1}$, a contradiction.

By the above, any local optimum of $\pi_{e_1}(\cdot)$ is strictly above t_{e_1} and consequently there exists a sufficiently small $\delta > 0$ such that increasing the toll of link e_1 from t_{e_1} to $t_{e_1} + \delta$, strictly increases the profit made by link e_1. Now consider the deviation for player k in which she increases the toll of e_1 by δ. By the previous discussion, she strictly gains more profit from e_1 but also she does not lose profit from the other links she operates, since by the complementarity condition each of her links may only gain flow. Thus, player k is not choosing a best response.

It remains to prove that $\bar{\boldsymbol{t}}$ is a Nash equilibrium. To reach a contradiction, assume $\bar{\boldsymbol{t}}$ is not a Nash equilibrium and let k be a player that by deviating from $\bar{\boldsymbol{t}}$ in some of her links, strictly gains more profit, i.e., for the profit of player k it is $\pi_k(\bar{\boldsymbol{t}}) < \pi_k(\boldsymbol{t}^1)$, where $\boldsymbol{t}^1$ is the resulting vector after player k's deviation. W.l.o.g., we may assume that for all links for which we have deviation from $\bar{\boldsymbol{t}}$, i.e., $t^1_e < \bar{t}_e$, it is $t^1_e > 0$, since if a link has $t^1_e = 0$ and $x_e(\boldsymbol{t}^1) = 0$ then setting $t^1_e > 0$ does not change the flow or the profits and thus we still have an improving deviation for player k, while if a link has $t^1_e = 0$ and $x_e(\boldsymbol{t}^1) > 0$ then player k will still be on an improving deviation if she sets $t^1_e = \delta$ for some small enough $\delta > 0$. She will still be at an improving deviation since by that small increase on the toll, e_1 still has positive flow and thus she gains strictly more profit: positive profit from e_1 (instead of zero) and at least the same profit from her other links since they don't lose flow, by the complementarity condition.

The underlying idea for the proof is the following. Under $\boldsymbol{t}^1$, because of Lemmas 1 (i) and 2, there exists a link for which player k can increase its toll and gain more profit from it and at the same time not lose profit from other links. The resulting toll vector gets closer to the cap vector $\bar{\boldsymbol{t}}$ and has strictly more profit, and as long as it does not get equal to $\bar{\boldsymbol{t}}$ we may repeat the procedure and get even closer to $\bar{\boldsymbol{t}}$ with even bigger profits. Formally, though, we will do it a bit different. Starting from the deviation $\boldsymbol{t}^1$, we will create a sequence of toll vectors $\boldsymbol{t}^i$ that will all correspond to deviations of player k and are such that for any $i \geq 2$ player k's profit under $\boldsymbol{t}^i$ is only higher than her profit under $\boldsymbol{t}^{i-1}$ (and $\boldsymbol{t}^1$ and $\bar{\boldsymbol{t}}$). Then we will show that this sequence converges to $\bar{\boldsymbol{t}}$ getting a contradiction since the above will imply $\pi_k(\bar{\boldsymbol{t}}) < \pi_k(\boldsymbol{t}^1) \leq \ldots \leq \pi_k(\boldsymbol{t}^i) \leq \ldots \leq \pi_k(\bar{\boldsymbol{t}})$.

Let E_k be the set of links that player k operates and consider any arbitrary cyclic order of the links of E_k. Also, let e^1 be an arbitrary link of E_k. To inductively create $\boldsymbol{t}^i$ from $\boldsymbol{t}^{i-1}$, for $i \geq 2$, let e^i be the link that follows e^{i-1} in the cyclic order of E_k. Consider the profit function $\pi_{e^i}(\cdot)$ of link e^i when the tolls on other links are fixed according to $\boldsymbol{t}^{i-1}$. Let $t^{i*}_{e^i}$ be the smallest local optimum of $\pi_{e^i}(\cdot)$ and set $t^i_{e^i} = \min\big\{\max\{t^{i-1}_{e^i}, t^{i*}_{e^i}\}, \bar{t}_{e^i}\big\}$. For any other link e of E^k $(e \neq e^i)$ set $t^i_e := t^{i-1}_e$. Note first that by construction, $\pi_k(\boldsymbol{t}^i) \geq \pi_k(\boldsymbol{t}^{i-1})$, since the only toll that possibly changes is that of e^i and it only increases towards the first maximizer of $\pi_{e^i}(\cdot)$, implying that the profit from e^i only increases and, by the complementarity condition, the flow and profit on any other link of E_k only increases. Additionally, note that for all i it is $\boldsymbol{t}^i \leq \bar{\boldsymbol{t}}$ and for all $i \geq 2$ it is $\boldsymbol{t}^{i-1} \leq \boldsymbol{t}^i$, since, at every step, the only link getting its toll changed is e^i, and it gets a toll in between $t^{i-1}_{e^i}$ and $\bar{t}_{e^i}$. Thus, $\boldsymbol{t}^i$ is an increasing sequence of toll vectors which additionally is upper bounded by $\bar{\boldsymbol{t}}$ and consequently it must converge to a toll vector, say $\boldsymbol{t}^0$, i.e., $\boldsymbol{t}^i \to \boldsymbol{t}^0$. Last, note that for all i, $\boldsymbol{t}^i \leq \boldsymbol{t}^0$.

We will prove that $\boldsymbol{t}^0 = \bar{\boldsymbol{t}}$. To reach a contradiction, assume otherwise and let $E^<_k = \{e \in E_k : t^0_e < \bar{t}_e\}$ be nonempty. By Lemma 1(i) there exists a link $f \in E^<_k$ such that $x_f(\boldsymbol{t}^0) \geq x_f(\bar{\boldsymbol{t}})$. Consider the profit function $\pi_f(\cdot)$ of link f when all other tolls are kept fixed (according to $\boldsymbol{t}^0$). Toll t^0_f cannot be a local optimum of $\pi_f(\cdot)$ since in that case by Lemma 2 (we can apply Lemma 2 since $t^0_f \geq t^1_f$ and for all e with $t^1_e < \bar{t}_e$ it is $t^1_e > 0$, implying $t^0_f > 0$) it would be $t^0_f \geq x_f(t^0_f) \cdot \ell'_f(x_f(t^0_f)) = x_f(\boldsymbol{t}^0) \cdot \ell'_f(x_f(\boldsymbol{t}^0)) \geq x_f(\bar{\boldsymbol{t}}) \cdot \ell'_f(x_f(\bar{\boldsymbol{t}})) = \hat{t}_f \geq \bar{t}_f$, contradicting $f \in E^<_k$. Additionally, any local optimum $\pi_f(\cdot)$ must be above t_f as otherwise, if there was a local optimum $t'_f < t^0_f$, then, by Lemma 1(i) it would be $x_f(t'_f) \geq x_f(t^0_f)$ and by a similar reasoning as above it would be $t'_f \geq \bar{t}_f$, a contradiction. Thus any local optimum for $\pi_f(\cdot)$ is strictly above t^0_f, say at $t^{0*}_f = t^0_f + \delta$, for $\delta > 0$.

On the other hand, consider all those times when while creating the sequence $\boldsymbol{t}^i$, link f was candidate for getting its toll changed (recall the cyclic order when choosing which tolls to update). Intuitively, since $\boldsymbol{t}^i \to \boldsymbol{t}^0$, there is some high enough j such that the toll vectors $\boldsymbol{t}^0$ and $\boldsymbol{t}^j$ are so close to each other so that the corresponding profit functions of link f almost coincide. More formally, for the δ defined above, since $\boldsymbol{t}^i \to \boldsymbol{t}^0$, there must be some j for which link f was candidate for getting its toll changed and is such that the smallest local optimum of the profit function of f under toll vector $\boldsymbol{t}^j$, say t^{j*}_f, is such that $|t^{0*}_f - t^{j*}_f| \leq \delta/2$. The latter implies that $t^{j*}_f > t^0_f$, since by above $t^{0*}_f = t^0_f + \delta$. But this readily gives a contradiction as for that j, based on the construction of the sequence (namely $t_f^j = \min\big\{\max\{t_f^{j-1}, t_f^{j*}\}, \bar{t}_f\big\}$), link f would get either t_f^{j*} or $\bar{t}_f$ as its toll which are both strictly greater than t_f^0 (recall, $t_f^{j-1} \le t_f^0$ by construction).

Thus, indeed $\boldsymbol{t}^0 = \bar{\boldsymbol{t}}$ and so $\boldsymbol{t}^i \to \bar{\boldsymbol{t}}$. Recall that the profit function $\pi_k(\cdot)$ of player k is a continuous function of tolls. This yields $\pi_k(\boldsymbol{t}^i) \to \pi_k(\bar{\boldsymbol{t}})$, which combined to that for all i, $\pi_k(\boldsymbol{t}^i) \ge \pi_k(\boldsymbol{t}^1) > \pi_k(\bar{\boldsymbol{t}})$, gives the desired contradiction: $\pi_k(\bar{\boldsymbol{t}}) > \pi_k(\bar{\boldsymbol{t}})$. Consequently, every player is on her best response when playing according to $\bar{\boldsymbol{t}}$. □

Proof of Theorem 6. We need only prove that $\bar{\boldsymbol{t}}$ is a strong Nash equilibrium. The proof is similar to that of Theorem 5. Assume there is a coalition of players that deviates form $\bar{\boldsymbol{t}}$ on some of their links, so that no player loses profit and some player, say player k, strictly gains profit. Since all links are complementary we may intuitively think that all deviated links belong to player k. We may get a contradiction similar to the proof of Theorem 5, since we may get a sequence of deviations $\boldsymbol{t}^i \to \bar{\boldsymbol{t}}$ such that, for every $i \ge 2$, no deviating player gets worse profit in $\boldsymbol{t}^i$ compared to $\boldsymbol{t}^{i-1}$ and for player k it is $\pi_k(\boldsymbol{t}^i) \ge \pi_k(\boldsymbol{t}^{i-1}) \ge \pi_k(\boldsymbol{t}^1) > \pi_k(\bar{\boldsymbol{t}})$, yielding $\pi_k(\bar{\boldsymbol{t}}) > \pi_k(\bar{\boldsymbol{t}})$. □

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