Optimal Power Allocation for a Time-Varying Wireless Channel under Heavy-Traffic Approximation

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***Abstract*— This paper studies the problem of minimizing the queueing delay for a time-varying channel with a single queue, subject to constraints on the average and peak power. First, by separating the time-scales of the arrival process, the channel process and the queueing dynamics it derives a heavy-traffic limit for the queue length in the form of a reflected diffusion process.**

Given a monotone function of the queue-length process that serves as a penalty, and constraints on the average and peak available power, it shows that the optimal power allocation policy is a channel-state based threshold policy. For each channel state j there corresponds a threshold value of the queue length, and it is optimal to transmit at peak power if the queue length exceeds this threshold, and not transmit otherwise.

Numerical results compare the optimal policy for the original Markovian dynamics to the threshold policy which is optimal for the heavy-traffic approximation, to conclude that that latter performs very well even outside the heavy-traffic operating regime.

***Index Terms*— power allocation, heavy-traffic, controlled diffusion, fading channel**

I. Introduction

With the widespread deployment of wireless and ad-hoc networks, the energy-efficiency of wireless transmission in a fading channel has attracted much attention. It is now well understood that a transmission scheme that takes advantage of the time-varying character of a channel can significantly improve the use of scarce energy resources. As an extreme case, the policy that transmits only when the channel is in the best state can achieve the best energy efficiency while resulting in arbitrary long delay. Thus, there is clearly a trade-off between energy efficiency and delay constraints.

The problem of energy-efficient scheduling over a fading wireless channel has been studied under different delay constraints in the recent past [1]–[3]. In [1], [2], the authors consider scheduling under a hard delay constraint, and maximize the throughput given energy and timing constraints. In [2], a finite horizon stochastic control formulation is used and a closed form solution to the dynamic programming equation is derived in some simplified cases. Berry and Gallager consider power control with delay constraints in an asymptotic sense [3]. They consider a single queue served by a fading channel. For a given data-arrival rate, the minimum power required to stabilize the queue can be computed directly from the capacity of the channel. However, with this minimum power, it is well known from queueing theory that the associated queueing delay is unbounded. The authors in [3] allocate an excess power ΔP and study the associated mean queuing delay D. They show that the optimal power control policy which takes both the channel state and the queue length into account results in an excess-power versus delay trade-off that behaves asymptotically as $\Delta P \propto \frac{1}{D^2}$. Further, they show that a single queue-length based threshold type policy achieves the same decay rate as the optimal policy (however, they do not show optimality of the threshold policy).

In recent years, the heavy-traffic approximation has been successfully applied to performance evaluation and control of communication networks. By heavy-traffic, we mean that the average fraction of time that the server is free is small, or equivalently, the traffic intensity of the server approaches 1. Largely due to this 'small idle time' assumption, the scaled queueing process can be well approximated by a reflected diffusion process. In [4], Buche and Kushner apply the heavy-traffic approximation to model the multi-user power allocation problem in time varying channels, and design an optimal control in the heavy-traffic region. They consider the scenario where a fixed amount of power is available at each time slot, and this power needs to be allocated to multiple users according to their queue length and current channel states. They show that the optimal policy is a switching curve.

A. Main Contributions

In this paper, we study a single queue with a time-varying channel having a finite number of channel states indexed by $j \in \{1, 2, \dots, N\}$. We impose both a peak power constraint $p_{\max}$, as well as an average power constraint $\bar{p}$ for power allocation. We work with the heavy-traffic limit for such a system under a fast channel variation assumption [4]–[6], whose dynamics are governed by a reflected Itô stochastic differential equation.

We consider the problem of minimizing the long-term average value of a function $c(x)$ which depends on the heavy-traffic queue-length process x, subject to the peak and average power constraints. We consider a continuous cost function $c(x)$ (where x corresponds to the heavy-traffic queue length), that satisfies either *(i)* $c(x)$ is strictly increasing and bounded, or *(ii)* $c(x)$ grows unbounded (i.e., $c(x) \to \infty$, as $x \to \infty$). For

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example, $c(x) = x$ corresponds to minimizing the average queue length (or equivalently, from Little's law, the mean delay). The main contributions of this paper are:

(i) We show that when c is monotone, then the optimal control that minimizes the long-term average cost subject to the power constraints is a *channel state based threshold policy*. Specifically, associated with each channel state j there is a queue-threshold $\hat{x}_j$, such that at any time t, the optimal policy transmits at *peak power* $p_{\max}$ over channel state j, if the queue length $x(t) > \hat{x}_j$, and does not transmit otherwise. Further, using Lagrange duality and exploiting the monotonicity property of c, we reduce the problem of determining the queue-thresholds $\{\hat{x}_j, j = 1, 2, \ldots, N\}$ to that of solving a set of algebraic equations. Throughout the analysis we strive not to rely as much on the one-dimensional (one queue) character of the problem, aiming to present an approach that can scale up to higher dimensions.

(ii) An interpretation of the heavy-traffic limit is the following: Given a data arrival rate, sufficient "equilibrium" power is first allocated such that the capacity of the channel matches the arrival rate. Then, an amount of *excess power* is allocated based on the channel state and queue length. With such an interpretation, a special case of our result when the equilibrium power is allocated according to channel state dependent water-filling [7] (and is strictly positive in each channel state), results in the queue-length threshold being channel state invariant. In other words, for any monotone cost function $c(x)$, we have $\hat{x}_j = \hat{x}$, independent of channel state j. Thus, by applying the cost function $c(x) = x$, in this special case, our results indicate that the single-threshold policy derived in [3] is in fact asymptotically optimal.

(iii) For a system not in heavy-traffic, we numerically compute the optimal policy using dynamic programming, and compare this with the threshold policy that is optimal in the heavy-traffic limit. These numerical results indicate that the threshold policy performs close to the optimal policy even when the system is not in heavy-traffic.

(iv) From a technical standpoint, this problem falls under the domain of ergodic control of diffusions with constraints, and we adopt the convex analytic approach of [8], [9]. The approach in [9] requires *both* the cost function as well as the constraint function (due to power constraints) to satisfy a *near-monotone* condition (see (17)). However, the constraint function is *not* near-monotone in our problem. Hence, since the results in [9] cannot be quoted, we first establish the existence of an optimal control within the class of stationary feedback controls. Next, using classical Lagrange multiplier theory, we show that the constrained problem is equivalent to an unconstrained one, namely minimizing the ergodic cost of the associated Lagrangian. We accomplish this by establishing that the near-monotone condition is satisfied for the Lagrangian (this result uses only the near-monotonicity of the cost function), and proceed to characterize the optimal policy for the unconstrained problem via the associated Hamilton Jacobi Bellman (HJB) equation. The solution to the original problem is then obtained by a straightforward application of Lagrange duality. We exhibit the structure of the optimal policy, and also establish that optimality holds over all non-anticipative policies, and not only over the stationary ones.

B. Paper Organization

The paper is organized as follows. Section II presents the Markovian model and the heavy-traffic model for the time-varying channel. In Section III we describe the optimal control problem and prove the existence of an optimal policy among stationary ones. In Section IV we introduce the equivalent unconstrained problem using Lagrange multiplier theory and characterize the ergodic control problem relative to the Lagrangian via the HJB equation. We also show that the optimal policy has a multi-threshold structure. In Section V we present an analytical solution of the HJB equation. In order to demonstrate the approach, we specialize to the problem of minimizing the mean delay, i.e., $c(x) = x$, and derive closed form expressions for one and two-state channels. In Section VI, we evaluate the performance of the optimal policy for the heavy-traffic model by applying it to a system which does not operate in the heavy-traffic region. Conclusions and some discussion on future directions are presented in Section VII.

II. The System Model and the Heavy-Traffic Limit

We consider a queuing system that consists of a transmitter operating over a fading channel (see Figure 1). Time is assumed to be divided into discrete slots, and the channel state process is an irreducible, aperiodic, finite state Markov chain $L(t)$ with N states having a stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$. The channel gain is denoted by g_j when the channel state $L(t) = j$, and the power P allocated at time t determines the service rate $r(P, j)$ of the queue. For example, given the power P, bandwidth W and channel gain g_j, $r(P, j) = W \log_2(1 + P g_j)$ is the Shannon capacity, the upper bound of the channel transmission rate. The service rate $r(P, j)$ can take different forms for practical systems depending on the details of modulation and coding.

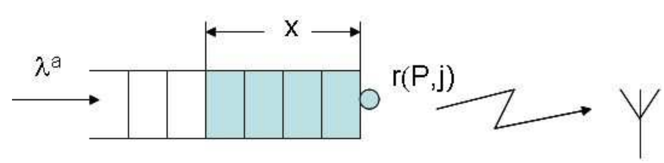

Fig. 1. A transmitter sends packets to a receiver through a time-varying wireless channel.

As is common in heavy-traffic analysis, we construct a sequence of queueing systems indexed by n, such that as $n \to \infty$, the transmitter idle time goes to zero in an appropriate manner (see (1) below). In the heavy-traffic approximation, there are two time scales: one is the time scale the real system works on; the other is the diffusion time scale, which is a slower scale. A small time period Δt in the diffusion time scale contains a large number of arrivals and departures, which is of order $\mathcal{O}(n\Delta t)$. For a wireless channel with time-varying characteristics, there is yet another time scale, i.e., the time scale of channel variation. We consider the fast channel variation model [5], [6], which assumes that the channel variation has a time scale faster than the diffusion time scale, but slower than the arrival process time scale, as shown in Figure 2. Thus, for the n-scaled queueing system, the channel process is $L(n^{-\kappa}t)$, where $\kappa \in (0,1)$. As a result, over an interval of time $n\Delta t$, the number of channel transitions is $\mathcal{O}(n^{1-\kappa}\Delta t)$, and the number of arrivals *within each channel state* (i.e, between any pair of channel transitions) is $\mathcal{O}(n^{\kappa}\Delta t)$. Thus, the total number of arrivals over the time interval $n\Delta t$ is $\mathcal{O}(n\Delta t)$.

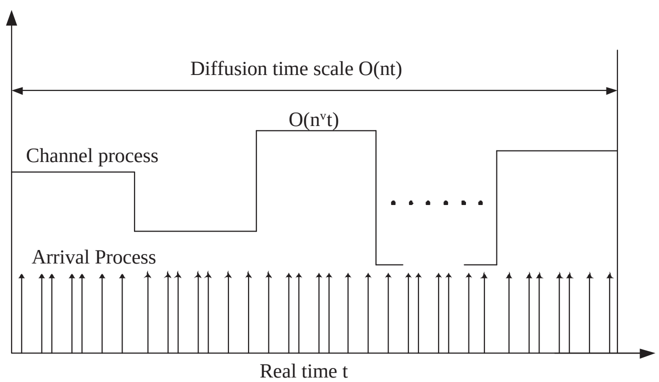

Fig. 2. The three time scales of the heavy-traffic model under the fast channel variation assumption.

Practically, this scaling fits into the scenario that the channel changes slowly compared to the packet arrival rate, i.e., a slowly fading channel such as an indoor wireless environment, or a low-mobile-velocity outdoor wireless environment [10]. For instance, with 1xEV-DO (the 3G wireless data service), a scheduling time-slot is 1.667 msec, which corresponds to the arrival time-scale. For a mobile user with velocity 6 mph, the channel coherence time, which corresponds to the time-scale of channel changes, is about 50 msec. Thus, the scaling we use in this paper seems applicable in these practical regimes.

We consider a sequence of queueing systems indexed by n, with the queue length $x^n(t)$, arrival process $A^n(t)$ and departure process $D^n(t)$, which can be controlled by transmission power. For the queueing system indexed by n, we denote the l-th inter-arrival time by ζ_l^n, and assume it satisfies the following assumption [11].

Assumption 2.1: The inter-arrival intervals $\{\zeta_l^n,\ l \in \mathbb{N}\}$ satisfy the following:

1) $\big\{|\zeta_l^n|^2,\ l,n \in \mathbb{N}\big\}$ is uniformly integrable.
2) For each n, $\{\zeta_l^n,\ l \in \mathbb{N}\}$ are independent. Moreover, there exist constants $\bar{\zeta}^n$, $\bar{\zeta}$, σ_a^2, such that
$$\mathbb{E}[\zeta_l^n] = \bar{\zeta}^n \xrightarrow[n\to\infty]{} \bar{\zeta}\,, \qquad \lim_{n\to\infty} \mathbb{E}\left[1 - \frac{\zeta_l^n}{\bar{\zeta}^n}\right]^2 = \sigma_a^2\,.$$
3) The inter-arrivals are independent of the channel process.

Note that if either ζ_l^n are identically distributed with finite variance, or ζ_l^n are deterministic but periodic, Assumption 2.1 is satisfied. The mean arrival rate for the n-th system is defined as $\lambda_n^a = 1/\bar{\zeta}^n$ and the limiting arrival rate λ^a is defined as $\lambda^a = 1/\bar{\zeta}$.

For the queue indexed by n, the service rate r is controlled by the transmission power P_n. Under the heavy-traffic approximation, we suppose that mean arrival rate converges to the service rate under the scaling,
$$\lim_{n\to\infty} \Big(\lambda_n^a - \mathbb{E}\big[r(P_n)\big]\Big) n^{\frac{1-\kappa}{2}} = \text{constant}\,. \tag{1}$$
for some $\kappa \in (0,1)$. Assuming (1) holds, we decompose the power allocation $P(q,j)$ for buffer size q, and channel state j into
$$P_n(q,j) = P_0(j) + n^{-\frac{1-\kappa}{2}} u_j(q)\,.$$
The "equilibrium" power $P_0(j)$ is allocated in such a manner that
$$\lambda^a = \sum_{j=1} r(P_0(j),j)\pi_j\,, \tag{2}$$

Remark 2.1: Note that the optimal allocation of the equilibrium power gives rise to a static optimization problem, namely, minimize the average power $\mathbb{E}[P]$ given the service rate $\mathbb{E}[r(P)] \geq \lambda^a$, where $\mathbb{E}[\,\cdot\,]$ is taken over the channel distribution. For a fading channel with additive white Gaussian noise (AWGN), water-filling is the optimal way for allocating power subject to (2) in an information theoretic sense [7]. In general, the equilibrium allocation can be computed numerically.

In this paper, we assume that the equilibrium power has been allocated, either by water-filling or by numerically determining the optimal allocation, and we address the problem of optimally allocating the residual power. Optimality here is in an asymptotic sense, i.e., pertains to the limiting system under heavy-traffic conditions. By expanding the service rate $r(P,j)$ around $P = P_0(j)$, using Taylor's series, we obtain
$$r(P,j) = r(P_0(j),j) + \frac{u_j}{n^{\frac{1-\kappa}{2}}}\frac{\partial r}{\partial P}(P_0(j),j) + o(n^{-\frac{1-\kappa}{2}})\,.$$
Let
$$r_0(j) := r(P_0(j),j)\,, \qquad \gamma_j := \frac{\partial r}{\partial P}(P_0(j),j)\,.$$
Then
$$r(P,j) \approx r_0(j) + n^{-\frac{1-\kappa}{2}}\gamma_j u_j\,. \tag{3}$$
Thus, $\lambda^a = \sum_{j=1} r_0(j)\pi_j$, and the incremental service rate gained from the residual amount of power u is $n^{-\frac{1-\kappa}{2}}b(u)$, where
$$b(u) = \sum_{j=1}^{N} \gamma_j \pi_j u_j\,.$$

Remark 2.2: We observe that if the equilibrium power $\{P_0(j)\}$ is allocated according to channel-state dependent water-filling [7], and if such an allocation results in $P_0(j) > 0$ for all channel states j, then $\gamma_i = \gamma_j$ for all i, j.

Next, defining $x^n(t) := n^{-\frac{(1+\kappa)}{2}}q(nt)$ and using the techniques in [4], we show in Appendix I that $x^n(t)$ converges weakly to a limiting queueing system as $n \to \infty$. The dynamics of the limiting queueing system are governed by the equation

$$x(t) = x(0) - \int_0^t b\big(u(s)\big)\,ds + \sigma W(t) + z(t)\,, \tag{4}$$

where $x(t)$ is the queue-length process, $W(t)$ is the standard Wiener process, σ is a positive constant, $z(t)$ is a nonincreasing process and grows only at those points t for which $x(t) = 0$, and $x(t) \geq 0$, for all $t \geq 0$. The process $z(t)$, which ensures that the queue-length $x(t)$ remains non-negative, is uniquely defined. For further details see [12, pp. 128, Theorem 6.1] and [13, pg. 178]. The corresponding Itô stochastic differential equation describing the heavy-traffic dynamics takes the form

$$dx(t) = -b\big(u(t)\big)dt + \sigma dW(t) + dz(t)\,. \tag{5}$$

III. The Optimal Control Problem for the Heavy-Traffic Model

The optimization problem of interest for the non-scaled queueing system is to minimize (pathwise, a.s.) the long-term average queueing length (and thus, from Little's law, the mean delay)

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T q(t)\,dt\,,$$

or more generally, to minimize the long-term average value of some penalty function $c : \mathbb{R}_+ \to \mathbb{R}$, i.e.,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T c\big(q(t)\big)\,dt\,,$$

subject to a constraint on the average available power of the form

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T P\big(q(t), L(t)\big)\,dt \leq P_{\text{avg}}\,.$$

It is well known from queueing theory, that if only the basic power P_0 is allocated, which matches the service rate to the arrival rate, then the resulting traffic intensity is equal to 1, and the queueing delay diverges. However, choosing the control term u appropriately can result in a bounded average queue length. In the heavy-traffic model described in Section II, once the channel model is provided, v is fixed, and only the excess power u can be used to control the queue. Thus the original optimization problem transforms to an analogous problem in the limiting system, namely,

$$\text{minimize} \qquad \limsup_{T\to\infty} \frac{1}{T}\int_0^T c\big(x(t)\big)\,dt\,, \quad \text{a.s.} \tag{6a}$$

$$\text{subject to} \qquad \limsup_{T\to\infty} \frac{1}{T}\int_0^T h\big(u(t)\big)\,dt \leq \bar{p}\,, \quad \text{a.s.} \tag{6b}$$

where

$$h(u) = h(u_1, \dots, u_N) = \sum_{j=1}^N \pi_j u_j\,.$$

The control variable u takes values in $U := \big[0, p_{\text{max}}\big]^N$, with p_{max} denoting the (excess) peak power, and $\bar{p}$ denoting the (excess) average power. Naturally, for the constraint in (6b) to be feasible $\bar{p} \leq p_{\text{max}}$.

The standard probabilistic framework for (5) is as follows. Let $(\Omega, \mathbb{P}, \mathfrak{F})$ be a complete probability space and $\{\mathfrak{F}_t\}$ be a right-continuous filtration of σ-algebras such that $\mathfrak{F}_t$ is complete with respect to the measure $\mathbb{P}$. The Wiener process is $\mathfrak{F}_t$-adapted, and for any $t,\ h \geq 0$ the random variable $W(t+h) - W(t)$ and σ-algebra $\mathfrak{F}_t$ are independent. Also, the initial condition $x(0)$ is an $\mathfrak{F}_0$-measurable random variable and has a finite second moment.

Definition 3.1: The minimization in (6) is over all control processes $u(t)$ which are progressively measurable with respect to the σ-algebras $\{\mathfrak{F}_t\}$. Such a process u is called an *admissible control* and the class of admissible controls is denoted by $\mathfrak{U}$. An admissible control which takes the form $u(t) = v\big(x(t)\big)$, for some measurable function $v : \mathbb{R}_+ \to U$ is called a *stationary* (Markov) control, and we denote this class by $\mathfrak{U}_{\text{s}}$.

Given a measurable function $v : \mathbb{R}_+ \to U$, the stochastic differential equation in (5) under the control $u(t) = v\big(x(t)\big)$ has a unique strong solution, which is a Feller-Markov process. Let $\mathbb{E}_x^v$ denote the expectation operator on the path space of the process, with initial condition $x(0) = x$, and T_t^v denote the Markov semigroup acting on the space of bounded continuous functions $\mathcal{C}_b(\mathbb{R}+)$, defined by $T_t^v f(x) = \mathbb{E}_x^v\big[f\big(x(t)\big)\big]$, $f \in \mathcal{C}_b(\mathbb{R}+)$. It is known that T_t^v has infinitesimal generator $\mathcal{L}^{v(x)}$ (see [14, pg. 366-367], [15]), where

$$\mathcal{L}^u := \frac{\sigma^2}{2}\frac{d^2}{dx^2} - b(u)\frac{d}{dx}\,, \quad u \in U\,.$$

The boundary at 0, imposes restrictions on the domain of $\mathcal{L}^u$ (see [14, pg. 366-367]).

The generator $\mathcal{L}$ can be readily used to compute functionals of the process. As asserted in [15, pg. 80], if f is a bounded measurable function on $\mathbb{R}_+$ then $\varphi(x,t) = \mathbb{E}_x^v\big[f\big(x(t)\big)\big]$ is a generalized solution of the problem

$$\begin{aligned} \frac{\partial\varphi}{\partial t}(x,t) &= \mathcal{L}^v\varphi(x,t)\,, \quad x \in (0,\infty)\,,\ t > 0\,, \\ \varphi(x,0) &= f(x)\,, \qquad \frac{\partial\varphi}{\partial x}(0,t) = 0\,. \end{aligned} \tag{7}$$

Also, Itô's formula can be applied as follows [16, pg. 500, Lemma 4], [17]: If $\varphi \in \mathcal{W}^{2,p}(\mathbb{R}_+)$ is a bounded function (here $\mathcal{W}$ stands for the Sobolev space) satisfying $\frac{d\varphi}{dx}(0) = 0$, then for $t \geq 0$,

$$\mathbb{E}_x^v\big[\varphi\big(x(t)\big)\big] - \varphi(x) = \mathbb{E}_x^v\left[\int_0^t \mathcal{L}^v\varphi\big(x(t)\big)\,dt\right]. \tag{8}$$

Definition 3.2: A control $v \in \mathfrak{U}_{\text{s}}$ is called *stable* if the resulting $x(t)$ is positive recurrent. We denote the class of stable controls by $\mathfrak{U}_{\text{ss}}$. A control $v \in \mathfrak{U}_{\text{s}}$ is called *bang-bang*, or *extreme*, if $v(x) \in \{0, p_{\text{max}}\}^N$, for almost all $x \in \mathbb{R}_+$. We refer to the class of extreme controls in $\mathfrak{U}_{\text{ss}}$ as *stable extreme* controls and denote it by $\mathfrak{U}_{\text{se}}$.

Let $\mathcal{P}(\mathbb{R}_+)$ denote the set of probability measures on the Borel σ-field of $\mathbb{R}_+$. Recall that a probability measure $\mu \in \mathcal{P}(R_+)$ is said to be invariant for process $x(t)$ under the control $v \in \mathfrak{U}_{\text{s}}$, if $\int T_t^v f\,d\mu = \int f\,d\mu$, for all $f \in \mathcal{C}_b(R_+)$, and $t \geq 0$. It is the case that if $v \in \mathfrak{U}_{\text{ss}}$, then the controlled process $x(t)$ has a unique invariant probability measure μ_v which is absolutely continuous with respect to the Lebesgue measure.

Let $\mathcal{C}_c^\infty(0,\infty)$ denote the class of smooth functions in $(0,\infty)$ with compact support. We make frequent use of the following characterization. A necessary and sufficient condition for a probability measure $\mu \in \mathcal{P}(R_+)$ to be an invariant probability measure of the controlled process $x(t)$ under $v \in \mathfrak{U}_{\mathrm{s}}$ is

$$\int_{\mathbb{R}_+} \mathcal{L}^v g(x)\,\mu(dx) = 0\,, \quad \forall g \in \mathcal{C}_c^\infty(0,\infty)\,. \tag{9}$$

Necessity of (9) is a straightforward application of (8) and the definition of an invariant measure. Borkar establishes sufficiency for diffusions without reflection, by employing the uniqueness of the Cauchy problem for the forward Kolmogorov equation [18, pg. 144, Lemma 1.2]. The boundary complicates matters for this approach, so we employ the following result, which we state in the d-dimensional setting. Let $D \subset \mathbb{R}^d$ be a domain and $\mathcal{L}$ a second order uniformly elliptic operator with bounded measurable coefficients in D, and with the second order coefficients Lipschitz continuous. If μ is a finite Borel measure on D satisfying $\int_D \mathcal{L}g(x)\,\mu(dx) = 0$, for all $g \in \mathcal{C}_c^\infty(D)$, then μ is absolutely continuous with respect to the Lebesgue measure, i.e., has density [19, Theorem 2.1]. Thus, if μ satisfies (9), then $\mu(dx) = f_v(x)\,dx$, and hence using the adjoint operator $(\mathcal{L}^v)^*$ we have

$$\int_{\mathbb{R}_+} g(x)(\mathcal{L}^v)^* f_v(x)\,dx = 0\,, \quad \forall g \in \mathcal{C}_c^\infty(0,\infty)\,,$$

which is equivalent to $(\mathcal{L}^v)^* f_v = 0$. Following the proof of [20, pg. 87, Proposition 8.2] and utilizing (7), we deduce that f_v is indeed the density of an invariant probability distribution. It follows from the preceding discussion that f_v is the density of an invariant probability measure μ_v if and only if it is a solution of the Fokker-Planck equation

$$\bigl(\mathcal{L}^v\bigr)^* f_v(x) = \frac{d}{dx}\left(\frac{\sigma^2}{2}\frac{df_v}{dx}(x) + b\bigl(v(x)\bigr)f_v(x)\right) = 0\,. \tag{10}$$

Moreover, solving (10), we deduce that $v \in \mathfrak{U}_{\mathrm{s}}$ is stable if and only if

$$A_v := \int_0^\infty \exp\biggl(-\frac{2}{\sigma^2}\int_0^x b\bigl(v(y)\bigr)\,dy\biggr)\,dx < \infty\,,$$

in which case the solution of (10) takes the form

$$f_v(x) = A_v^{-1}\exp\biggl(-\frac{2}{\sigma^2}\int_0^x b\bigl(v(y)\bigr)\,dy\biggr)\,. \tag{11}$$

We work under the assumption that c has the following monotone property:

Assumption 3.1: The function c is continuous and either it is asymptotically unbounded, i.e., $\liminf_{x\to\infty}\ c(x) = \infty$, or if c is bounded then it is strictly increasing. In the latter case we define

$$c_\infty := \lim_{x\to\infty}\ c(x)\,.$$

The analysis and solution of the optimization problem proceeds as follows: We first show that optimality is achieved for (6) relative to the class of stationary controls. Next, in Section IV using the theory of Lagrange multipliers we formulate an equivalent unconstrained optimization problem. We show that an optimal control for the unconstrained problem can be characterized via the HJB equation. This accomplishes two tasks. First, it enables us to study the structure of the optimal policies. Second, we show that this control is optimal among all controls in $\mathfrak{U}$. An analytical solution of the HJB equation is presented in Section V. A list of symbols is included in Table. I for quick reference.

TABLE I
TABLE OF SYMBOLS

Symbol	Definition	First Appearance
$\mathfrak{U}$ ($\mathfrak{U}_{\mathrm{s}}$)	admissible (stationary) controls	Def. 3.1
$\mathfrak{U}_{\mathrm{ss}}$ ($\mathfrak{U}_{\mathrm{se}}$)	stable stationary (extreme) controls	Def. 3.2
$\mathcal{P}(X)$	probability measures on X	Sec. III
$\mathcal{G}$	set of ergodic occupation measures	Sec. III-A
$\mathcal{M}$	set of invariant probability measures	Sec. III-A
$H(\bar{p})$	subset of $\mathcal{G}$ with power bound $\bar{p}$	(13)

A. Existence of Optimal Stationary Controls

In this subsection, we show that if the optimization problem in (6) is restricted to stationary controls, then there exists $v \in \mathfrak{U}_{\mathrm{se}}$ which is optimal.

Due to the presence of the constraint in (6b), the study of the optimization problem in (6) is more amenable by convex analytic arguments. We follow the approach in [8], [9]. However, we take advantage of the fact that the set of power levels U is convex and avoid transforming the problem to the relaxed control framework. Instead, we view U as the space of product probability measures on $\{0, p_{\max}\}^N$. This is simply stating that for each j, u_j may be represented as a convex combination of the '0' power-level and the peak power $p_{\max}$. In other words, U is viewed as a space of relaxed controls relative to the discrete control input space $\{0, p_{\max}\}^N$. This has the following advantage: by showing that optimality is attained in the set of precise controls, we assert the existence of a control in $\mathfrak{U}_{\mathrm{se}}$ which is optimal.

Let $\mathcal{M} \subset \mathcal{P}(\mathbb{R}_+)$ denote the set of all invariant probability measures μ_v of the process $x(t)$ under the controls $v \in \mathfrak{U}_{\mathrm{ss}}$. Let $\widetilde{U} := \{0, p_{\max}\}^N$. The generic element of $\widetilde{U}$ takes the form $\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_N)$, with $\tilde{u}_i \in \{0, p_{\max}\}$, $i = 1,\dots,N$. There is a natural isomorphism between U and the space of product probability measures on $\widetilde{U}$ which we denote by $\mathcal{P}_\otimes(\widetilde{U})$. This is viewed as follows. Let δ_p denote the Dirac probability measure concentrated at $p \in \mathbb{R}_+$. For $u \in U$, we associate the probability measure $\tilde{\eta}_u \in \mathcal{P}_\otimes(\widetilde{U})$ defined by

$$\tilde{\eta}_u(\tilde{u}) := \bigotimes_{i=1}^N \left[\left(1 - \frac{u_i}{p_{\max}}\right)\delta_0(\tilde{u}_i) + \frac{u_i}{p_{\max}}\delta_{p_{\max}}(\tilde{u}_i)\right]\,,$$

for $\tilde{u} \in \widetilde{U}$. Similarly, given $v \in \mathfrak{U}_{\mathrm{ss}}$ we define $\eta_v : \mathbb{R}_+ \to \mathcal{P}_\otimes(\widetilde{U})$ and and $\nu_v \in \mathcal{P}(\mathbb{R}_+ \times \widetilde{U})$ by

$$\begin{aligned}\eta_v(x, d\tilde{u}) &:= \tilde{\eta}_{v(x)}(d\tilde{u})\\ \nu_v(dx, d\tilde{u}) &:= \mu_v(dx)\eta_v(x, d\tilde{u})\,,\end{aligned}$$

where $\mu_v \in \mathcal{M}$ is the invariant probability measure of the process under the control $v \in \mathfrak{U}_{\mathrm{ss}}$. The set of *ergodic* *occupation* measures is defined as $\mathcal{G} := \{\nu_v : v \in \mathfrak{U}_{\text{ss}}\}$. It follows by (9) that $\nu \in \mathcal{G}$ if and only if

$$\int_{\mathbb{R}_+} \mathcal{L}^{\tilde{u}} g(x)\, \nu(dx, d\tilde{u}) = 0\,, \quad \forall g \in \mathcal{C}_c^\infty(0,\infty)\,. \tag{12}$$

Due to the linearity of $u \mapsto h(u)$, we have the following identity (which we choose to express as an integral rather than a sum, despite the fact that $\widetilde{U}$ is a finite space):

$$h\bigl(v(x)\bigr) = \int_{\widetilde{U}} h(\tilde{u})\eta_v(x, d\tilde{u})\,, \quad v \in \mathfrak{U}_{\text{ss}}\,,$$

As a point of clarification, 'h' inside this integral is interpreted as the restriction of h on $\widetilde{U}$. The analogous identity holds for $b(u)$.

In this manner we have defined a model whose input space $\widetilde{U}$ is discrete, and for which the original input space U provides an appropriate convexification. Note however that $U \sim \mathcal{P}_\otimes(\widetilde{U})$ is not the input space corresponding to the relaxed controls based on $\widetilde{U}$. The latter is $\mathcal{P}(\widetilde{U})$, which is isomorphic to a 2^N-simplex in $\mathbb{R}^{2^N-1}$, whereas $\mathcal{P}_\otimes(\widetilde{U})$ is isomorphic to a cube in $\mathbb{R}^N$. We select $\mathcal{P}_\otimes(\widetilde{U})$ as the input space mainly because it is isomorphic to U. Since there is a one to one correspondence between the extreme points of $\mathcal{P}_\otimes(\widetilde{U})$ and $\mathcal{P}(\widetilde{U})$, had we chosen to use the latter, the analysis and results would have remained unchanged. Even though we are not using the standard relaxed control setting, since $\mathcal{P}_\otimes(\widetilde{U})$ is closed under convex combinations and limits, the theory goes through without any essential modifications.

For $\bar{p} \in (0, p_{\text{max}}]$, let

$$H(\bar{p}) := \left\{ \nu \in \mathcal{G} : \int_{\mathbb{R}_+ \times \widetilde{U}} h(\tilde{u})\, \nu(dx, d\tilde{u}) \le \bar{p} \right\}. \tag{13}$$

Then $H(\bar{p})$ is a closed, convex subset of $\mathcal{G}$. It is easy to see that it is also nonempty, provided $\bar{p} > 0$. Indeed, let $x' \in \mathbb{R}_+$ and consider the policy $v_{x'}$ defined by

$$(v_{x'})_i = \begin{cases} 0\,, & x \le x' \\ p_{\text{max}}\,, & x > x'\,, \end{cases} \qquad i = 1, \dots, N\,.$$

Under this policy, the diffusion process in (5) is positive recurrent and its invariant probability measure has a density $f_{x'}$ which is a solution of (10). Let

$$\alpha_k := \frac{2p_{\text{max}}}{\sigma^2} \sum_{i=1}^{k} \gamma_i \pi_i\,, \quad k = 1, \dots, N\,. \tag{14}$$

The solution of (10) takes the form

$$f_{x'}(x) = \frac{\alpha_N e^{-\alpha_N (x - x')^+}}{1 + \alpha_N x'}\,.$$

where $(y)^+ := \max(y, 0)$. Then

$$\int_{\mathbb{R}_+} h\bigl(v(x)\bigr) f_{x'}(x)\, dx = \frac{p_{\text{max}}}{1 + \alpha_N x'}\,,$$

and it follows that $\nu_{v_{x'}} \in H(\bar{p})$, provided

$$x' \ge \frac{1}{\alpha_N}\Bigl(\frac{p_{\text{max}}}{\bar{p}} - 1\Bigr)\,.$$

Thus, the optimization problem in (6) when restricted to stationary, stable controls is equivalent to

$$\text{minimize over } \nu \in H(\bar{p}) : \quad \int_{\mathbb{R}_+ \times \widetilde{U}} c(x)\, \nu(dx, d\tilde{u})\,. \tag{15}$$

We also define

$$J^*(\bar{p}) := \inf_{\nu \in H(\bar{p})} \int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\nu\,. \tag{16}$$

We proceed as follows. It is well known that $\mathcal{G}$ and $\mathcal{M}$ are convex and that their extreme points $\mathcal{G}_e$ and $\mathcal{M}_e$ correspond to controls in $\mathfrak{U}_{\text{se}}$. It is shown in [8], [9] that, under a near-monotone assumption on both the running cost c and h the infimum in (16) is attained in $H(\bar{p})$. This near-monotone condition amounts to

$$\liminf_{x \to \infty}\; c(x) > J^*(\bar{p}) \tag{17a}$$

$$\inf_{\tilde{u} \in \widetilde{U}}\; h(\tilde{u}) > \bar{p}\,. \tag{17b}$$

Clearly (17b) does not hold, and hence the results in [8], [9] cannot be quoted to assert existence. So we show directly in Theorem 3.3 that (15) attains a minimum in $H(\bar{p})$, and more specifically that this minimum is attained in $\mathfrak{U}_{\text{se}}$.

Concerning the extreme points of $\mathcal{G}$, the following lemma is a variation of [8, Lemma 3.5].

Lemma 3.1: Let $A \subset \mathbb{R}_+$ be a bounded Borel set of positive Lebesgue measure. Suppose that v', $v'' \in \mathfrak{U}_{\text{s}}$ differ a.e. on A and agree on A^c, and that for some $v_0 \in \mathfrak{U}_{\text{ss}}$ and measurable $r : \mathbb{R}_+ \to [0,1]$, which satisfies $r(x) \in (0,1)$, for almost all $x \in A$, we have

$$v_0(x) = r(x)v'(x) + \bigl(1 - r(x)\bigr)v''(x)\,. \tag{18}$$

Then, there exist $\hat{v}'$, $\hat{v}'' \in \mathfrak{U}_{\text{ss}}$ which differ a.e. on A and agree on A^c, such that

$$\nu_{v_0} = \tfrac{1}{2}(\nu_{\hat{v}'} + \nu_{\hat{v}''})\,.$$

In particular ν_{v_0} is not an extreme point of $\mathcal{G}$.

Since, every $v \in \mathfrak{U}_{\text{ss}} \setminus \mathfrak{U}_{\text{se}}$ can be decomposed as in (18) satisfying the hypotheses of Lemma 3.1, we obtain the following corollary.

Corollary 3.2: If $\nu_v \in \mathcal{G}_e$ then $v \in \mathfrak{U}_{\text{se}}$.

The main result of this section is contained in the following theorem whose proof can be found in Appendix II.

Theorem 3.3: Under Assumption 3.1, for any $\bar{p} \in (0, p_{\text{max}}]$, there exists $v^* \in \mathfrak{U}_{\text{se}}$ such that ν_{v^*} attains the minimum in (15).

IV. Lagrange multipliers and the HJB equation

In order to study the stationary optimal policies for (15), we introduce a parameterized family of unconstrained optimization problems that is equivalent to the problem in (6) in the sense that stationary optimal policies for the former are also optimal for the latter and vice-versa. We show that optimal policies for the unconstrained problem can be derived from the associated HJB equation. Hence, by studying the HJB equation we characterize the stationary optimal policies (15). We show that these are of a multi-threshold type and this enables us to reduce the optimal control problem to that of solving a system of $N+1$ algebraic equations. Furthermore, we show that optimality is achieved over the class of all admissible policies $\mathfrak{U}$, and not only over $\mathfrak{U}_{\mathrm{s}}$.

With $\lambda \in \mathbb{R}_+$ playing the role of a Lagrange multiplier, we define

$$\begin{aligned} L(x,u,\bar p,\lambda) &:= c(x) + \lambda\bigl(h(u) - \bar p\bigr) \\ \tilde J(v,\bar p,\lambda) &:= \limsup_{T\to\infty} \frac{1}{T}\int_0^T L\bigl(x(t), v(t), \bar p, \lambda\bigr)\,dt \\ \tilde J^*(\bar p,\lambda) &:= \inf_{v\in\mathfrak{U}_{\mathrm{ss}}} \tilde J(v,\bar p,\lambda)\,. \end{aligned} \tag{19}$$

The choice of the optimization problem in (19) is motivated by the fact that $J^*(\bar p)$, defined in (16) is a convex, decreasing function of $\bar p$. This is rather simple to establish. Let $\bar p'$, $\bar p'' \in (0, p_{\mathrm{max}}]$ and denote by ν', ν'' the corresponding ergodic occupation measures that achieve the minimum in (15). Then, if $\delta \in [0,1]$, $\nu_0 := \delta\nu' + (1-\delta)\nu''$ satisfies $\int h\,d\nu_0 = \delta\bar p' + (1-\delta)\bar p''$, and since ν_0 is suboptimal for the optimization problem in (15) with power constraint $\delta\bar p' + (1-\delta)\bar p''$, we have

$$J^*(\delta\bar p' + (1-\delta)\bar p'') \le \int c\,d\nu_0 = \delta J^*(\bar p') + (1-\delta) J^*(\bar p'')\,.$$

A separating hyperplane which is tangent to the the graph of the function $J^*(\cdot)$ at a point $\bigl(\bar p_0, J^*(\bar p_0)\bigr)$, with $\bar p_0 \in (0, p_{\mathrm{max}}]$ takes the form

$$\bigl\{(\bar p, J) : J + \lambda_{\bar p_0}(\bar p - \bar p_0) = J^*(\bar p_0)\bigr\}\,,$$

for some $\lambda_{\bar p_0} \in \mathbb{R}_+$ (see Figure 3).

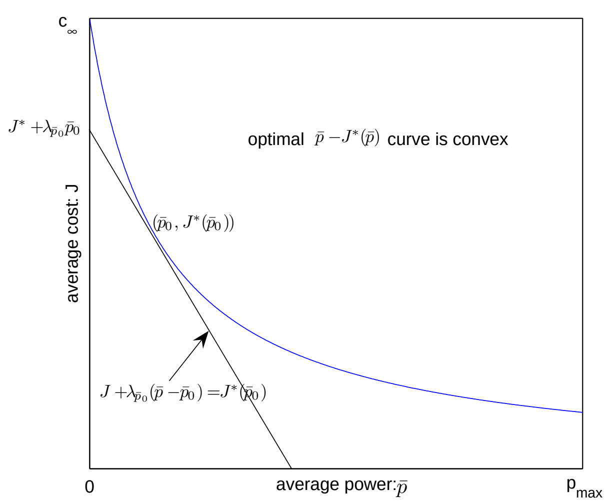

Fig. 3. Convexity of $\bar p \mapsto J^*(\bar p)$ and the separating hyperplane through $(\bar p_0, J^*(\bar p_0))$.

Standard Lagrange multiplier theory yields the following (see [21, pg. 217, Thm. 1]):

Theorem 4.1: Let $\bar p_0 \in (0, p_{\mathrm{max}}]$. There exists $\lambda_{\bar p_0} \in \mathbb{R}_+$, such that the minimization problem in (15), over $H(\bar p_0)$ as well as the problem

$$\text{minimize}: \quad \int_{\mathbb{R}_+\times\tilde U} L(x,\tilde u,\bar p_0,\lambda_{\bar p_0})\,\nu(dx,d\tilde u) \tag{20}$$

over $\nu \in \mathcal{G}$, both attain the same minimum value $J^*(\bar p_0) = \tilde J^*(\bar p_0, \lambda_{\bar p_0})$, at some $\nu_0 \in H(\bar p_0)$. In particular,

$$\int_{\mathbb{R}_+\times\tilde U} h(\tilde u)\,\nu_0(dx,d\tilde u) = \bar p_0\,.$$

Characterizing the optimal policy via the HJB equation associated with the unconstrained problem in (20), is made possible by first showing that under Assumption 3.1 the cost $L(x,u,\bar p,\lambda)$ is near-monotone (see (22) below), and then employing the results in [18]. It is not difficult to show that under Assumption 3.1

$$\lim_{\bar p\to 0} J^*(\bar p) = \lim_{x\to\infty} c(x)\,. \tag{21}$$

Indeed, for $\bar p \in (0, p_{\mathrm{max}}]$, suppose $v \in \mathfrak{U}_{\mathrm{ss}}$ such that $\nu_v \in H(\bar p)$. Letting $\gamma_{\mathrm{max}} := \max_i\,\{\gamma_i\}$, and using (11) we obtain

$$\begin{aligned} \bar p &\ge \int_0^\infty h\bigl(v(x)\bigr) f_v(x)\,dx \\ &\ge \gamma_{\mathrm{max}}^{-1}\int_0^\infty b\bigl(v(x)\bigr) f_v(x)\,dx \\ &= \frac{\sigma^2}{2\gamma_{\mathrm{max}} A_v}\,. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^\infty c(x) f_v(x)\,dx &\ge \min_{x\ge\frac{1}{\sqrt{\bar p}}}\,\bigl\{c(x)\bigr\}\int_{\frac{1}{\sqrt{\bar p}}}^\infty f_v(x)\,dx \\ &= \min_{x\ge\frac{1}{\sqrt{\bar p}}}\,\bigl\{c(x)\bigr\}\left(1 - \frac{A_v^{-1}}{\sqrt{\bar p}}\right) \\ &\ge \min_{x\ge\frac{1}{\sqrt{\bar p}}}\,\bigl\{c(x)\bigr\}\left(1 - \frac{2\gamma_{\mathrm{max}}}{\sigma^2}\sqrt{\bar p}\right)\,. \end{aligned}$$

Hence,

$$J^*(\bar p) \ge \min_{x\ge\frac{1}{\sqrt{\bar p}}}\,\bigl\{c(x)\bigr\}\left(1 - \frac{2\gamma_{\mathrm{max}}}{\sigma^2}\sqrt{\bar p}\right)$$

and (21) follows. We need the following lemma, whose proof is contained in Appendix II.

Lemma 4.2: Let Assumption 3.1 hold and suppose c is bounded. Then for any $\bar p \in (0, p_{\mathrm{max}}]$, we have

$$J^*\bigl(\tfrac{\bar p}{2}\bigr) < \tfrac12\bigl(J^*(\bar p) + c_\infty\bigr)\,.$$

We are now ready to establish the near-monotone property of L. First, we introduce some new notation. For $\bar p \in (0, p_{\mathrm{max}}]$, let

$$\Lambda(\bar p) := \bigl\{\lambda \in \mathbb{R}_+ : J^*(\bar p') \ge J^*(\bar p) + \lambda(\bar p - \bar p')\,, \\ \forall \bar p' \in (0, p_{\mathrm{max}}]\bigr\}$$

and

$$\Lambda := \bigcup_{\bar p\in(0,p_{\mathrm{max}}]} \Lambda(\bar p)\,.$$

Remark 4.1: It follows from the definition of $\Lambda(\bar p)$ that

$$\inf_{\nu\in\mathcal{G}} \int_{\mathbb{R}_+\times\tilde U} \bigl[c(x) + \lambda h(\tilde u)\bigr]\,\nu(dx,d\tilde u) = J^*(\bar p) + \lambda\bar p\,,$$

for all $\lambda \in \Lambda(\bar p)$. Also, it is rather straightforward to show that $\Lambda = [0, \bar\lambda)$ for some $\bar\lambda \in \mathbb{R}_+ \cup \{\infty\}$.

Lemma 4.3: Let Assumption 3.1 hold. Then, for all $\bar{p} \in (0, p_{\max}]$ and $\lambda \in \Lambda$,

$$\liminf_{x\to\infty} \inf_{\tilde{u}\in\tilde{U}} L(x,\tilde{u},\bar{p},\lambda) > \tilde{J}^*(\bar{p},\lambda)\,. \tag{22}$$

Proof: If c is asymptotically unbounded, (22) always follows. Otherwise, fix $\bar{p} \in (0, p_{\max}]$ and $\lambda \in \Lambda$. Let $\bar{p}' \in (0, p_{\max}]$ be such that $\lambda \in \Lambda(\bar{p}')$. By convexity

$$J^*\bigl(\tfrac{\bar{p}'}{2}\bigr) \geq J^*(\bar{p}') + \lambda\tfrac{\bar{p}'}{2}\,.$$

Thus, using Lemma 4.2, we obtain

$$J^*(\bar{p}') + \lambda\bar{p}' < c_\infty\,. \tag{23}$$

Hence, by (21) and (23),

$$\begin{aligned}\liminf_{x\to\infty} \inf_{\tilde{u}\in\tilde{U}} L(x,\tilde{u},\bar{p},\lambda) + \lambda\bar{p} &= \liminf_{x\to\infty} c(x)\\ &> J^*(\bar{p}') + \lambda\bar{p}'\\ &= \tilde{J}^*(\bar{p},\lambda) + \lambda\bar{p}\,,\end{aligned}$$

and the proof is complete. ■

A. The Structure of the Optimal Policy

Using the theory in [18, Chapter IV.3], we can characterize optimality via the HJB equation. This is summarized as follows:

Theorem 4.4: Let Assumption 3.1 hold. Fix $\bar{p} \in (0, p_{\max}]$ and $\lambda_{\bar{p}} \in \Lambda(\bar{p})$. Then there exists a unique solution pair (V,β), with $V \in \mathcal{C}^2(\mathbb{R}_+,\mathbb{R})$ and $\beta \in \mathbb{R}$, to the HJB

$$\min_{\tilde{u}\in\tilde{U}} \bigl[\mathcal{L}^{\tilde{u}}V(x) + L(x,\tilde{u},\bar{p},\lambda_{\bar{p}})\bigr] = \beta\,, \tag{24a}$$

subject to the boundary condition

$$\frac{dV}{dx}(0) = 0\,, \tag{24b}$$

and also satisfying

(a) $V(0) = 0$
(b) $\inf_{x\in\mathbb{R}_+} V(x) > -\infty$
(c) $\beta \leq \tilde{J}^*(\bar{p},\lambda_{\bar{p}})$.

Moreover, if v^* is a measurable selector of the minimizer in (24a), then $v^* \in \mathfrak{U}_{\text{se}} \subset \mathfrak{U}_{\text{ss}}$, and v^* is an optimal policy for (20), or equivalently, for (15). Also, $\beta = \tilde{J}^*(\bar{p},\lambda_{\bar{p}}) = J^*(\bar{p})$ (the second equality follows by Theorem 4.1).

Following [18, Chapter IV.1] we can show that the stationary policy v^* in Theorem 4.4 is optimal among all admissible controls $\mathfrak{U}$, and hence is a minimizer for (6). This is done as follows: For a control $v \in \mathfrak{U}$ define the process $\{\varphi^v_t\,,\ t \geq 0\}$ of empirical measures as a $\mathcal{P}(\mathbb{R}_+ \times \widetilde{U})$-valued process satisfying, for all $g \in \mathcal{C}_b(\mathbb{R}_+ \times \widetilde{U})$,

$$\varphi^v_t(A,B) = \frac{1}{t}\int_0^t I_A\bigl(x(s)\bigr)\eta_v\bigl(x(s),B\bigr)\,ds\,.$$

Suppose that $v \in \mathfrak{U}$ is such that, for $\bar{p} \in (0, p_{\max}]$,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t h\bigl(v(s)\bigr)\,ds \leq \bar{p}\,, \quad \text{a.s.} \tag{25}$$

Following the approach in [18, Chapter IV.1], utilizing the near-monotone property asserted in Lemma 4.3 and the characterization of $\mathcal{G}$ in (12), we first deduce that any subsequence $\{t_n\}$, $t_n \to \infty$, contains a further subsequence $\{t'_n\}$ along which $\varphi^v_{t'_n}$ converges weakly, as $n \to \infty$, to some $\nu \in \mathcal{G}$. Thus

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\int_0^t &L\bigl(x(s),v(s),\bar{p},\lambda_{\bar{p}}\bigr)\,ds\\ &\geq \int_{\mathbb{R}_+\times\tilde{U}} L\bigl(x,\tilde{u},\bar{p},\lambda_{\bar{p}}\bigr)\,\nu(dx,d\tilde{u})\\ &\geq \tilde{J}^*(\bar{p},\lambda_{\bar{p}})\,, \quad \text{a.s.}\end{aligned} \tag{26}$$

Then, (25)–(26) imply that under the policy v

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t c\bigl(x(s)\bigr)\,ds \geq \tilde{J}^*(\bar{p},\lambda_{\bar{p}}) = J^*(\bar{p})\,, \quad \text{a.s.} \tag{27}$$

Optimality of $v^* \in \mathfrak{U}_{\text{se}}$ then follows by (25) and (27), and we have the following theorem.

Theorem 4.5: Under Assumption 3.1, for any $\bar{p} \in (0, p_{\max}]$, there exists $v^* \in \mathfrak{U}_{\text{se}}$ which attains the minimum in (6) over all controls in $\mathfrak{U}$.

If $\Lambda(\bar{p})$ and $J^*(\bar{p})$ were known, then one could solve (24) and derive the optimal policy. Since this is not the case, we embark on a different approach. We write (24) as

$$\min_{\tilde{u}\in\tilde{U}} \bigl[\mathcal{L}^{\tilde{u}}V(x) + c(x) + \lambda_{\bar{p}}h(\tilde{u})\bigr] = \beta + \lambda_{\bar{p}}\,\bar{p}\,. \tag{28}$$

By Theorem 4.4, $J^*(\bar{p})$ is the smallest value of β for which there exists a solution pair (V,β) to (24), satisfying (b). This yields the following corollary:

Corollary 4.6: Let Assumption 3.1 hold. For $\lambda \in \Lambda$, consider the HJB equation

$$\min_{\tilde{u}\in\tilde{U}} \bigl[\mathcal{L}^{\tilde{u}}V(x) + c(x) + \lambda h(\tilde{u})\bigr] = \varrho\,, \tag{29a}$$

subject to the boundary condition

$$\frac{dV}{dx}(0) = 0\,, \tag{29b}$$

and define

$$\mathcal{Q}_\lambda := \Bigl\{(V,\varrho) \text{ solves (29) and } \inf_{x\in\mathbb{R}_+} V(x) > -\infty\Bigr\} \tag{30a}$$

$$\varrho_\lambda := \min\bigl\{\varrho : (V,\varrho) \in \mathcal{Q}_\lambda\bigr\}\,. \tag{30b}$$

Then

$$\varrho_\lambda = \min_{v\in\mathfrak{U}_{\text{ss}}} \int_{\mathbb{R}_+\times\tilde{U}} \bigl[c(x) + \lambda h(\tilde{u})\bigr]\,\nu_v(dx,d\tilde{u})\,. \tag{31}$$

Furthermore, if $\bar{p}$ is a point in $(0, p_{\max}]$ such that $\lambda \in \Lambda(\bar{p})$, then $\varrho_\lambda = J^*(\bar{p}) + \lambda\bar{p}$, and if v^*_λ is a measurable selector of the minimizer in (29a) with $\varrho = \varrho_\lambda$, then v^*_λ is a stationary optimal policy for (20).

The minimizer in (29a) satisfies

$$\min_{\tilde{u}\in\tilde{U}} \Bigl[-b(\tilde{u})\frac{dV}{dx} + \lambda h(\tilde{u})\Bigr] = \min_{\tilde{u}\in\tilde{U}} \sum_j (\lambda - \gamma_j\frac{dV}{dx})\pi_j\tilde{u}_j\,.$$

Thus the optimal control v^*_λ takes the following simple form: for $i = 1, \dots, N$ and $x \in \mathbb{R}_+$,

$$(v^*_\lambda)_i(x) = \begin{cases} 0\,, & \text{if} \quad \gamma_i \frac{dV}{dx}(x) < \lambda \\ p_{\max}\,, & \text{if} \quad \gamma_i \frac{dV}{dx}(x) \geq \lambda\,. \end{cases} \tag{32}$$

Thus, provided $\frac{dV}{dx}$ is monotone, the optimal control v^*_λ is of multi-threshold type, i.e., for each channel state j there is a queue-threshold $\hat{x}_j$, such that at any time t, the optimal policy transmits at peak power $p_{\max}$ over channel state j, if the queue length $x(t) > \hat{x}_j$, and does not transmit otherwise.

Further, from Remark 2.2, it follows that if the equilibrium power $\{P_0(j)\}$ is allocated according to channel-state dependent water-filling with strictly positive equilibrium power allocations for each channel state, the multi-threshold policy collapses to a *single-threshold* policy (since $\gamma_i = \gamma_j$, for all i, j). In other words, there is a state-independent queue-threshold $\hat{x}$, such that at any time t, the optimal policy transmits at peak power $p_{\max}$, if the queue length $x(t) > \hat{x}$, and does not transmit otherwise.

The following lemma asserts the monotonicity of $\frac{dV}{dx}$, under the additional assumption that c is non-decreasing.

Lemma 4.7: Suppose c satisfies Assumption 3.1, and is non-decreasing on $[0, \infty)$. Then every $(V, \varrho) \in \mathcal{Q}_\lambda$ satisfies

(a) $\frac{dV}{dx}$ is non-decreasing;
(b) If c is unbounded, then $\frac{dV}{dx}$ is unbounded.

Proof: Equation (29a) takes the form

$$\frac{\sigma^2}{2}\frac{d^2V}{dx^2}(x) = \sum_j \pi_j p_{\max}\Big[\gamma_j \frac{dV}{dx}(x) - \lambda\Big]^+ + \varrho - c(x)\,, \tag{33}$$

where the initial condition is given by (29b). Since c is non-decreasing, then by (31), $\varrho > c(0)$. Suppose that for some $x' \in \mathbb{R}_+$, $\frac{d^2V}{dx^2}(x') = -\varepsilon < 0$. Let $x'' = \inf\big\{x > x' : \frac{d^2V}{dx^2}(x) \geq 0\big\}$. Since by Theorem 4.4 $\frac{d^2V}{dx^2}$ is continuous, it must hold $x'' > x'$. Suppose $x'' < \infty$. Since $\frac{d^2V}{dx^2} < 0$ on $[x', x'')$ and $\varrho - c(x)$ is non-increasing, (33) implies that $\frac{d^2V}{dx^2}(x'') \leq \frac{d^2V}{dx^2}(x') < 0$. Thus we are led to a contradiction, and it follows that $\frac{d^2V}{dx^2}(x) \leq -\varepsilon < 0$, for all $x \in [x', \infty)$, implying that V is not bounded below. It is clear from (33) that since $\frac{d^2V}{dx^2} \geq 0$, then $\frac{d^2V}{dx^2}(x) \to \infty$, as $x \to \infty$, provided c is not bounded. ■

The proof of Lemma 4.7 shows that if (V, ϱ) solves (29), then V is bounded below, if and only if $\frac{d^2V}{dx^2}(x) \geq 0$, for all $x \in \mathbb{R}_+$. Thus $\mathcal{Q}_\lambda$ defined in (30a), has an alternate characterization given in the following corollary.

Corollary 4.8: Suppose c satisfies Assumption 3.1, and is non-decreasing on $[0, \infty)$. Then, for all $\lambda \in \Lambda$,

$$\mathcal{Q}_\lambda = \Big\{(V, \varrho) \text{ solves (29) and } \tfrac{d^2V}{dx^2} \geq 0\,, \text{ on } \mathbb{R}_+\Big\}\,.$$

Comparing (29) and (28), a classical application of Lagrange duality (see [21, pg. 224, Thm. 1]) yields the following:

Lemma 4.9: If c satisfies Assumption 3.1, and is non-decreasing on $[0, \infty)$, then, for any $\bar{p} \in (0, p_{\max}]$ and $\lambda_{\bar{p}} \in \Lambda(\bar{p})$, we have:

$$\varrho_{\lambda_{\bar{p}}} - \lambda_{\bar{p}}\,\bar{p} = \max_{\lambda \geq 0}\big\{\varrho_\lambda - \lambda\bar{p}\big\} = J^*(\bar{p})\,. \tag{34}$$

Moreover, if λ_0 attains the maximum in $\lambda \mapsto \varrho_\lambda - \lambda\bar{p}$ then $\varrho_{\lambda_0} = J^*(\bar{p}) + \lambda_0\bar{p}$, which implies that $\lambda_0 \in \Lambda(\bar{p})$.

Remark 4.2: Lemma 4.9 furnishes a method for solving (15). This can be done as follows: With λ viewed as a parameter, we first solve for ϱ_λ which is defined in (30b). Then, given $\bar{p}$, we obtain the corresponding value of the Lagrange multiplier via the maximization in (34). The optimal control can then be evaluated using (32), with $\lambda = \lambda_{\bar{p}}$. Section V-A contains an example demonstrating this method.

V. Solution of the HJB equation

In this section we present an analytical solution of the HJB equation (29). We deal only with the case where the cost function c is non-decreasing and asymptotically unbounded. However, the only reason for doing so is in the interest of simplicity and clarity. If c is bounded the optimal policy may have less than N threshold points, but other than the need to introduce some extra notation, the solution we outline below for unbounded c, holds virtually unchanged for the bounded case. Also, without loss of generality, we assume that $\gamma_1 > \dots > \gamma_N > 0$.

We parameterize the policies in (32) by a collection of points $\{\hat{x}_1, \dots, \hat{x}_N\}$ in $\mathbb{R}_+$. In other words, if V is the solution (33), then $\hat{x}_i$ is the least positive number such that $\frac{dV}{dx}(\hat{x}_i) \geq \gamma_i^{-1}$. Thus, if we define

$$\mathcal{X}^N := \big\{\hat{x} = (\hat{x}_1, \dots, \hat{x}_N) \in \mathbb{R}^N_+ : \hat{x}_1 < \dots < \hat{x}_N\big\}\,,$$

then for each $\hat{x} \in \mathcal{X}^N$, there corresponds a multi-threshold policy $v_{\hat{x}}$ of the form

$$(v_{\hat{x}})_i(x) = \begin{cases} p_{\max}\,, & \text{if} \quad x \geq \hat{x}_i \\ 0\,, & \text{otherwise.} \end{cases} \qquad 1 \leq i \leq N\,. \tag{35}$$

To facilitate expressing the solution of (33), we need to introduce some new notation. For $i = 1, \dots, N$, define

$$\tilde{\pi}_i := \sum_{j=1}^{i} \pi_i\,, \quad \tilde{\gamma}_i := \sum_{j=1}^{i} \pi_i\gamma_i\,, \quad \Gamma_i := \frac{\tilde{\gamma}_i}{\gamma_i} - \tilde{\pi}_i\,.$$

Note that from (14), we obtain the identity

$$\alpha_i = \frac{2p_{\max}}{\sigma^2}\tilde{\gamma}_i\,, \quad i = 1, \dots, N\,.$$

For $x, z \in \mathbb{R}_+$, with $z \leq x$, we define the functions

$$F_0(\varrho, x) := \varrho x - \int_0^x c(y)\,dy\,,$$

and for $i = 1, \dots, N$,

$$F_i(\varrho, x, z) := [\varrho + \lambda p_{\max}\Gamma_i]\big(1 - e^{\alpha_i(z-x)}\big) - \alpha_i \int_z^x e^{\alpha_i(z-y)}c(y)\,dy\,,$$

$$G_i(\varrho, x, z) := \varrho + \lambda p_{\max}\Gamma_i - \alpha_i \int_z^x e^{\alpha_i(z-y)}c(y)\,dy - e^{\alpha_i(z-x)}c(x)\,.$$

Using the convention $\hat{x}_{N+1} \equiv \infty$, we write the solution of (33) as

$$\frac{dV}{dx}(x) = \frac{2}{\sigma^2} F_0(\varrho, x)\,, \quad 0 \le x < \hat{x}_1\,, \tag{36a}$$

and for $x \in [\hat{x}_i, \hat{x}_{i+1})$, $i = 1, \dots, N$,

$$\frac{dV}{dx}(x) = \frac{2}{\sigma^2 \alpha_i} e^{\alpha_i (x - \hat{x}_i)} F_i(\varrho, x, \hat{x}_i) + \frac{\lambda}{\gamma_i}\,. \tag{36b}$$

In addition, the following boundary conditions are satisfied

$$F_0(\varrho, \hat{x}_1) - \frac{\lambda \sigma^2}{2\gamma_1} = 0\,, \tag{37a}$$

and for $i = 1, \dots, N-1$,

$$F_i(\varrho, \hat{x}_{i+1}, \hat{x}_i) = \lambda p_{\max} \tilde{\gamma}_i e^{\alpha_i(\hat{x}_i - \hat{x}_{i+1})} \Bigl(\frac{1}{\gamma_{i+1}} - \frac{1}{\gamma_i}\Bigr)\,. \tag{37b}$$

Also, for $i = 1, \dots, N$, we have

$$\frac{d^2 V}{dx^2}(x) = \frac{2}{\sigma^2} e^{\alpha_i (x - \hat{x}_i)} G_i(\varrho, x, \hat{x}_i)\,, \quad x \in (\hat{x}_i, \hat{x}_{i+1})\,.$$

Since c is monotone, the map

$$x \mapsto \alpha_i \int_z^x e^{\alpha_i (z-y)} c(y)\, dy + e^{\alpha_i (z-x)} c(x) \tag{38}$$

is non-decreasing. Moreover, using the fact that c is either asymptotically unbounded (or strictly monotone increasing, when bounded), an easy calculation yields

$$G_i(\varrho, x, z) > \lim_{x \to \infty} G_i(\varrho, x, z)\,. \tag{39}$$

Suppose $\hat{x} \in \mathcal{X}^N$, are the threshold points of a solution (V, ϱ) of (33). It follows from (39) that $\lim_{x\to\infty} G_N(\varrho, x, \hat{x}_N) \ge 0$ is a necessary and sufficient condition for $\frac{d^2V}{dx^2}(x) \ge 0$, for all $x \in (\hat{x}_N, \infty)$. This condition translates to

$$\varrho + \lambda p_{\max} \mathit{\Gamma}_N - \alpha_N \int_{\hat{x}_N}^{\infty} e^{\alpha_N (\hat{x}_N - y)} c(y)\, dy \ge 0\,. \tag{40}$$

The arguments in the proof of Lemma 4.7 actually show that (40) is sufficient for $\frac{d^2V}{dx^2}$ to be non-negative on $\mathbb{R}_+$. We sharpen this result by showing in Lemma 5.1 below that (40) implies that $\frac{d^2V}{dx^2}$ is strictly positive on $\mathbb{R}_+$.

Lemma 5.1: Suppose $\hat{x} \in \mathcal{X}^N$ satisfies (37). If (40) holds, then $\varrho > c(\hat{x}_1)$ and $G_i(\varrho, x, \hat{x}_i) > 0$, for all $x \in [\hat{x}_i, \hat{x}_{i+1}]$, $i = 0, \dots, N-1$.

Proof: We argue by contradiction. If $\varrho \le c(\hat{x}_1)$, then $G_1(\varrho, \hat{x}_i, \hat{x}_i) \le 0$, hence it is enough to assume that $G_i(\varrho, x, \hat{x}_i) \le 0$, for some $x \in [\hat{x}_i, \hat{x}_{i+1}]$ and $i \in \{1, \dots, N-1\}$. Then, since (38) is non-decreasing,

$$G_i(\varrho, \hat{x}_{i+1}, \hat{x}_i) \le 0\,. \tag{41}$$

Therefore, since

$$\begin{aligned} F_i(\varrho, x, \hat{x}_i) = G_i(\varrho, x, \hat{x}_i) + e^{\alpha_i(\hat{x}_i - x)} c(x) \\ - [\varrho + \lambda p_{\max} \mathit{\Gamma}_i] e^{\alpha_i (\hat{x}_i - x)}\,, \end{aligned} \tag{42}$$

combining (37b) and (41)–(42), we obtain

$$c(\hat{x}_{i+1}) - \varrho - \lambda p_{\max} \mathit{\Gamma}_i \ge \lambda \tilde{\gamma}_i \Bigl(\frac{1}{\gamma_{i+1}} - \frac{1}{\gamma_i}\Bigr)\,,$$

which simplifies to

$$c(\hat{x}_{i+1}) - \varrho + \lambda p_{\max} \tilde{\pi}_i \ge \lambda p_{\max} \frac{\tilde{\gamma}_i}{\gamma_{i+1}}\,. \tag{43}$$

Since

$$\frac{\tilde{\gamma}_i}{\gamma_{i+1}} - \tilde{\pi}_i = \frac{\tilde{\gamma}_{i+1}}{\gamma_{i+1}} - \tilde{\pi}_{i+1} = \mathit{\Gamma}_{i+1}\,,$$

(43) yields

$$\varrho + \lambda p_{\max} \mathit{\Gamma}_{i+1} \le c(\hat{x}_{i+1})\,. \tag{44}$$

Using the monotonicity of $x \mapsto G_{i+1}(\varrho, x, \hat{x}_{i+1})$ together with (44), we get $G_{i+1}(\varrho, x, \hat{x}_{i+1}) \le 0$, for all $x \in [\hat{x}_{i+1}, \hat{x}_{i+2}]$, and iterating this argument, we conclude that $G_N(\varrho, x, \hat{x}_N) \le 0$, for all $x \in (\hat{x}_N, \infty)$, thus contradicting (40). ■

Combining Corollary 4.8 with Lemma 5.1, yields the following.

Corollary 5.2: Suppose (V, ϱ) satisfies (36)–(37), for some $\hat{x} \in \mathcal{X}^N$ and $\lambda \in \mathit{\Lambda}$. Then $(V, \varrho) \in \mathcal{Q}_\lambda$, if and only if (40) holds.

For $\lambda \in \mathit{\Lambda}$, define

$$\mathcal{R}_\lambda := \{\varrho \in \mathbb{R}_+ : (V, \varrho) \in \mathcal{Q}_\lambda\}\,.$$

For each $\varrho \in \mathcal{R}_\lambda$, equations (37) define a map $\varrho \mapsto \hat{x}$, which we denote by $\hat{x}(\varrho)$.

Lemma 5.3: Let $\lambda \in \mathit{\Lambda}$ and suppose $\varrho_0 \in \mathcal{R}_\lambda$. With ϱ_λ as defined in (30b), and denoting the left-hand side of (40) by $G_N(\varrho, \infty, \hat{x}_N)$, the following hold:

(a) If $\varrho' > \varrho_0$, then $\varrho' \in \mathcal{R}_\lambda$ and $G_N\bigl(\varrho', \infty, \hat{x}(\varrho')\bigr) > 0$.
(b) If $G_N\bigl(\varrho_0, \infty, \hat{x}(\varrho_0)\bigr) > 0$, then $\varrho_0 > \varrho_\lambda$.
(c) $\mathcal{R}_\lambda = [\varrho_\lambda, \infty)$, and ϱ_λ is the only point in $\mathcal{R}_\lambda$ which satisfies $G_N\bigl(\varrho_\lambda, \infty, \hat{x}(\varrho_\lambda)\bigr) = 0$.

Proof: Part (a) follows easily from (33). Denoting by V_0 and V' the solutions of (33) corresponding to ϱ_0 and ϱ', respectively, a standard argument shows that

$$\frac{d^2(V' - V_0)}{dx^2}(x) \ge \varrho' - \varrho_0 > 0\,, \quad \forall x \in \mathbb{R}_+\,,$$

implying

$$\frac{dV'}{dx}(x) \ge \frac{dV_0}{dx}(x)\,, \quad \forall x \in \mathbb{R}_+\,. \tag{45}$$

Hence, since by the definition of $\mathcal{Q}_\lambda$, V_0 is bounded below, the same holds for V', in turn implying that $(V', \varrho') \in \mathcal{Q}_\lambda$. By (45), $\hat{x}(\varrho') \le \hat{x}(\varrho_0)$, and since $\hat{x}_N \mapsto G_N(\varrho, \infty, \hat{x}_N)$ is non-increasing and $\varrho' > \varrho_0$, we obtain $G_N\bigl(\varrho', \infty, \hat{x}(\varrho')\bigr) > 0$.

Concerning (b), we write (37) in the form $\tilde{F}(\varrho, \hat{x}) = 0$, with $\tilde{F} : \mathbb{R}_+^{N+1} \to \mathbb{R}_+^N$. The map $\tilde{F}$ is continuously differentiable and as a result of Lemma 5.1 its Jacobian $D_{\hat{x}} F$ with respect to $\hat{x}$ has full rank at $\bigl(\varrho_0, \hat{x}(\varrho_0)\bigr)$. By the Implicit Function Theorem, there exists an open neighborhood $W(\varrho_0)$ and a continuous map $\hat{x} : W(\varrho_0) \to \mathbb{R}_+$, such that $\tilde{F}\bigl(\varrho, \hat{x}(\varrho)\bigr) = 0$, for all $\varrho \in W(\varrho_0)$. Using the continuity of G_N, we may restrict $W(\varrho_0)$ further so that $G_N\bigl(\varrho, \infty, \hat{x}(\varrho)\bigr) > 0$, for all $\varrho \in W(\varrho_0)$. Hence $W(\varrho_0) \subset \mathcal{R}_\lambda$, implying that $\varrho_0 > \varrho_\lambda$.

Part (c) follows directly from (a) and (b). ■

Combining Corollary 4.6 and Lemma 5.1, we obtain the following characterization of the solution to the HJB equation (29).

Theorem 5.4: Let c be non-decreasing and asymptotically unbounded. Then, the threshold points $(\hat{x}_1, \dots, \hat{x}_N) \in \mathcal{X}^N$ of the stationary optimal policy in (35) and the optimal value $\varrho_\lambda > 0$, are the (unique) solution of the set of $N+1$ algebraic equations which is comprised of the equations in (37) and $G_N\big(\varrho_\lambda, \infty, \hat{x}(\varrho_\lambda)\big) = 0$.

A. Example: Minimizing the Mean Delay

We specialize the optimization problem to the case $c(x) = x$, which corresponds to minimizing the mean delay.

First consider the case $N = 1$, letting $\alpha \equiv \alpha_1$ and $\hat{x} \equiv \hat{x}_1$. Solving (29) we obtain

$$\frac{dV}{dx}(x) = \frac{2\varrho}{\sigma^2}x - \frac{x^2}{\sigma^2}\,, \quad x \le \hat{x}\,,$$

with

$$\hat{x} = \varrho - \sqrt{\varrho^2 - \tfrac{\lambda\sigma^2}{\gamma}}\,. \tag{46}$$

Also, for $x \ge \hat{x}$,

$$\frac{dV}{dx}(x) = \frac{2e^{\alpha(x-\hat{x})}}{\sigma^2\alpha}\Big(\varrho - \hat{x} - \frac{1}{\alpha}\Big) + \frac{2}{\sigma^2\alpha}\Big(\varrho - \lambda p_{\max} + x + \frac{1}{\alpha}\Big)\,.$$

Therefore, for $x > \hat{x}$,

$$\frac{d^2V}{dx^2}(x) = \frac{2}{\sigma^2}\Big(\varrho - \hat{x} - \frac{1}{\alpha}\Big)e^{\alpha(x-\hat{x})} + \frac{2}{\sigma^2\alpha}\,. \tag{47}$$

It follows from (47) that

$$\varrho_\lambda = \hat{x} + \frac{1}{\alpha}\,. \tag{48}$$

By (46) and (48),

$$\varrho_\lambda = \sqrt{\tfrac{1}{\alpha^2} + \tfrac{\lambda\sigma^2}{\gamma}}\,. \tag{49}$$

Let $\bar{p} \in (0, p_{\max}]$ be given. Applying Lemma 4.9, we obtain from (49)

$$\lambda_{\bar{p}} = \frac{p_{\max}}{2\alpha\bar{p}^2} - \frac{1}{2\alpha p_{\max}}\,.$$

and

$$J^*(\bar{p}) = \frac{1}{2\alpha}\Big(\frac{p_{\max}}{\bar{p}} + \frac{\bar{p}}{p_{\max}}\Big)\,.$$

Moreover, the threshold point of the optimal policy is given by

$$\hat{x} = \frac{1}{\alpha}\Big(\frac{p_{\max}}{\bar{p}} - 1\Big)\,. \tag{50}$$

Now consider the case $N = 2$. We obtain:

$$\frac{dV}{dx}(x) = \frac{2\varrho}{\sigma^2}x - \frac{x^2}{\sigma^2}\,, \quad x \le \hat{x}_1 \tag{51a}$$

$$\frac{dV}{dx}(x) = \frac{2}{\sigma^2\alpha_1}\Big(\varrho - \hat{x}_1 - \frac{1}{\alpha_1}\Big)\big[e^{\alpha_1(x-\hat{x}_1)} - 1\big] + \frac{2(x-\hat{x}_1)}{\sigma^2\alpha_1} + \frac{\lambda}{\gamma_1}\,, \quad \hat{x}_1 \le x < \hat{x}_2\,, \tag{51b}$$

and for $x \ge \hat{x}_2$,

$$\frac{dV}{dx}(x) = \frac{2}{\sigma^2\alpha_2}\Big(\varrho - \hat{x}_2 - \frac{1}{\alpha_2} + \lambda p_{\max}\pi_1\frac{\gamma_1-\gamma_2}{\gamma_2}\Big) \times \big[e^{\alpha_2(x-\hat{x}_2)} - 1\big] + \frac{2(x-\hat{x}_2)}{\sigma^2\alpha_2} + \frac{\lambda}{\gamma_2}\,. \tag{51c}$$

Since $\frac{dV}{dx}(\hat{x}_1) = \frac{\lambda}{\gamma_1}$, we obtain by (51a),

$$\hat{x}_1 = \varrho - \sqrt{\varrho^2 - \tfrac{\lambda\sigma^2}{\gamma_1}}\,. \tag{52}$$

By (51c), $\frac{d^2V}{dx^2}(x) \ge 0$, for all $x > \hat{x}_2$, if and only if

$$\varrho - \hat{x}_2 - \frac{1}{\alpha_2} + \lambda p_{\max}\pi_1\frac{\gamma_1-\gamma_2}{\gamma_2} \ge 0\,.$$

Also, since $\frac{dV}{dx}(\hat{x}_2) = \frac{\lambda}{\gamma_2}$, we obtain from (51b),

$$\Big(\varrho - \hat{x}_1 - \frac{1}{\alpha_1}\Big)\big[e^{\alpha_1(\hat{x}_2-\hat{x}_1)} - 1\big] + \hat{x}_2 - \hat{x}_1 = \frac{\sigma^2\lambda\alpha_1}{2}\Big(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\Big)\,.$$

We apply Theorem 5.4 to compute the optimal policy. Define $\hat{x}_1(\varrho)$ by (52) and

$$\hat{x}_2(\varrho) := \hat{x}_1(\varrho) + \sqrt{\varrho^2 - \tfrac{\lambda\sigma^2}{\gamma_1}} - \tfrac{1}{\alpha_2} + \lambda p_{\max}\pi_1\tfrac{\gamma_1-\gamma_2}{\gamma_2}\,.$$

Then ϱ_λ is the solution of

$$\Big(\sqrt{\varrho^2 - \tfrac{\lambda\sigma^2}{\gamma_1}} - \tfrac{1}{\alpha_1}\Big)e^{\alpha_2(\hat{x}_2(\varrho)-\hat{x}_1(\varrho))} + \Big(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_2}\Big) = 0\,.$$

In Figure 4 we plot the optimal threshold points for a two state channel ($N = 2$) as a function of $\bar{p}$. The parameters are selected as $\pi = (0.5, 0.5)$, $\gamma = (2, 1)$, $\sigma = 1$ and $p_{\max} = 1$.

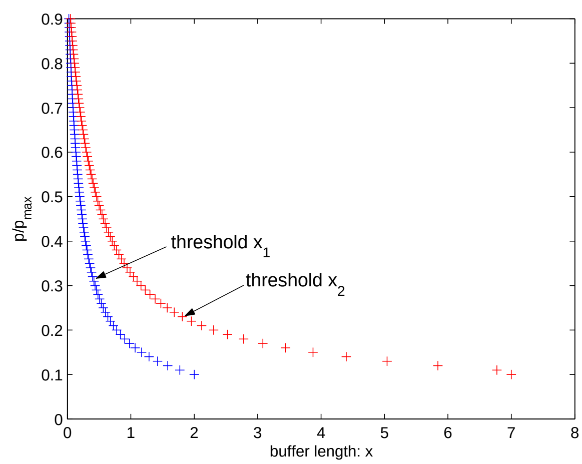

Fig. 4. Optimal threshold points as a function of $\bar{p}$.

VI. Numerical Results

We have considered the optimal power allocation problem in a time-varying channel under the heavy-traffic approximation. In the heavy-traffic region, the queueing process is modeled as a controlled diffusion process. The policy which minimizes the delay subject to a long-term average power constraint is multi-threshold and can be computed by the procedure outlined in Theorem 5.4. In this section, we compare the performance of the optimal policy under the heavy-traffic approximation with the optimal policy for the original non-scaled system. The latter is computed numerically in [3].

In [3], under the Poisson assumption on the arrival process, the power allocation problem is formulated as a discrete-time Markov decision process (MDP) with the state variable (X, g), where X is the buffer state, g is the channel state, and the action $P(X, g)$ is the transmitting power. With $A(t)$ denoting the arrival process, the queueing process is described by

$$X(t) = \min\left\{\max\left\{X(t-1) + A(t) - D(t), 0\right\}, L\right\},$$

where L is the buffer size, and the departure process $D(t)$ is controlled by the power allocation $P(X, g)$.

In our simulations, we consider the power allocation in a two-state Markov channel with stationary distribution $\pi = [0.8, 0.2]$ and corresponding channel gains $g = [0.9, 0.3]$. The arrival process is a Poisson process with expectation $\lambda^a = 5$, and the service rate r depends on the power allocation P according to $r(P, g) = 10\ln\left(1 + \frac{Pg}{10}\right)$.

Importantly, we comment here that the threshold based policy *does not* necessarily need a Poisson assumption for the proof of asymptotic optimality. For any sequence of arrival processes which converges to a Wiener process in the heavy-traffic limit, the threshold-based policy is asymptotically optimal. However, we do not know what the optimal policy is in the non-asymptotic regime with general arrivals. Thus, in our simulations, we compare the threshold-based policy with the optimal policy (obtained in [3]) with Poisson arrivals.

The numerical computation of the optimal policy of MDP in [3] is facilitated by standard methods, such as policy iteration and value iteration [22]. The optimal policies under different power constraints, are simulated to yield different average queue length drawn as the solid line in Figure 5. Note that the optimal policy under the heavy-traffic approximation is a single-threshold one. The optimal threshold as a function of the average power constraint can be obtained by (50). By using the threshold policies corresponding to different power constraints, a simulated power - queue length curve is plotted in Figure 5 with cross marks. The dotted line at the bottom in Figure 5 is the minimum power ($P_{\min} = 7.7$) required for the arrival rate to match the service rate (see (2)). By the affine relation between mean delay and mean queue length through Little's law with the constant of proportionality being the arrival rate, Figure 5 can be interpreted as a delay-power tradeoff curve. As can be seen in Figure 5, the two power-delay trade-off curves are very close, and they get even closer as the average queue length approaches $+\infty$, or equivalently, as the average power approaches $P_{\min}$, i.e., the heavy-traffic regime.

In terms of computational effort, in order to obtain the optimal policy of the discrete-time Markov decision process in [3] by value iteration or policy iteration, the complexity grows in proportion to the buffer size L, the number of channel states, the number of power levels, and the iteration steps needed, whereas the algorithm in Theorem 5.4 has complexity proportional to the number of channel states. With limited performance degradation, the multi-threshold policy has much simpler structure and lower computational complexity than the optimal control, and this makes it very promising for practical deployment.

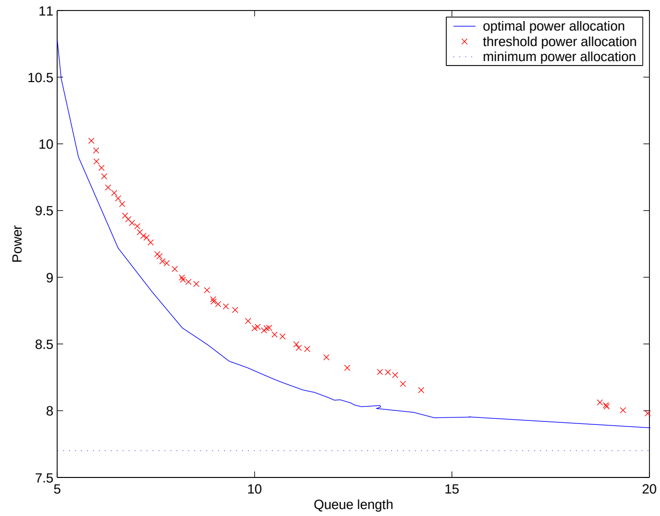

Fig. 5. Power-delay trade-off curve comparison.

VII. Conclusion

We studied the optimal power allocation of a single queue with a time-varying channel concerning both queueing delay and power efficiency. Under a fast channel variation assumption, i.e., if the channel state changes much faster than the queueing dynamics, we consider the heavy-traffic limit and associate a monotone cost function with the limiting queue-length process. We first show the existence of the optimal stationary Markov policy, and then show that this is a channel-state based threshold policy. In other words, for each channel state j, there is a queue-length threshold. The optimal policy transmits at peak power over channel state j only if the queue length exceeds the threshold, and does not transmit otherwise.

Implementing the optimal policy requires knowing the arrival rate and channel statistics. A possible extension of this work is to study adaptive schemes, which can adjust the parameter settings based on the service rate and current channel state.

The tools developed here could also be applied to study other resource allocation and control problems in wireless networks. For example, one could investigate the optimal scheduler for a multi-class queue and multiple servers with time-varying channels.

Extending the results to multiple queues is hardly straightforward. The main difficulty is that the reflection direction is not fixed but depends on the control policy. This complicates the optimization problem. Concerning existence of an invariant measure and explicit solutions for the density for the multi-dimensional problem see [23], [24]. These problems are under current investigation.

Appendix I
The Heavy-Traffic Limit

We apply the methodology in [4, Section III], with a slightly different scaling, and obtain the heavy-traffic limit. We consider a sequence of single-queue systems with time-varying channel process $L^n(t) = L(n^{-\kappa}t)$ and define the scaled queue size by

$$x^n(t) = n^{-\frac{1+\kappa}{2}}q(nt)\,.$$

Let

$$A^n(t) := n^{-\frac{1+\kappa}{2}} \times \text{number of arrival bits by time } nt$$
$$D^n(t) := n^{-\frac{1+\kappa}{2}} \times \text{number of bits transmitted by time } nt\,.$$

Then the queue dynamics can be described by

$$x^n(t) = x^n(0) + A^n(t) - D^n(t)\,,$$

where the service process $D^n(t)$ is coupled with the power allocation and the channel process. Using (3), we obtain

$$\begin{aligned} D^n(t) &= \frac{1}{n^{\frac{1+\kappa}{2}}}\int_0^{nt}\sum_{j=1}^N I_{\{L^n(s)=j\}} r(P_n,j) I_{\{x^n(s)>0\}}\,ds \\ &= \frac{1}{n^{\frac{1+\kappa}{2}}}\int_0^{nt}\sum_{j=1}^N \biggl(r_0(j) + \frac{\gamma_j u_j}{n^{\frac{1+\kappa}{2}}}\biggr) \\ &\qquad \times I_{\{L(n^{-\kappa}s)=j\}} I_{\{x^n(s)>0\}}\,ds \\ &= \frac{1}{n^{\frac{1-\kappa}{2}}}\int_0^{tn^{1-\kappa}}\sum_{j=1}^N \biggl(r_0(j) + \frac{\gamma_j u_j}{n^{\frac{1-\kappa}{2}}}\biggr) \\ &\qquad \times I_{\{L(s')=j\}} I_{\{x^n(s')>0\}}\,ds'\,. \end{aligned} \tag{53}$$

Let

$$M^{d,n}(t) := \frac{1}{n^{\frac{1-\kappa}{2}}}\int_0^{tn^{1-\kappa}}\sum_{j=1}^N I_{\{L(s')=j\}} r_0(j)\,ds' - \lambda^a n^{\frac{1-\kappa}{2}} t\,. \tag{54}$$

By (2), we have

$$M^{d,n}(t) = \frac{1}{n^{\frac{1-\kappa}{2}}}\int_0^{tn^{1-\kappa}}\sum_{j=1}^N \Bigl[I_{\{L(s')=j\}} - \pi_j\Bigr] r_0(j)\,ds'\,.$$

By Donsker's theorem [25], $M^{d,n}$ converges weakly to a Wiener process w^d with a finite variance σ_d^2, as $n\to\infty$. At the same time, the centered process of arrivals $M^{a,n}(t) := A^n(t) - \lambda^a n^{\frac{1-\kappa}{2}} t$ also converges weakly to a Wiener process w^a with variance σ_a^2. Furthermore, by Assumption 2.1, w^d and w^a are independent. Let

$$B^{d,n}(t) := \frac{1}{n^{1-\kappa}}\int_0^{tn^{1-\kappa}}\sum_{j=1}^N I_{\{L(s')=j\}}\gamma_j u_j\,ds'\,. \tag{55}$$

Then,

$$\begin{aligned} B^{d,n}(t) \xrightarrow[n\to\infty]{} &\int_0^t \sum_{j=1}^N \pi_j\gamma_j u_j(s)\,ds \\ &= \int_0^t b\bigl(u(s)\bigr)\,ds\,, \quad \text{a.s.,} \end{aligned}$$

by functional law of large numbers (FLLN) [12]. The scaled idle time for the queue with channel state j is

$$T^n(j,t) = \frac{1}{n^{\frac{1+\kappa}{2}}}\int_0^{nt} I_{\{L^n(s)=j\}} I_{\{x^n(s)=0\}}\,ds\,. \tag{56}$$

Thus, we define

$$z^n(t) := \sum_{j=1}^N r_0(j) T^n(j,t)\,, \tag{57}$$

which can be viewed as the scaled number of bits in the queue that could have been transmitted with the power allocation $P_0(j)$. By (53)–(57),

$$D^n(t) = \lambda^a n^{\frac{1-\kappa}{2}} t + M^{d,n}(t) + B^{d,n}(t) - z^n(t)\,.$$

Thus,

$$\begin{aligned} x^n(t) &= x(0) + \bigl(\lambda^a n^{\frac{1-\kappa}{2}} t + M^{a,n}(t)\bigr) \\ &\qquad - \bigl(\lambda^a n^{\frac{1-\kappa}{2}} t + M^{d,n}(t) + B^{d,n}(t) - z^n(t)\bigr) \\ &= x(0) - B^{d,n}(t) + M^{a,n}(t) \\ &\qquad - M^{d,n}(t) + z^n(t)\,, \end{aligned} \tag{58}$$

Note that $z^n(t)$ is also the *reflection term* of the process $x^n(t)$ (e.g., see [11]), satisfying,

$$\begin{aligned} z^n(t) = \max\Bigl\{0, -\min_{s\le t}\bigl[x^n(0) - B^{d,n}(s) \\ + M^{a,n}(s) - M^{d,n}(s)\bigr]\Bigr\}\,. \end{aligned} \tag{59}$$

By the weak convergence of $M^{a,n}(t)$, $M^{d,n}(t)$ to their continuous limits on the right side of (59), $z^n(t)$ thus converges weakly to $z(t)$, where

$$\begin{aligned} z(t) = \max\biggl\{0, -\min_{s\le t}\biggl[x(0) - \int_0^s b\bigl(u(s')\bigr)\,ds' \\ + w^a(s) - w^d(s)\biggr]\biggr\}\,. \end{aligned}$$

Thus (58) converges weakly to (4) by the preceding discussion, where $\sigma^2 = \sigma_a^2 + \sigma_d^2$.

APPENDIX II
PROOFS OF THEOREM 3.3 AND LEMMA 4.2

We start with some preliminary discussion. Let $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ denote the one point compactification of $\mathbb{R}_+$ and let $\bar{\mathcal{G}}$ denote the closure of $\mathcal{G}$ in $\mathcal{P}(\bar{\mathbb{R}}_+ \times \widetilde{U})$. Since $\mathcal{P}(\bar{\mathbb{R}}_+ \times \widetilde{U})$ is compact, so is $\bar{\mathcal{G}}$, and hence any sequence of probability measures $\{\nu_k : k \in \mathbb{N}\}$ in $\mathcal{G}$ contains a subsequence which converges weakly in $\bar{\mathcal{G}}$. Furthermore, using the criterion in (9) one can show (see [18]) that any $\nu \in \bar{\mathcal{G}}$ can be decomposed as follows: there exists $\delta \in [0,1]$ and probability measures $\nu' \in \mathcal{G}$ and $\nu'' \in \mathcal{P}(\{\infty\}\times\widetilde{U})$ such that for any Borel set $B \subset \bar{\mathbb{R}}_+ \times \widetilde{U}$,

$$\nu(B) = \delta\nu'\bigl(B\cap(\mathbb{R}_+\times\widetilde{U})\bigr) + (1-\delta)\nu''\bigl(B\cap(\{\infty\}\times\widetilde{U})\bigr)\,. \tag{60}$$

We also make use of the following lemma.

Lemma 2.1: Let $\mathfrak{M}_{\mathrm{s}}(\mathbb{R}_+ \times \widetilde{U})$ denote the space of finite signed measures on $\mathbb{R}_+\times\widetilde{U}$, and let $H_1,\dots,H_n$ be half spaces of the form

$$H_i = \biggl\{\nu \in \mathfrak{M}_{\mathrm{s}}(\mathbb{R}_+\times\widetilde{U}) : \int g_i\,d\nu \le k_i\biggr\}\,,$$

where $g_i : \mathbb{R}_+ \times \widetilde{U} \to \mathbb{R}_+$ are continuous, and $k_i \in \mathbb{R}_+$, $i = 1,\dots,k$. Suppose $H_i \neq \varnothing$, for $i = 1,\dots,k$, and let $H = H_1\cap\dots\cap H_k$. Then $(\mathcal{G}\cap H)_e \subset \mathcal{G}_e$.

The proof of Lemma 2.1 is contained in [9], [26], and relies on the following: It is shown in [26] that the convex set $\mathcal{G}$, when viewed as a subset of $\mathfrak{M}_{\mathrm{s}}(\mathbb{R}_+ \times \widetilde{U})$, does not have any finite dimensional faces other than its extreme points. Since H is the intersection of a finite collection of closed half-spaces in $\mathfrak{M}_{\mathrm{s}}(\mathbb{R}_+ \times \widetilde{U})$, it has finite co-dimension in $\mathfrak{M}_{\mathrm{s}}(\mathbb{R}_+ \times \widetilde{U})$. Hence, there are no extreme points in $\mathcal{G} \cap H$, other than the ones in $\mathcal{G}_e$.

An application of Choquet's Theorem (see [18]), together with Corollary 3.2 and Lemma 2.1 yield the following.

Lemma 2.2: Let $\nu \in \mathcal{G} \cap H(\bar{p})$. Then there exists $v \in \mathfrak{U}_{\mathrm{se}}$ such that $\nu_v \in H(\bar{p})$ and

$$\int_{\mathbb{R}_+ \times \widetilde{U}} c(x)\, \nu_v(dx, d\tilde{u}) \le \int_{\mathbb{R}_+ \times \widetilde{U}} c(x)\, \nu(dx, d\tilde{u})\,.$$

We now prove Theorem 3.3 and Lemma 4.2.

Proof of Theorem 3.3: First suppose c is unbounded. Fix $\bar{p} \in (0, p_{\max}]$ and let $\{\nu_k\}$ be a sequence in $H(\bar{p})$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\nu_k \to J^*(\bar{p})\,. \tag{61}$$

Since c was assumed asymptotically unbounded, it follows that the sequence $\{\nu_k\}$ is tight in $\mathcal{P}(\mathbb{R}_+ \times \widetilde{U})$ and hence converges weakly to some ν^* in $\mathcal{P}(\mathbb{R}_+ \times \widetilde{U})$. Clearly, in view of (60), $\nu^* \in \mathcal{G}$. On the other hand, since h is continuous and bounded, and $\nu_k \to \nu^*$, weakly, we obtain

$$\int_{\mathbb{R}_+ \times \widetilde{U}} h\, d\nu^* = \lim_{k\to\infty} \int_{\mathbb{R}_+ \times \widetilde{U}} h\, d\nu_k \le \bar{p}\,.$$

Hence, $\nu^* \in H(\bar{p})$. Since the map $\nu \mapsto \int c\, d\nu$ is lower-semicontinuous on $\mathcal{G}$, we have

$$\begin{aligned}\int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\nu^* &\le \liminf_{k\to\infty} \int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\nu_k \\ &= J^*(\bar{p})\,,\end{aligned}$$

and thus ν^* attains the infimum in (15).

Now suppose c is bounded. As before, let $\{\nu_k\}$ be a sequence in $\mathcal{G}$ satisfying (61) and let $\tilde{\nu}$ be a limit point of $\{\nu_k\}$ in $\bar{\mathcal{G}}$. Dropping to a subsequence if necessary, we suppose without changing the notation that $\nu_k \to \tilde{\nu}$ in $\bar{\mathcal{G}}$, and we decompose $\tilde{\nu}$ as in (60), i.e.,

$$\tilde{\nu} = \delta\, \tilde{\nu}' + (1-\delta)\tilde{\nu}''\,,$$

with $\tilde{\nu}' \in \mathcal{G}$, $\tilde{\nu}'' \in \mathcal{P}(\{\infty\} \times \widetilde{U})$, and $\delta \in [0,1]$. Then, on the one hand

$$\delta \int_{\mathbb{R}_+ \times \widetilde{U}} h\, d\tilde{\nu}' \le \liminf_{k\to\infty} \int_{\mathbb{R}_+ \times \widetilde{U}} h\, d\nu_k \le \bar{p}\,, \tag{62}$$

while on the other, since c has a continuous extension on $\bar{\mathbb{R}}_+$ (this is a simple consequence of the fact that $\lim_{x\to\infty} c(x)$ exists, and the definition of the topology of the one-point compactification [27]),

$$\begin{aligned}J^*(\bar{p}) &= \lim_{k\to\infty} \int_{\bar{\mathbb{R}}_+ \times \widetilde{U}} c\, d\nu_k \\ &= \delta \int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\tilde{\nu}' + (1-\delta)c_\infty\,.\end{aligned} \tag{63}$$

Note that since by Assumption 3.1 c is not a constant, $J^*(\bar{p}) < c_\infty$, and hence, by (63), $\delta > 0$. Let $\tilde{v} \in \mathfrak{U}_{\mathrm{ss}}$ be the control associated with $\tilde{\nu}'$ and $f_{\tilde{v}}$ be the corresponding density of the invariant probability measure. Let $\hat{x} \in \mathbb{R}_+$ have the value

$$\hat{x} = \frac{1-\delta}{\delta f_{\tilde{v}}(0)}\,,$$

and $v^* \in \mathfrak{U}_{\mathrm{ss}}$ defined by

$$v^*(x) = \begin{cases} 0, & \text{if } x \le \hat{x} \\ \tilde{v}(x - \hat{x}), & \text{otherwise.}\end{cases}$$

The corresponding density is

$$f_{v^*}(x) = \begin{cases} \delta f_{\tilde{v}}(0), & \text{if } x \le \hat{x} \\ \delta f_{\tilde{v}}(x - \hat{x}), & \text{otherwise.}\end{cases}$$

By (62),

$$\begin{aligned}\int_{\mathbb{R}_+} h\big(v^*(x)\big) f_{v^*}(x)\, dx &= \delta \int_{\hat{x}}^\infty h\big(v^*(x)\big) f_{\tilde{v}}(x - \hat{x})\, dx \\ &= \delta \int_{\mathbb{R}_+ \times \widetilde{U}} h\, d\tilde{\nu}' \\ &\le \bar{p}\,.\end{aligned}$$

By construction $f_{v^*}(x) \ge \delta f_{\tilde{v}}(x)$, for all $x \in \mathbb{R}_+$. Hence,

$$\int_{\mathbb{R}_+} c(x)\big[f_{v^*}(x) - \delta f_{\tilde{v}}(x)\big]\, dx \le (1-\delta)c_\infty\,. \tag{64}$$

By (63)–(64),

$$\begin{aligned}\int_{\mathbb{R}_+} c(x) f_{v^*}(x)\, dx &\le \delta \int_{\mathbb{R}_+} c(x) f_{\tilde{v}}(x)\, dx + (1-\delta)c_\infty \\ &= J^*(\bar{p})\,.\end{aligned}$$

Therefore, $v^* \in \mathfrak{U}_{\mathrm{ss}}$ is optimal for (15). By Lemma 2.2, v^* may be selected in $\mathfrak{U}_{\mathrm{se}}$. ■

Proof of Lemma 4.2: For $\bar{p} \in (0, p_{\max}]$, let $\nu^{(\bar{p})} \in H(\bar{p})$ be an optimal ergodic measure, i.e.,

$$\int_{\mathbb{R}_+ \times \widetilde{U}} c\, d\nu^{(\bar{p})} = J^*(\bar{p})\,.$$

Denote by $v^{(\bar{p})} \in \mathfrak{U}_{\mathrm{ss}}$ the associated optimal control, and let $f_{v^{(\bar{p})}}$ stand for the density of the invariant probability measure. Set $\hat{x} = [f_{v^{(\bar{p})}}(0)]^{-1}$, and define $v^* \in \mathfrak{U}_{\mathrm{ss}}$ by

$$v^*(x) := \begin{cases} 0, & \text{if } x \le \hat{x} \\ v^{(\bar{p})}(x - \hat{x}), & \text{otherwise.}\end{cases}$$

We compute the density of the invariant probability measure as

$$f_{v^*}(x) = \begin{cases} \frac{f_{v^{(\bar{p})}}(0)}{2}, & \text{if } x \le \hat{x} \\ \frac{f_{v^{(\bar{p})}}(x-\hat{x})}{2}, & \text{otherwise.}\end{cases}$$

Then,

$$\int_{\mathbb{R}_+} h\big(v^*(x)\big) f_{v^*}(x)\, dx = \frac{\bar{p}}{2}\,.$$

Observe that $f_{v^*}(x) \ge \frac{1}{2} f_{v^{(\bar{p})}}(x)$, for all $x \in \mathbb{R}_+$. Hence, since $c(x) < c_\infty$, for all $x \in \mathbb{R}_+$, we obtain

$$\begin{aligned}J^*\big(\tfrac{\bar{p}}{2}\big) - \tfrac{1}{2} J^*(\bar{p}) &\le \int_{\mathbb{R}_+} c(x)\big[f_{v^*}(x) - \tfrac{1}{2} f_{v^{(\bar{p})}}(x)\big]\, dx \\ &< \tfrac{1}{2} c_\infty\,,\end{aligned}$$

which yields the desired result. ■

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